# SOLUTIONS TO HW #11

## Chapter 13

**18.** Well,

$$(4+i)(4+4i) = 4.4 + (4+4.4)i + 4.4.i^2 = 1 + 4 + (4+1)i = 0.$$

Since neither of $(4+i$ and $(4+4i)$ are zero, they are zero divisors.

**22.** **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function with some $\alpha \in \mathbb{R}$ so that $f(\alpha) = 0$.

Define $g : \mathbb{R} \to RR$ by $g(x) = 0$ if $x \neq \alpha$ and $g(\alpha) = 0$. Then for all $x \in \mathbb{R}$ we have $(f.g)(x) = 0$. If $f$ is not the zero function, it is a zero divisor.

Conversely, suppose that $f(\alpha) \neq 0$ for all $\alpha \in \mathbb{R}$, but that $(f.g)$ is the zero function. This implies that $g(\alpha) = 0$ for all $\alpha \in \mathbb{R}$, so $g$ is the zero function, and so $f$ is not a zero divisor.

Therefore, the set of zero divisors is the set of functions which have a zero, but are not identically zero.

**(b)** Suppose that for some $n \in \mathbb{N}$ we have $f^n$ is the zero function. Then for all $x \in \mathbb{R}$ we have $(f(x))^n = 0$. But this means that $f(x) = 0$ for all $x \in \mathbb{R}$, and so $f$ is the zero function.

Therefore, the only nilpotent element is the zero function.

**(c)** If $f$ is not a zero divisor, we have to show that it is a unit.

Suppose that $f$ is a nonzero fiunction which is not a zero divisor. Then we know from Part (a) that $f(\alpha) \neq 0$ for all $\alpha \in \mathbb{R}$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \frac{1}{f(x)}$. We know that this is well-defined, and that for all $x \in \mathbb{R}$ we have $(g.f)(x) = \frac{1}{f(x)}f(x) = 1$, so $g.f$ is the function which is identically 1, which is the multiplicative identity. Therefore $f$ is a unit.

## Chapter 14

**4.** Let $S = \{(m,m) \mid m \in \mathbb{Z}\} \subset \mathbb{Z} \oplus \mathbb{Z}$.

It is straightforward to see that $S$ is a subring of $\mathbb{Z} \oplus \mathbb{Z}$.

However,

$$(1,0) = (1,1).(1,0) \notin S,$$

so $S$ is not an ideal.

**8.** Let $R$ be a ring, let $\Lambda$ a set, and consider a set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of $R$. Let $I = \bigcap_{\lambda\in\Lambda} I_\lambda$.

Then since $0 \in I_\lambda$ for all $\lambda$, we have $0 \in I$.

Now suppose $a, b \in I$ and $r \in R$.

Then, $a, b \in I_\lambda$ for all $\lambda$. Hence, since $I_\lambda$ is an ideal, we must have, for all $\lambda \in \Lambda$,

$$a + b \in I_\lambda, -a \in I_\lambda, ab \in I_\lambda, ar, ra \in I_\lambda.$$

This means that $a + b, -a, ab, ar, ra \in I$, so $I$ is an ideal as required.

**10.** Well, $0 = 0 + 0 \in A + B$.

Suppose that $x, y \in A + B$ and $r \in R$. Then $x = a_1 + b_1$ and $y = a_2 + b_2$ for $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Since $A$ is an ideal, we have

$$a_1 + a_2, -a_1, a_1a_2, a_1b_2, a_1r, ra_1 \in A$$

and, since $B$ is an ideal,

$$b_1 + b_2, -b_1, b_1b_2, b_1a_2, b_1r, rb_1 \in B.$$

Now,

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2) \in A + B,$$
$$-(a_1 + b_1) = (-a_1) + (-b_1) \in A + B,$$
$$(a_1 + b_1)(a_2 + b_2) = (a_1a_2 + a_1b_2) + (b_1a_2 + b_1b_2) \in A + B,$$
$$(a_1 + b_1)r = (a_1r) + (b_1r) \in A + B,$$

and

$$r(a_1 + b_1) = (ra_1) + (rb_1) \in A + B.$$

This proves that $A + B$ is an ideal of $R$.

**12.** First, $0 = 0.0 \in AB$.

Now suppose that $a_1, \ldots, a_n, c_1, \ldots, c_m \in A$ and $b_1, \ldots, b_n, d_1, \ldots, d_m \in B$, and that $r \in R$. For each $i$ we have

$$b_ir \in B, -a_i, ra_i \in A, b_i(\sum_{j=1}^{m} c_jd_j) \in B.$$

Then

$$\begin{aligned}
(\sum_{i=1}^{n} a_ib_i) + (\sum_{j=1}^{m} c_jd_j) &= a_1b_1 + \ldots + a_nb_n + c_1d_1 + \ldots + c_md_m \in AB \\
-(\sum_{i=1}^{n} a_ib_i) &= \sum_{i=1}^{n}(-a_i)b_i \in AB \\
(\sum_{i=1}^{n} a_ib_i)(\sum_{j=1}^{m} c_id_i) &= \sum_{i=1}^{n} a_i(b_i(\sum_{j=1}^{m} c_jd_j)) \in AB \\
(\sum_{i=1}^{n} a_ib_i)r &= \sum_{i=1}^{n} a_i(b_ir) \in AB \\
r(\sum_{i=1}^{n} a_ib_i) &= \sum_{i=1}^{n}(ra_i)b_i \in AB.
\end{aligned}$$

This proves that $AB$ is an ideal.

**14.** Since $A$ is an ideal, for all $a \in A$, $b \in B$ we have $ab \in A$. Since $B$ is an ideal, for all $a \in A$ and $b \in B$ we have $ab \in B$. Consider $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$. Then, for each $i$ we have $a_ib_i \in A \cap B$. Since $A \cap B$ is an ideal, we have

$$\sum_{i=1}^{n} a_ib_i \in A \cap B.$$

This proves that $AB \subseteq A \cap B$.