1 The Principle of Mathematical Induction

Theorem 1 (Principle of Mathematical Induction)**.** *Suppose that d is a natural number. Suppose that $A(n)$ is a mathematical statement which depends on a natural number n. Suppose further that the following two statements are true:*

1. *$A(d)$;*
2. *For all natural numbers $k \geq d$, if $A(k)$ is true then $A(k+1)$ is true.*

Then $A(n)$ is true for every natural number $n \geq d$.

The special case where $d = 1$ is the following:

Theorem 2. *Suppose that $A(n)$ is a mathematical statement which depends on a natural number n. Suppose further that the following two statements are true:*

1. *$A(1)$;*
2. *For all natural numbers k, if $A(k)$ is true then $A(k+1)$ is true.*

Then $A(n)$ is true for every natural number n.

Exercise 3. *Which of the following are 'mathematical statements which depend on a natural number'? Why or why not?*

$$(1)\ \mathrm{Q}(n) : n. \qquad (2)\ \mathrm{R}(x) : x = 2x \qquad (3)\ \mathrm{S}(a) : \int_0^a x^2 dx$$

$$(4)\ \mathrm{T}(m) : \sum_{i=1}^{m} i = \frac{m(m+1)}{2}. \qquad (5)\ \mathrm{U}(n) : \sum_{n=1}^{100} n = 5050.$$

Exercise 4. *Prove the Principle of Mathematical Induction (PMI), assuming the Well-Ordering Principle.*

A *proof by induction* involves setting up a statement that we want to prove for all natural numbers[1], and then proving that the two statements (1) and (2) hold for that statement.

Exercise 5. *Let $D(n)$ be a mathematical statement which depends on a natural number n. Write a proof that $D(n)$ is true for all natural numbers n, using the PMI. This proof will have some holes in it (specific to actual statement $D(n)$). Specify what goes in these holes to make it a correct proof.*

[1] This must be a 'mathematical statement depending on a natural number'

Notation 6. *Suppose that $Z(n)$ is a mathematical statement involving a natural number n, and that we want to prove that $Z(n)$ holds for all natural numbers n using the PMI. The work involved in prove that (1) holds for $Z(n)$ is called the* base case. *The work is to prove that $Z(1)$ is true.*

Exercise 7. *For each of the following statements, suppose you want to prove them true for all natural numbers using the PMI. Write the proof from the beginning until you've proved the base case.*

(1) $\mathrm{B}(n)$: *Every group of n horses is the same color.* (2) $\mathrm{C}(m) : m < 2$.

$$(3)\ \mathrm{F}(j) : \sum_{i=1}^{j} i^3 = \frac{j^2(j+1)^2}{4}.$$

(4) $\mathrm{G}(k)$: *If a and b are integers, n is a natural number and $a \equiv b \bmod n$ then $a^k \equiv b^k \bmod n$.*

Proposition 8. *For any natural number m we have*

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}.$$

Proposition 9. *For any natural number n we have*

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proposition 10. *For any natural number j we have*

$$\sum_{i=1}^{j} i^3 = \frac{j^2(j+1)^2}{4}.$$

Proposition 11. *For every integer $n \geq 5$, $2^n > n^2$.*

Proposition 12. *Suppose that n is a natural number, and that $a_1, a_2, \ldots, a_n$ are natural numbers. Let p be a prime. If $p|(a_1 a_2 \cdots a_n)$ then there exists an i with $1 \leq i \leq n$ such that $p|a_i$.*

Definition 13. *For a non-negative integer n, define the quantity $n!$ (read as 'n factorial') as follows:*

$$0! = 1$$

and for all natural numbers n,

$$n! = n.(n-1)!.$$

Proposition 14. *For every integer $n \geq 4$, $n! \geq 2^n$.*

Proposition 15. *For every non-negative integer n and for every real number $x \neq 1$,*

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Proposition 16. *For every integer $n \geq 1$ and for any real numbers $x_1, x_2, \ldots, x_n$,*

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|.$$

[For this last proposition, you may assume that for any real numbers a and b, the "triangle inequality" holds: $|a+b| \leq |a| + |b|$.]

Proposition 17. *There is an odd-numbered collection of people standing in a snowy field. No two pairs of people are the same distance apart. At the same instant, everyone throws a snowball at someone else. The person they throw at is the one closest to them.*

Prove that there is always someone who does not have a snowball thrown at them.

Proposition 18. *A triomino is an L-shaped domino tile as pictured:*

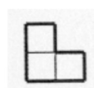

Figure 1: A triomino.

A grid of squares can be tiled with triominos *if one can place a collection of triominos onto the grid so that each square is covered by exactly one triomino.*

Let B_n be an $n \times n$ grid of squares which has one square on the corner removed.

Prove that for every natural number k the board B_{2^k} can be tiled by triominos.