SOLUTIONS TO HW #7

Chapter 9

16. The order of $R_{60}Z(D_6)$ in $D_6/Z(D_6)$ is 3, since
$(R_{60}Z(D_6))^2 = R_{60}R_{60}Z(D_6) = R_{120}Z(D_6)$, while
$(R_{60}Z(D_6))^3 = R_{60}R_{60}R_{60}Z(D_6) = R_{180}Z(D_6) = R_0Z(D_6)$, the identity element in the factor group.

18. By Lagrange's Theorem, since $|Z_{60}| = 60$ and $|\langle 15\rangle| = 4$,

$$|Z_{60}/\langle 15\rangle| = \frac{|Z_{60}|}{|\langle 15\rangle|} = \frac{60}{4} = 15.$$

22. Notice that the set $\{(n,0) | n \in \mathbb{Z}\}$ is an infinite subset of $(\mathbb{Z}\oplus\mathbb{Z})/\langle(2,2)\rangle$, since $(n,0)+\langle(2,2)\rangle = (m,0)+\langle(2,2)\rangle$ iff $n = m$. Hence this factor group is of infinite order. We now show that this group cannot be cyclic.

Claim. An infinite cyclic group cannot contain an element of finite order.
Proof. Suppose G is an infinite cyclic group, say $G = \langle a\rangle$, and $b \in G$ is an element of finite order, say $|b| = n$. Then since a generates G,

$$b = \underbrace{a + a + \cdots + a}_{k} = ka$$

for some $k \in \mathbb{Z}$, so that

$$0 = \underbrace{b + \cdots + b}_{n} = nb = n(ka) = (nk)a = \underbrace{a + \cdots + a}_{nk}$$

or in other words, a has finite order. But a generates an infinite cyclic group, so this is a contradiction. This proves the claim.

To show that $(\mathbb{Z}\oplus\mathbb{Z})/\langle(2,2)\rangle$ is not cyclic, then, we need only to produce an element of finite order. Note that $((1,1)+\langle(2,2)\rangle)+((1,1)+\langle(2,2)\rangle) = (2,2)+\langle(2,2)\rangle = (0,0)+\langle(2,2)\rangle$. This completes the proof.

Chapter 10

6. Let $\varphi : G \to G$ be the homomorphism defined in the problem. Then we examine what φ does to a general element of G:

$$\int (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)dx = a_nx^{n+1}/(n+1) + a_{n-1}x^n/n + \cdots + a_1x^2/2 + a_0x + C$$

But the initial conditions $(0,0)$ force $C = 0$. So if F is the antiderivative of f under these initial conditions, then $\varphi(f) = F$.
Now let $f_1 = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and $f_2 = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$ and suppose (without loss of generality) that $n > m$. So by the initial conditions given, $\varphi(f_1)$ and $\varphi(f_2)$ will not have a constant term. Similarly, $\varphi(f_1 + f_2)$ will not have a constant term

either. By the properties of integrals, all the other terms will coincide. More specifically, if we denote F_1 and F_2 to be the respective antiderivatives under these initial conditions of f_1 and f_2, then

$$\varphi(f_1 + f_2) = F_1 + F_2 = \varphi(f_1) + \varphi(f_2).$$

Hence, φ is a homomorphism. The kernel of φ is the zero polynomial, since we cannot integrate any other polynomial to obtain the zero polynomial.

Now let $\psi : G \to G$ be the map that sends a polynomial f to its integral $\int f$ under the initial conditions $(0, 1)$. Then, doing the calculations on a general polynomial, one finds that $C = 1$. If we consider f_1 and f_2 as above, we have

$$\begin{aligned} \psi(f_1 + f_2) &= \psi\big(a_n x^n + \cdots + a_{m+1}x^{m+1} + (a_m + b_m)x^m + \cdots + (a_1 + b_1)x + (a_0 + b_0)\big) \\ &= a_n x^{n+1}/(n+1) + \cdots + (a_1 + b_1)x^2/2 + (a_0 + b_0)x + 1 \end{aligned}$$

while

$$\begin{aligned} \psi(f_1) + \psi(f_2) &= \psi(a_n x^n + \cdots + a_1 x + a_0) + \psi(b_m x^m + \cdots + b_1 x + b_0) \\ &= (a_n x^{n+1}/(n+1) + a_{n-1}x^n/n + \cdots + a_1 x^2/2 + a_0 x + 1) \\ &\quad + (b_m x^{m+1}/(m+1) + b_{m-1}x^m/m + \cdots + b_1 x^2/2 + b_0 x + 1) \\ &= a_n x^{n+1}/(n+1) + \cdots + (a_1 + b_1)x^2/2 + (a_0 + b_0)x + 2. \end{aligned}$$

This means that, if ψ is a homomorphism, then $1 = 2$, which is absurd. Hence ψ is not a homomorphism.

12. Define the homomorphism $f : Z_n \to Z_k$ as $f(a \bmod n) = a \bmod k$. First we check that this is well defined. Let $x, y \in Z_n \cong \mathbb{Z}/n\mathbb{Z}$ be equal, *i.e.*, $x \equiv y \bmod n$, *i.e.*, $n|(x - y)$. But we are given that $k|n$, so $k|(x - y)$, which means that $x \equiv y \bmod k$, or $x = y$ in $Z_k \cong \mathbb{Z}/k\mathbb{Z}$. This means that $f(x + n\mathbb{Z}) = x + k\mathbb{Z} = y + k\mathbb{Z} = f(y + n\mathbb{Z})$, so the map is well-defined.

Note that f is onto, since if $a + k\mathbb{Z} \in \mathbb{Z}/k\mathbb{Z}$, then there is an equivalence class $a + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ for which $f(a + n\mathbb{Z}) = a + k\mathbb{Z}$.

f is a homomorphism, since

$$f\big((a+n\mathbb{Z})+(b+n\mathbb{Z})\big) = f\big((a+b)+n\mathbb{Z}\big) = (a+b)+k\mathbb{Z} = (a+k\mathbb{Z})+(b+k\mathbb{Z}) = f(a+n\mathbb{Z})+f(b+n\mathbb{Z}).$$

$$\begin{aligned} \ker(f) &= \{a + n\mathbb{Z} | f(a + n\mathbb{Z}) = 0 + k\mathbb{Z}\} \\ &= \{a + n\mathbb{Z} | a \equiv 0 \bmod k\} \\ &= \langle k \rangle \end{aligned}$$

Hence, by the First Isomorphism Theorem, $Z_n / \langle k \rangle \cong Z_k$.

14. Define a function $\varphi : Z_{12} \to Z_{10}$ as $x \mapsto 3x$. Then if φ were a homomorphism,
$0 = \varphi(\underbrace{1 + \cdots + 1}_{12}) = \underbrace{\varphi(1) + \cdots + \varphi(1)}_{12} = \underbrace{3 + \cdots + 3}_{12} = 36 = 6$
in Z_{10}, which is a contradiction.