Graded Homework 4: Due Wednesday, May 1 2019 at the beginning of class

1. (10 points) Let a and b be natural numbers. A natural number m is a *least common multiple* of a and b if:

   (a) $a|m$ and $b|m$; and

   (b) For any n so that $a|n$ and $b|n$ we have $m \leq n$.

   Prove that a and b have a least common multiple.

   [HINT: You could define a set of common multiples, prove it is non-empty and use the Well-Ordering Principle.]

2. (10 points) Prove Proposition 9 from the Induction Worksheet.

   Proposition 9. *For any natural number n we have*

   $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. (10 points)

   (a) Prove Proposition 7 from the worksheet "Rings and Fields, II"

   Proposition 7. *Let $\mathbb{F}$ be a field, and let $a, b \in \mathbb{F}$. If $ab = 0_{\mathbb{F}}$, then $a = 0_{\mathbb{F}}$ or $b = 0_{\mathbb{F}}$.*

   (b) Prove Theorem 55 from the Number Theory Worksheet.

   Theorem 55. *Let $n > 1$ be a natural number. Then there exists a prime p so that $p|n$.*

4. (10pts) Let X be the set $\mathbb{Z} \times \mathbb{N}$ consisting of pairs (u, v) where u is an integer and v is a natural number. Define a relation on X by $(x, y) \sim (r, s)$ if $xs - yr = 0$.

   (a) Prove that this is an equivalence relation.

   (b) Prove that in each equivalence class $[(a, b)]$ there is a unique element (p, q) so that for any $(x, y) \in [(a, b)]$ we have $q \leq y$.

   (c) Fix an equivalence class $[(a, b)]$, and find the element (p, q) as in the previous part (where the second element is smallest). Prove that

   $$[(a, b)] = \{(pn, qn) \mid n \in \mathbb{N}\}.$$

   (HINT: When thinking about this question, you might want to think about a pair $(a, b) \in \mathbb{Z} \times \mathbb{N}$ as the fraction $\frac{a}{b}$. This might help you figure out what's going on, but you **SHOULD NOT** use fractions in your proofs.)