SOLUTIONS TO HW #8

Chapter 8

2. Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has seven subgroups of order 2.

Solution: We can list the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ explicitly, and there are 8 of them:

$([0],[0],[0]),([1],[0],[0]),([0],[1],[0]),([1],[1],[0]),([0],[0],[1]),([1],[0],[1]),([0],[1],[1]),([1],[1],[1])$.

Now $[0]+[0] = [1]+[1] = [0]$ in $\mathbb{Z}_2$, and the direct sum construction is defined with operation:

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2).$$

Thus, if we take any $a = (x, y, z)$ element in $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ then we must have $a^2 = (x, y, z)(x, y, z) = (x + x, y + y, z + z) = ([0], [0], [0])$, the identity of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

There are seven elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of order 2 (every element except e), and for each such a there is a subgroup of order 2, namely $\{e, a\}$. This gives seven different subgroups.

However, this is all of the subgroups of order 2, since a subgroup of order 2 has e and one other element.

4. Show that $G \oplus H$ is abelian if and only if G and H are abelian.

Solution: Suppose that G and H are abelian, and that $(g_1, h_1), (g_2, h_2) \in G \oplus H$. Then

$$\begin{aligned} (g_1, h_1)(g_2, h_2) &= (g_1g_2, h_1h_2) \\ &= (g_2g_1, h_2h_1) \text{ since } G \text{ and } H \text{ are abelian,} \\ &= (g_2, h_2)(g_1, h_1). \end{aligned}$$

Therefore $G \oplus H$ is abelian.

Conversely, suppose $G \oplus H$ is abelian and let $g_1, g_2 \in G, h_1, h_2 \in H$. Then

$$\begin{aligned} (g_1g_2, h_1h_2) &= (g_1, h_1)(g_2, h_2) \\ &= (g_2, h_2)(g_1, h_1) \text{ since } G \oplus H \text{ is abelian} \\ &= (g_2g_1, h_2h_1). \end{aligned}$$

Therefore, $g_1g_2 = g_2g_1$, and G is abelian, and $h_1h_2 = h_2h_1$, so H is abelian.

12. The dihedral group D_n of order $2n$ $(n \geq 3)$ has a subgroup of n rotations and a subgroup of order 2. Explain why D_n cannot be isomorphic to the external direct product of two such groups.

Solution: The rotation subgroup of D_n is abelian (we've seen this in class many times), and the subgroup of order 2 is abelian (since we know that the only group of order 2, up to isomorphism, is the cyclic group of order 2).

Therefore, the direct product of the rotation subgroup and a group of order 2 is abelian, by Question 4. But if $n \geq 3$, then D_n is not abelian. Therefore, D_n cannot be a direct product of these two groups.

Chapter 9

4 Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $\mathrm{GL}(2, \mathbb{R})$?

Solution:

Let $x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$. Then

$$x^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let

$$y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H.$$

We calculate:

$$\begin{aligned} x^{-1}yx &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Therefore, $x^{-1}yx \notin H$. But, if H were normal then $x^{-1}Hx \subseteq H$ for any $x \in \mathrm{GL}(2, \mathbb{R})$. So H is not normal in $\mathrm{GL}(2, \mathbb{R})$.

10 Prove that a factor group of a cyclic group is cyclic.

Solution: Suppose that $G = \langle a \rangle$ and that $H \trianglelefteq G$. An element of G/H has the form gH for some $g \in H$. Each element g can be written as a^k for some k. Now

$$a^k H = (aH)^k,$$

(as can be seen by an easy inductive proof, and the definition of the product in G/H.)

Therefore $G/H = \langle aH \rangle$ is cyclic, as required.

14 What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$.

Solution: $\langle 8 \rangle$ has order 3 in $\mathbb{Z}_{24}$, so $\mathbb{Z}_{24}/\langle 8 \rangle$ is cyclic of order 8, generated by $[1] + \langle 8 \rangle$. Now $\langle 8 \rangle = \{[0], [8], [16]\} \subset \mathbb{Z}_{24}$. We can calculate explicitly:

$$\begin{aligned} [14] + \langle 8 \rangle &\neq \langle 8 \rangle \text{ (since } 14 \notin \langle 8 \rangle) \\ ([14] + \langle 8 \rangle)^2 &= [28] + \langle 8 \rangle \\ &= [4] + \langle 8 \rangle \neq \langle 8 \rangle \\ ([14] + \langle 8 \rangle)^3 &= [42] + \langle 8 \rangle \\ &= [18] + \langle 8 \rangle \neq \langle 8 \rangle \\ ([14] + \langle 8 \rangle)^4 &= [56]\langle 8 \rangle \\ &= [8] + \langle 8 \rangle = \langle 8 \rangle \end{aligned}$$

Therefore, $[14] + \langle 8 \rangle$ has order 4 in $\mathbb{Z}_{24}/\langle 8 \rangle$.