

HOMEWORK #4
DUE 1PM, FEBRUARY 6, 2009

- (1) Define a relation between groups by saying $G \sim H$ if there is an isomorphism $d : G \rightarrow H$. Prove that this relation is an equivalence relation.
- (2) Suppose that G and H are groups and that $d : G \rightarrow H$ is an isomorphism.
 - (a) Prove that G is abelian if and only if H is abelian.
 - (b) Prove that if $g \in G$ then the order of g is the same as the order of $d(g)$.
 - (c) For $m, n \in \mathbb{N}$, let D_n be the dihedral group of order $2n$, and C_m be the equivalence classes of integers modulo m . Prove that D_2 and C_4 are not isomorphic. Prove also that D_3 and C_6 are not isomorphic. Are D_n and C_{2n} ever isomorphic?
- (3) Let $n \in \mathbb{N}$ and let $\mathbb{Z}/n\mathbb{Z}$ be the set of equivalence classes of integers modulo n . Define multiplication in $\mathbb{Z}/n\mathbb{Z}$ coming from multiplication in \mathbb{Z} , and prove that this multiplication is well-defined.

NOTE: In this question and the next one I use the notation $\mathbb{Z}/n\mathbb{Z}$ rather than C_n because we are not thinking of this set as a group. In fact, we're thinking of it as a 'ring', a concept we'll see later in the semester, but you don't need to know about rings to answer these questions.

- (4) Let $\text{SL}(2, 2)$ be the set of 2×2 matrices with coefficients in $\mathbb{Z}/2\mathbb{Z}$ and determinant 1:

$$\text{SL}(2, 2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/2\mathbb{Z}, ad - bc = 1 \right\}.$$

(The '1' in $\mathbb{Z}/2\mathbb{Z}$ is $[1]$, the equivalence class containing 1.)

- (a) How many elements does $\text{SL}(2, 2)$ have? List them and write out a multiplication table for $\text{SL}(2, 2)$.
- (b) Is $\text{SL}(2, 2)$ abelian? (Prove your answer.)
- (c) Find an n and an isomorphism $d : \text{SL}(2, 2) \rightarrow D_n$.