Math 330 - Abstract Algebra I **Spring 2009**

PRACTICE QUESTIONS FOR FINAL - WORKED SOLUTIONS

These questions are intended to represent approximately how difficult and long the Final Exam will be, and also to indicate some of the types of questions that might arise. They should not be contrued as a complete list of the topics that are examinable.

(1) Let

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ab - cd \neq 0 \right\},$$

which is a group with the usual operation of multiplication.

Let H be the subset of matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $a, b, d \in \mathbb{R}$ and $ad \neq 0$.

(a) Prove that H is a subgroup of $\mathrm{GL}_2(\mathbb{R};$

(b) What is the identity of H?

(c) Give an explicit formula for the inverse of $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.

Solution:

(b) The identity element of $\mathrm{GL}_2(\mathbb{R})$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and this is also the identity element of H.

(c) The inverse of

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

is

$$\begin{pmatrix} \frac{1}{a} & -\frac{b}{ad} \\ 0 & \frac{1}{d} \end{pmatrix}$$

as can be easily seen by multiplying these two matrices together.

(a) We just checked that H contains the identity element.

Also, the above formula shows that if $A \in H$ then $A^{-1} \in H$. Finally, we make the following calculation to see that H is closed under multiplication:

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ 0 & d_1d_2 \end{pmatrix}$$

Note that since $a_1d_1 \neq 0$ and $a_2d_2 \neq 0$ we have $(a_1a_2)(d_1d_2) = 0$. Therefore, the product of two matrices in H is also in H, as required.

(2) Consider the following two elements of S_7:

$$\begin{array}{c|ccccccc} x & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \alpha & 3 & 4 & 7 & 1 & 2 & 5 & 6 \end{array}$$

and

$$\begin{array}{c|ccccccc} x & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \beta & 2 & 3 & 1 & 4 & 5 & 7 & 6 \end{array}$$

  (a) For each of the following elements of S_7, write them as a product of disjoint cycles: $\alpha, \beta, \alpha\beta, \beta^{-1}\alpha\beta, \alpha^{-1}\beta^{-1}\alpha\beta$;

  (b) For each the elements in part (a), calculate their order;

  (c) For each of the elements in part (a), say whether they are even or odd.

Solutions:

(a):

We have:

$$\begin{array}{rcl} \alpha & = & (1376524) \\ \beta & = & (123)(67) \\ \alpha\beta & = & (14)(275) \\ \beta^{-1} & = & (132)(67) \\ \beta^{-1}\alpha\beta & = & (1432675) \\ \alpha^{-1} & = & (1425673) \\ \alpha^{-1}\beta^{-1}\alpha\beta & = & (1276354) \end{array}$$

(b): The orders are as follows:

α has order 7

β has order 6

$\alpha\beta$ has order 6

$\beta^{-1}\alpha\beta$ has order 7

$\alpha^{-1}\beta^{-1}\alpha\beta$ has order 7

(c): A seven cycle is a product of 6 2-cycles, so α, $\beta^{-1}\alpha\beta$ and $\alpha^{-1}\beta^{-1}\alpha\beta$ are all even.

A 3-cycle is a product of 2 2-cycles, so each of β and $\alpha\beta$ are odd.

(3) Let G and H be groups, and let $\phi : G \to H$ be a homomorphism which is also onto. Let $Z(G) = \{g \in G \mid \text{ for all } g_0 \in G, g_0 g = g g_0\}$ be the center of G, and $Z(H)$ the center of H.

  (a) Prove that $\phi(Z(G)) \subseteq Z(H)$.

(b) Give an example of groups G and H and a homomorphism $\phi : G \to H$ which is *not* surjective so that $\phi(Z(G)) \nsubseteq Z(H)$. [Hint: $Z(S_3) = \{1\}$]

Solution:

(a): Suppose that $h \in H$ and $g \in Z(G)$. We have to prove that $h\phi(g) = \phi(g)h$. Well, ϕ is a surjective homomorphism, so there is some $g_0 \in G$ so that $\phi(g_0) = h$. Therefore, we have

$$\begin{aligned} h\phi(g) &= \phi(g_0)\phi(g) \\ &= \phi(g_0 g) \\ &= \phi(g g_0) \\ &= \phi(g)\phi(g_0) \\ &= \phi(g)h, \end{aligned}$$

as required.

(b): Consider $G = \mathbb{Z}/3\mathbb{Z}$, and $H = S_3$. Define $\phi : G \to H$ by $\phi(1+3\mathbb{Z}) = (123)$. Since G is abelian, $Z(G) = G$. However, as noted in the Hint, $Z(S_3) = \{1\}$. Now, $\phi(G) = \langle e, (123), (132)\rangle \nsubseteq \{1\} = Z(S_3)$.

(4) (Gallian, 14.18, p.269) Suppose that in the ring $\mathbb{Z}$ the ideal $\langle 35\rangle$ is a proper ideal of J, and J is a proper ideal of I. What are the possibilities for J? What are the possibilities for I?

Solutions:

All ideals of $\mathbb{Z}$ are of the form $m\mathbb{Z}$. We have $m\mathbb{Z} \subseteq n\mathbb{Z}$ if and only if $n \mid m$. Therefore, the ideals that contain $\langle 35\rangle = 35\mathbb{Z}$ are:

$\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}$, and $35\mathbb{Z}$. The first three have $35\mathbb{Z}$ as a proper ideal.

It is not hard to see that we must have $I = \mathbb{Z}$, whereas the possibilities for J are $5\mathbb{Z}$ and $7\mathbb{Z}$.

(5) (Gallian, 13.24, p.255) Let d be a positive integer. Prove that $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$ is a field.

Solution:

We consider two separate cases. If d has a square root in $\mathbb{Q}$ then $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}$, which is certainly a field.

Therefore, suppose that d does not have a square root in $\mathbb{Q}$. It is straightforward to check that $\mathbb{Q}[\sqrt{2}]$ is commutative:

$$\begin{aligned} (a_1 + b_1\sqrt{d})(a_2 + b_2\sqrt{d} &= (a_1a_2 + db_1b_2) + (a_1b_2 + a_2b_1)\sqrt{d} \\ &= (a_2a_1 + db_2b_1) + (a_2b_1 + a_1b_2)\sqrt{d} \\ &= (a_2 + b_2\sqrt{d})(a_1 + b_1\sqrt{d}) \end{aligned}$$

Also, $1 = 1 + 0\sqrt{d} \in \mathbb{Q}[\sqrt{d}]$, so $\mathbb{Q}[\sqrt{d}]$ is a commutative ring with unity.

We therefore need to check that every nonzero element of $\mathbb{Q}[\sqrt{d}]$ has a multiplicative inverse. So, let $a + b\sqrt{d}$ be so that $(a, b) \neq (0, 0)$. Then $a^2 - b^2d \neq 0$, since then either $b^2 = 0$, in which case a and b are both zero (which we're assuming is not the case), or else $(a/b)^2 = d$, so d has a square root in $\mathbb{Q}$, which we are also assuming is not the case.

Thus, $a^2 - b^2d \neq 0$ and so $a - b\sqrt{d} \neq 0$ also (since $a^2 - b^2d = (a + b\sqrt{d})(a - b\sqrt{d})$. Now, we have

$$(a + b\sqrt{d}).\frac{1}{a^2 - b^2d}(a - b\sqrt{d}) = 1,$$

so the multiplicative inverse of $a + b\sqrt{d}$ is

$$\frac{a}{a^2 - b^2d} - \frac{b}{a^2 - b^2d}\sqrt{d}.$$

Finally, note that $a^2 - b^2d \in \mathbb{Q}$, so $\frac{a}{a^2-b^2d}, \frac{-b}{a^2-b^2d} \in \mathbb{Q}$, which means that every element of $\mathbb{Q}[\sqrt{d}]$ has a multiplicative inverse in $\mathbb{Q}[\sqrt{d}]$, so it is a field, as required.