Math 330 - Abstract Algebra I **Spring 2009**

SPRING BREAK PRACTICE PROBLEMS - WORKED SOLUTIONS

(1) Suppose that G is a group, $H \leq G$ is a subgroup and $K \trianglelefteq G$ is a normal subgroup. Prove that $H \cap K \trianglelefteq H$.

Solution:

We know (from the first midterm, if you like) that $H \cap K \leq H$. Therefore, we need to prove that for all $h \in H$ we have $h(H \cap K)h^{-1} = H \cap K$.

Suppose that $x \in H \cap K$. Then $hxh^{-1} \in H$, since H is a subgroup, and $hxh^{-1} \in K$ since $x \in K$ and K is a normal subgroup. Therefore $hxh^{-1} \in H \cap K$. This shows that

$$h(H \cap K)h^{-1} \subseteq H \cap K.$$

On the other hand, we can multiply this equation on the left by h^{-1} and on the right by h to get

$$H \cap K = h^{-1}h(H \cap K)h^{-1}h \subseteq h^{-1}(H \cap K)h.$$

Replacing h by h^{-1} in the above argument we get

$$H \cap K \subseteq h(H \cap K)h^{-1},$$

so $h(H \cap K)h^{-1} = H \cap K$ and $H \cap K \trianglelefteq H$ as required.

(2) Let H be a subgroup of S_n. Let H_0 be the set of H consisting of even permutations.

(a) Prove that H_0 is a subgroup of H;
(b) Suppose that $y \in H \setminus H_0$. Prove that $H = H_0 \cup yH_0$. Deduce that either $H = H_0$ or $|H : H_0| = 2$.
(c) Prove that $H_0 \trianglelefteq H$.

Solutions:

(a): Well, $H_0 = H \cap A_n$, since A_n is the set of all even permutations in S_n. The intersection of two subgroups is a subgroup.

(b): Suppose that there is an element $x \in H \setminus H_0$. We need to prove that in this case $|H : H_0| = 2$.

Well, if $y \in H \setminus H_0$, then $y^{-1}x$ is an even permutation, so $y^{-1}x \in H_0$. That is to say that $yH_0 = xH_0$. Therefore, the only cosets of H_0 in H are H_0 and xH_0. This means that $|H : H_0| = 2$, as required.

(c): This follows from Question 1, since $A_n \trianglelefteq S_n$. Alternatively, we can apply the result that a subgroup of index 2 is always normal (as is a subgroup of index 1, of course).

(3) Let A be a set and $G \leq \mathrm{Sym}(A)$. Suppose that $x \in A$.

(a) Let $g \in G$. Prove that $\mathrm{Stab}_G(g.x) = g\mathrm{Stab}_G(x)g^{-1}$;

(b) Suppose that $\mathrm{Stab}_G(x) \trianglelefteq G$. Prove that for all $y \in \mathrm{Orb}_G(x)$ we have $\mathrm{Stab}_G(y) = \mathrm{Stab}_G(x)$.

Solutions:

(a): Suppose that $h \in \mathrm{Stab}_G(x)$. Then

$$gyg^{-1}.(g.x) = gy(g^{-1}g).x = gy.x = g.x,$$

so $gyg^{-1} \in \mathrm{Stab}_G(g.x)$. Therefore, $g\mathrm{Stab}_G(x)g^{-1} \subseteq \mathrm{Stab}_G(g.x)$.

Conversely, suppose that $z \in \mathrm{Stab}_G(g.x)$. Then the same argument as above proves that $(g^{-1}zg).x = x$, so $g^{-1}zg \in \mathrm{Stab}_G(x)$. This proves that $z = g(g^{-1}zg)g^{-1} \in g\mathrm{Stab}_G(x)g^{-1}$, and so

$$\mathrm{Stab}_G(g.x) \subseteq g\mathrm{Stab}_G(x)g^{-1},$$

and these two sets must be equal, as required.

(b): Suppose that $\mathrm{Stab}_G(x) \trianglelefteq G$. Then for all $g \in G$ we have $g\mathrm{Stab}_G(x)g^{-1} = \mathrm{Stab}_G(x)$ (by the definition of normal subgroups).

Now let $y \in \mathrm{Orb}_G(x)$, and let $h \in G$ be such that $y = h.x$. By Part (a), we have

$$\mathrm{Stab}_G(y) = h\mathrm{Stab}_G(x)h^{-1},$$

but we've just noted that this is still equal to $\mathrm{Stab}_G(x)$, since this subgroup is normal.

(4) Let $A \subseteq \mathbb{R}^2$ be an equilateral triangle centered at the origin, and let $X = A \times [-1, 1]$ be a triangular prism in $\mathbb{R}^3$. Let G be the group of rotations in $\mathbb{R}^3$ which fix the origin and send X to itself (as a set).

(a) Let $a = (0, 0, 1)$ be the point in the middle of the 'top' triangle of X. What is the order of the stabiliser of a in G? What is the size of $\mathrm{Orb}_G(a)$? What is the size of G?

(b) Let $(x, y) \in A$ be one of the corners of A, and consider the point $b = (x, y, 0)$ be the corresponding point in X. What is the size of $\mathrm{Orb}_G(b)$?

(c) Show that if $g \in G \smallsetminus \{1\}$ then there is some $y \in \mathrm{Orb}_G(x)$ so that $g.y \neq y$.

(d) Find a dihedral group and a symmetric group, each isomorphic to G.

Solutions:

(a): There are three rotations which fix the point a. Also, the orbit of a is $\{a, (0, 0, -1)\}$, which has size 2. The Orbit-Stabilizer Theorem says that G has size $3.2 = 6$.

(b): The size of $\mathrm{Orb}_G(b)$ is 3. This is because the

(c): There was a typo in this question. Sorry about this. It was meant to say $y \in \mathrm{Orb}_G(b)$, not $\mathrm{Orb}_g(x)$.

In this case we can say that if $g \notin \mathrm{Stab}_G(b)$ then $g.b \neq b$. However, if $g \in \mathrm{Stab}_G(b)$ then g is the rotation which fixes b and interchanges the top and bottom triangle. This rotation fixes b and interchanges the other two vertices in the triangle $A \times \{0\}$.

(d): The symmetric group must be S_3 and the dihedral group must be D_3, just by counting. We already know that these two groups are isomorphic. To see that G is isomorphic to S_3, note that we know that any element of G permutes the three elements of $\mathrm{Orb}_G(b)$, and that any nontrivial element of G is a nontrivial permutation.

Therefore, we get a one-to-one homomorphism from G to the symmetric group on $\mathrm{Orb}_G(b)$, which has size 6 and is isomorphic to S_3. This homomorphism must be onto, since G and S_3 both have size 6.

(5) Prove the Second and Third Isomorphism Theorems (see Gallian, Exercises 39, 40, Chapter 10, p. 213).

I think these are proved in Gallian. Let me know if this isn't correct. (I'm away and don't have my copy of Gallian with me).

(6) Gallian, Ch.6: # 10, # 12. Ch. 10: # 10, #16, #18.

#6.10: Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all g in G is an automorphism if and only if G is abelian.

Solution: Suppose that α is a homomorphism, and let $g, h \in G$. Then

$$\begin{aligned} gh &= \alpha(g^{-1})\alpha(h^{-1}) \\ &= \alpha(g^{-1}h^{-1}) \\ &= \alpha((hg)^{-1}) \\ &= hg, \end{aligned}$$

so G is abelian, as required.

Conversely, suppose that G is abelian. Then, for any $g, h \in G$ we have

$$\begin{aligned} \alpha(gh) &= (gh)^{-1} \\ &= h^{-1}g^{-1} \\ &= g^{-1}h^{-1}, \text{ since } G \text{ is abelian} \\ &= \alpha(g)\alpha(h), \end{aligned}$$

so α is a homomorphism, as required.

#6.12: Find two groups G and H so that $G \not\cong H$ but $\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$.

Solution:

We know that $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The two automorphisms are the identity map and the map $x \to -x$.

Also, $\mathrm{Aut}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. We did this in class in the general case of a cyclic group, but the two automorphisms are the identity map and the map $x \to x^2$.

Clearly $\mathbb{Z} \not\cong \mathbb{Z}/3\mathbb{Z}$.

#10.10: Let G be a subgroup of some dihedral group. For each x in G, define

$$\phi(x) = \begin{cases} +1, & \text{if } x \text{ is a rotation} \\ -1, & \text{if } x \text{ is a reflection} \end{cases}$$

Prove that ϕ is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel of ϕ?

Solution: Let α_1, α_2 be rotations and γ_1, γ_2 be reflections. We know that $\alpha_1\alpha_2$ is a rotation, that $\alpha_i\gamma_j$ and $\gamma_i\alpha_j$ are reflections and that $\gamma_1\gamma_2$ is a rotation.

Therefore:

$$\begin{aligned} \phi(\alpha_1\alpha_2) &= 1 = 1.1 = \phi(\alpha_1)\phi(\alpha_2) \\ \phi(\gamma_1\gamma_2 &= 1 = (-1)(-1) = \phi(\gamma_1)\phi(\gamma_2) \\ \phi(\gamma_i\alpha_j) &= -1 = (-1).1 = \phi(\gamma_i)\phi(\alpha_j) \\ \phi(\alpha_i\gamma_j) &= -1 = 1.(-1) = \phi(\alpha_i)\phi(\gamma_j). \end{aligned}$$

This check shows that ϕ is a homomorphism.

The kernel of ϕ is the set of elements mapping to 1, which is the set of rotations in G.

10.16: Prove that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Solution: These groups have the same order (16), so an onto homomorphism would be a one-to-one homomorphism, and would have to be an isomorphism.

However, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ has an element of order 8, and $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ does not have any element of order 8, so the two groups are not isomorphic.

10.18: Can there be a homomorphism from $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ onto $\mathbb{Z}_8$? Can there be a homomorphism from $\mathbb{Z}_{16}$ onto $\mathbb{Z}_2 \oplus \mathbb{Z}_2$? Explain your answers.

Solutions:

There is no homomorphism from $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ because any element in the image of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ under a homomorphism has order dividing 4, and $\mathbb{Z}_8$ has an element of order 8.

Any quotient group of a cyclic group is cyclic. Therefore, the image of $\mathbb{Z}_{16}$ under a homomorphism is a cyclic group, and so cannot be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.