Math 330 - Abstract Algebra I **Spring 2009**

SOLUTIONS TO HW #6

Chapter 6

24. Let $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ and

$$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Q} \right\}.$$

Show that G and H are isomorphic under addition. Prove that G and H are closed under multiplication. Does your isomorphism preserve multiplication as well as addition?

Solution:

Let $R : G \to H$ be the map defined by $R(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$. Clearly, R is a bijection between G and H. We now prove that it preserves the group structure.

The identity element of G is 0, which can be represented as $0 + 0\sqrt{2}$. Then,

$$R(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is the identity of H.

Now, $(a + b\sqrt{2})^{-1} = -a - b\sqrt{2}$, and

$$\begin{aligned} R(-a - b\sqrt{2}) &= \begin{bmatrix} -a & -2b \\ -b & -a \end{bmatrix} \\ &= \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}^{-1} \end{aligned}$$

Finally, for $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ we have

$$\begin{aligned} R((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) &= R((a_1 + a_2) + (b_1 + b_2)\sqrt{2}) \\ &= \begin{bmatrix} a_1 + a_2 & 2(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix} \\ &= R(a_1 + b_1\sqrt{2}) + R(a_2 + b_2\sqrt{2}) \end{aligned}$$

Thus R is an isomorphism as required.

We now turn to multiplication.

$$\begin{aligned} (a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) &= a_1a_2 + (a_1b_2 + b_1a_2)\sqrt{2} + 2b_1b_2 \\ &= (a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2} \end{aligned}$$

On the other hand,

$$\begin{aligned} R(a_1 + b_1\sqrt{2}).R(a_2 + b_2\sqrt{2}) &= \begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1a_2 + 2b_1b_2 & 2a_1b_2 + 2b_1a_2 \\ b_1a_2 + b_2a_1 & 2b_1b_2 + a_1a_2 \end{bmatrix} \\ &= R((a_1a_2 + 2b_1b_2) + (a_1b_2 + b_1a_2)\sqrt{2}) \\ &= R((a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2})) \end{aligned}$$

So indeed R does preserve multiplication.

28. Let $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$. Show that the mapping

$$\phi : (a_1, a_2, \ldots, a_n) \to (-a_1, -a_2, \ldots, -a_n)$$

is an automorphism of the group $\mathbb{R}^n$ under component-wise addition. This automorphism is called *inversion*. Describe the action of ϕ geometrically.

Solution:

It is clear that $\phi \circ \phi$ is the identity, so ϕ has an inverse and must be a bijection. Now, we have

$$\phi(0, \ldots, 0) = (0, \ldots, 0)$$

and $(a_1, \ldots, a_n)^{-1} = (-a_1, \ldots, -a_n)$, so

$$\begin{aligned} \phi((a_1, \ldots, a_n)^{-1}) &= \phi(-a_1, \ldots, -a_n) \\ &= (a_1, \ldots, a_n) \\ &= (-a_1, \ldots, -a_n)^{-1} \\ &= \phi(a_1, \ldots, a_n)^{-1}. \end{aligned}$$

Finally,

$$\begin{aligned} \phi\left((a_1, \ldots, a_n) + (b_1, \ldots, b_n)\right) &= \phi(a_1 + b_1, \ldots, a_n + b_n) \\ &= (-(a_1 + b_1), \ldots, -(a_n + b_n)) \\ &= (-a_1, \ldots, -a_n) + (-b_1, \ldots, -b_n) \\ &= \phi(a_1, \ldots, a_n) + \phi(b_1, \ldots, b_n). \end{aligned}$$

This proves that ϕ is an automorphism of $\mathbb{R}^n$.

Geometrically, the effect of ϕ as as follows:

Let $x = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and let p be the line in $\mathbb{R}^n$ through the origin and x. Then $\phi(x)$ is the point on p which lies on p the same distance from the origin as x but on the other side.

In $\mathbb{R}^3$, when restricted to the unit sphere, $\phi(x)$ is the antipodal point of x.

32. Show that the mapping $a \to \log_{10} a$ is an isomorphism from $\mathbb{R}^+$ under multiplication to $\mathbb{R}$ under addition.

Solution: First, $\log_{10} : \mathbb{R}^+ \to \mathbb{R}$ is a bijection (since $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = 10^x$ is the inverse function).

Now,

$$\log_{10}(1) = 0,$$

so the identity of $\mathbb{R}^+$ is sent to the identity of $\mathbb{R}$. Now, the inverse of a in $\mathbb{R}^+$ is $\frac{1}{a}$ and the inverse of b in $\mathbb{R}$ is $-b$. We have

$$\log_{10}(\frac{1}{a}) = -\log_{10}(a),$$

so inverses are preserved by $\log_{10}$.

Finally,

$$\log_{10}(ab) = \log_{10}(a) + \log_{10}(b)$$

so $\log_{10}$ preserves the group operation, and so it is an isomorphism.

38. In Aut($\mathbf{Z_9}$), let α_i denote the autmorphism that sends 1 to i where $\gcd(i, 9) = 1$. Write α_5 and α_8 as permutations of $\{0, 1, 2, \ldots, 8\}$ in disjoint cycle form.

Solution: The map $\alpha_5 : \mathbf{Z_9} \to \mathbf{Z_9}$ is multiplication modulo 9. Thus we can make the following table.

x	0	1	2	3	4	5	6	7	8
$\alpha_5(x)$	0	5	1	6	2	7	3	8	4

Which can be written in cycle form as

$$(0)(1\ 5\ 7\ 8\ 4\ 2)(3\ 6).$$

Similarly, α_8 is multiplication modulo 8, and we can make a table.

x	0	1	2	3	4	5	6	7	8
$\alpha_8(x)$	0	8	7	6	5	4	3	2	1

which can be written in cycle form as

$$(0)(1\ 8)(2\ 7)(3\ 6)(4\ 5).$$

Chapter 7.

6. Let n be a positive integer. Let $H = \{0, \pm n, \pm 2n, \pm 3n, \ldots\}$. Find all left cosets of H in $\mathbb{Z}$. How many are there?

Solution: Let $x, y \in \mathbb{Z}$. Then $xH = yH$ if and only if $x - y \in H$. Since H consists of all multiples of n, we see that $x - y \in H$ if and only if $x - y$ is divisible by n. That is to say that xH and yH are the same coset if and only if $x \equiv y (\text{mod } n)$. Therefore the cosets of H in $\mathbb{Z}$ are the same as the equivalence classes of integers modulo n.

There are n equivalence classes, which are represented by the integers $0, \ldots, n-1$. They are

$$0H, 1H, \ldots, (n-1)H.$$

8. Suppose that a has order 15. Find all the left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

Solution:

$|\langle a^5 \rangle| = 3$, and $|\langle a \rangle| = 15$, so $|\langle a \rangle : \langle a^5 \rangle| = 5$, by Lagrange's Theorem. So there are 5 cosets.

Now, $x\langle a^5 \rangle = y\langle a^5 \rangle$ if and only if $x^{-1}y \in \langle a^5 \rangle$. Therefore, the five cosets are

$$\langle a^5 \rangle, a\langle a^5 \rangle, a^2\langle a^5 \rangle, a^3\langle a^5 \rangle, a^4\langle a^5 \rangle.$$

(It is easy to see using the above criterion that these are distinct cosets, and we know that there are five, so this is all of them.)