FinM 331/Stat 339 Financial Data Analysis

(Applied Statistical Analysis of Financial Data in MATLAB)

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Lecture 7

6:30-9:30 pm, 16 February 2009, Ryerson 251 in Chicago

7:30-10:30 pm, 16 February 2009 at UBS Stamford

8:30-11:30 am, 17 February 2009, #02-01 Spring Singapore

7. NonParametric Regression: Options, Calibration and Implied Volatility

7.0 Vanilla, European Options:[a]

- ***7.1 Black-Scholes European Call and Put Options:***

With delta-hedging to eliminate the risk due to volatility terms and arbitrage-free conditions restricting portfolio growth to the risk-free rate, r, we are effectively dealing with the modified, underlying stock price $S(t)$ diffusion SDE,

$$dS(t) = S(t)(r dt + \sigma dW(t)), \quad S(0) = S_0,$$

where both r and the volatility σ are assumed to be constant, although that is not necessary. Black and Scholes ('73) derived a solution for a ***European style call option*** contract to buy the stock at a ***strike price*** K at a specified contract maturity date paying the ***call option contract price***

[a] In part, adapted from Carmona ('04) Ch. 4.; Hull (6th Ed., '06); D. Higham ('04); Hanson's *Applications in Financial Engineering*, Chapter 10; CBOE's Stock Options brochure ***http://www.cboe.com/LearnCenter/pdf/understanding.pdf*** .

or ***premium*** $C(S(t), t; K, T, r, \sigma)$ at current time t when underlying price $S(t)$, all governed by the final gain or strike payoff function

$$C(S(T), T; K, T, r, \sigma) = G(S(T), K) = \max[S(T) - K, 0].$$

Presumably, $G(S_0, K) = 0$, i.e., $S_0 < K$, otherwise there would be no incentive to sell the contract to the buyer, the buyer betting that the stock price will rise over K and if $G(S(T), K) = 0$ the rational buyer would walk away from the contract since the stock could be purchased more cheaply in the market.

Note that index options[a] are different, mainly that there is a cash settlement replacing the opportunity to buy stock at K and thus is closer to real betting.

Black and Scholes' well-known solution formula uses the solution of the SDE (also a backward problem for a PDE) and is

$$C^{(\mathrm{bs})}(S, t; K, T, r, \sigma) = SF_X^{(\mathrm{n})}(d_1; 0, 1) - e^{-r(T-t)}KF_X^{(\mathrm{n})}(d_2; 0, 1),$$

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[a]See CBOE's Understanding Index Options brochure: *http://www.cboe.com/LearnCenter/pdf/understandingindexoptions.pdf*.

where

$$\begin{aligned} d_1 &= d_1(S/K, T-t, r, \sigma) \\ &\equiv \left(\log(S/K) + (r+\sigma^2/2)(T-t)\right) / \left(\sigma\sqrt{T-t}\right); \\ d_2 &= d_2(S/K, T-t, r, \sigma) \\ &= d_1(S/K, T-t, r, \sigma) - \sigma\sqrt{T-t}, \end{aligned}$$

noting the natural dependence on the ***time to maturity*** $\tau \equiv T - t$, also called the ***time-to-go***, and the ***moneyness***, the ratio $M \equiv Se^{r(T-t)}/K$, so that $\log(M) = \log(S/K) + r(T-t)$. Note that $Se^{r(T-t)}$ is the ***future value*** of the current price S to maturity from t compounded at the risk-ree rate r or alternately $e^{-r(T-t)}K$ is the ***present value*** of the strike price K, available at maturity T, but discounted at the risk-free rate r back to present time t. If $M = 1$ then the option is ***at the money (ATM)***, else if $M > 1$ then it is ***in the money (ITM)***, else $M < 1$ then it is ***out the money (OTM)***. Note that at exercise, $M = S(T)/K$, so then ITM or ATM mean $K \leq S(T)$. Also, ITM is not the same as in the profit "***ITP***", since that requires $\text{Profit} = S - K - \text{Premium} > 0$.

Thus, for financial and numerical purposes, we may define a more computational finance form of the call option price

$$\widetilde{C}^{(\mathrm{bs})}(M,\tau;\sigma) = C^{(\mathrm{bs})}\left(e^{-r\tau}KM, T-\tau; K, T, r, \sigma\right) / \left(e^{-r\tau}K\right)$$
$$= MF_X^{(\mathrm{n})}(d_1;0,1) - F_X^{(\mathrm{n})}(d_2;0,1),$$

where $d_1 = \log(M)/\widetilde{\sigma} + \widetilde{\sigma}/2$ and $\widetilde{\sigma} \equiv \sqrt{\sigma^2\tau}$ which is the scaled volatility. You can verify that the scaled call price goes to the correct limit as $\tau \longrightarrow 0^+$.

The corresponding ***European put option*** is a contract to sell stock to the contract maker at K at T under an asymmetric version of the payoff,

$$\mathcal{P}(S(T),T;K,T,r,\sigma) = G(-S(T),-K) = \max[K-S(T),0],$$

with solution,

$$\mathcal{P}^{(\mathrm{bs})}(S,t;K,T,r,\sigma) = -SF_X^{(\mathrm{n})}(-d_1;0,1)+e^{-r(T-t)}KF_X^{(\mathrm{n})}(-d_2;0,1),$$

connected by a maximum and replicated portfolio derivation called the ***put-call parity***,

$$\mathcal{P}^{(\mathrm{bs})}(S,t) + S = C^{(\mathrm{bs})}(S,t) + e^{-r(T-t)}K,$$

suppressing parameter arguments.

However, as we have previously discussed, the Black-Scholes model, despite its extensive service in quantitative finance for over 35 years, has many deficiencies, like unrealistic constant coefficients (though Merton's ('73) justification paper generalized it to variable coefficients and many other things), lack of flat tails subsequent poor risk assessment, skewness, jumps, stochastic volatilities, etc.

- ***7.1 Market Calibration and Implied Volatility:***

One work-around the deficiencies with Black-Scholes formula, is to find a volatility that better fits market values of the instrument of interest, say the European call option. Hence, given market data $C^{(\mathrm{mkt})}(K_i, T_j)$ for a discrete number of strikes K_i and maturities T_j for any given option, the financial engineer will make an estimate of the volatility, and possibly other parameters, that is implied by option market rather than the underlying stock market.

When the underlying stock price data is used to to estimate the underlying volatility, then the ***log-return*** $\mathrm{LR}_i \equiv \log(S_{i+1}/S_i)$ is used, with estimated mean

$$\overline{\mathrm{LR}} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{LR}_i$$

and unbiased estimated volatility

$$\widehat{\sigma}^{(\mathrm{hist})} = \sqrt{\frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(\mathrm{LR}_i - \overline{\mathrm{LR}})^2}$$

is called the ***historical volatility***; note that in the difference approximation to the asset SDE (SΔE), $\mathrm{E}[\mathrm{LR}_i] = (\widehat{\mu} - \widehat{\sigma}^2/2)\Delta t$ rather than the risk-neutral $(r - \widehat{\sigma}^2/2)\Delta t$.

However, the call market prices are not usually given directly, but, for instance in the delayed quotes at the ***Chicago Board of Options Exchange (CBOE)***[a], they are given in terms of the latest bid and ask quotes, so usually one takes the ***average of the bid and the ask quotes*** for $C^{(\mathrm{mkt})}(K_i, T_j)$ for each contract pair (K_i, T_j).

The option market implied estimate is the so-called ***Black-Scholes***

[a] CBOE *Delayed Market Quotes* page is found for downloadat the URL: ***http://www.cboe.com/delayedquote/QuoteTableDownload.aspx***.

implied volatility (IV), $\sigma^{(\mathrm{iv})}$, by solving the ***inverse problem***[a],

$$\widetilde{C}^{(\mathrm{bs})}\left(M_{i,j}, T_j - t; \sigma^{(\mathrm{iv})}_{i,j}\right) = C^{(\mathrm{mkt})}(K_i, T_j),$$

where $M_{i,j} = e^{r(T_j - t)}S/K_i$ for fixed r and t, or the latter perhaps fixed in a small interval.

One problem in estimating volatility or variance is that they can not be directly observed but must be deduced from other observations like stock or option prices. There are also many methods for estimating implied volatility including Newton's method, maximum likelihood, kernel methods, splines and Monte Carlo methods.

However, there is not that much strike-maturity data, so pooled data is sometimes used, e.g., short maturity, medium maturity and long maturity options, or long-run historical data. Getting historical data has been harder to get, e.g., for European options, in the public domain, unless available in a company or business school.

[a]Note that $\mathbf{vega} = `\nu` = \partial C^{(\mathrm{bs})}/\partial\sigma > 0$, a volatility sensitivity measure, see D. Highham ('04), *An Introduction to Financial Option Valuation*, p. 101 and 132. Hence, the inverse should exist for Black-Scholes.

- ***7.2 Risk-Neutral Option Pricing and Implied Volatility:***

While a relatively simple solution to European call or put option pricing problem with delta ($\boldsymbol{\Delta}^{(\mathrm{bs})} \equiv \partial C^{(\mathrm{bs})}/\partial S$) hedging, the multiple sources of randomness in jump-diffusions or stochastic-volatility jump-diffusions do not allow for delta hedging to eliminate the purely diffusive risks. However, a risk-neutral formulation of the discounted, expected, conditional payoff simulates the principal properties of delta hedging. In addition, the arbitrage-free condition must be used by setting the instantaneous mean rate to the risk-free rate, $\mathrm{E}[dS(t)|S(t)=s]/(sdt)=r$, e.g., $\mu=r$ for linear diffusions or $\mu+\lambda\overline{\nu}=r$ for linear compound-jump-diffusions. Thus, for linear diffusions and more general cases, the ***current risk-neutral (RN) European style call or put option prices*** are given by

$$\begin{bmatrix} C \\ \mathcal{P} \end{bmatrix}^{(\mathrm{rn})}\left(s,t;K,T,r,\vec{\theta}\right)=e^{-r(T-t)}\mathrm{E}^{(\mathrm{rn})}[G(\pm S(T),\pm K)|S(t)=s],$$

where again $G(S,K)=\max[\pm(S-K),0]\equiv[\pm(S-K)]_+$ is the payoff function and $\vec{\theta}$ is the vector of other model parameters.

As an example of a genuine risk-neutral options model, consider the risk-neutral version of the constant-coefficient, compound-Poisson, jump-diffusion SDE asset price model underlying the option,

$$dS^{(\mathrm{rn})}(t) = S(t)((r-\lambda\overline{\nu})dt + \sigma dW(t)) + (1-\delta_{dP,0})\sum_{j=1}^{dP(t;Q)} \nu(Q_j)S(T_j^-),$$

where $\nu(Q) = \exp(Q) - 1$, $\overline{\nu} = \mathrm{E}_Q[\nu(Q)]$ and the required risk-neutral property is $\mathrm{E}[dS(t)|S(t)] = rS(t)dt$. Converting to the log-return variable $Y(t) = \log(S(t))$ using the hybrid independent continuous and jump process stochastic chain, leads to a state independent right-hand-side,

$$dY^{(\mathrm{rn})}(t) = (r - \sigma^2/2 - \lambda\overline{\nu})dt + \sigma dW(t) + (1-\delta_{dP,0})\sum_{j=1}^{dP(t;Q)} Q_j,$$

where $Q = \log(1 + \nu(Q))$ has been used. Integrating from current time t to final, contract exercise time T, with $\tau = T - t$, and exponentiating yields,

$$S^{(\mathrm{rn})}(T) = S(t)\exp\left((r-\sigma^2/2-\lambda\overline{\nu})\tau + \sigma W(\tau) + (1-\delta_{P(\tau),0})\sum_{j=1}^{P(\tau)} Q_j\right).$$

Since the diffusion and time-homogeneous Poisson processes are stationary with increments depending only on the time step, here τ, then $W(T) - W(t) = W(\tau)$ and $P(T) - P(t) = P(\tau)$. Next the scaled diffusion form, $W(\tau) = \sqrt{\tau} Z$ where Z is a mean-zero, variance-one normal RV. For notational simplicity, the $P(\tau) = k$ jump-sum is $\mathcal{S}_k \equiv (1 - \delta_{k,0}) \sum_{j=1}^{k} Q_j$ of jump-amplitudes.

Substituting this formula for $S(T)$ into the risk-neutral formula for the call option price, along with normal density, Poisson distribution and IID RV expectation, yields,

$$C^{(\mathrm{rn})}\left(s, t; K, T, r, \vec{\theta}\right) = e^{-r\tau} \sum_{k=0}^{\infty} p_k(\lambda\tau) \mathrm{E}_{\mathcal{S}_k}\left[\int_{-\infty}^{\infty} dz f_Z^{(\mathrm{n})}(z; 0, 1) \right.$$
$$\left. \cdot \max\left[s\, e^{(r - \sigma^2/2 - \lambda\overline{\nu})\tau + \sigma\sqrt{\tau} z + \mathcal{S}_k} - K, 0\right]\right]$$

Next, the payoff maximum operator will be eliminated by finding the break even point (BEP) for the payoff by finding the zero Z_0 of the first argument

$$s\,e^{(r-\sigma^2/2-\lambda\overline{\nu})\tau+\sigma\sqrt{\tau}Z_0+\mathcal{S}_k}-K=0,$$

whose solution is

$$Z_0 \stackrel{\text{alg}}{=} -\left(\log(s/K)+(r-\sigma^2/2-\lambda\overline{\nu})\tau+\mathcal{S}_k\right)/\sqrt{\sigma^2\tau}$$
$$-d_2+\left(\lambda\overline{\nu}\tau+\mathcal{S}_k\right)/\sqrt{\sigma^2\tau}\equiv -d_{2,k}\equiv -d_{1,k}+\sigma\sqrt{\tau},$$

borrowing from Black and Scholes normal argument notation, since we are following the risk-neutral procedure for the Black-Scholes formula modified for jumps.

Substitution back into the current version of risk-neutral call option price, cutting off the left tail of the integral,

$$C^{(\mathrm{rn})}\left(s,t;K,T,r,\vec{\theta}\right) = e^{-r\tau}\sum_{k=0}^{\infty} p_k(\lambda\tau)\mathrm{E}_{\mathcal{S}_k}\left[\int_{-d_{2,k}}^{\infty} dz\, f_Z^{(\mathrm{n})}(z;0,1)\left(s\, e^{(r-\sigma^2/2-\lambda\overline{\nu})\tau+\sigma\sqrt{\tau}z+\mathcal{S}_k} - K\right)\right]$$

$$\stackrel{\mathrm{alg}}{=} \sum_{k=0}^{\infty} p_k(\lambda\tau)\mathrm{E}_{\mathcal{S}_k}\Big[s\, e^{-(\sigma^2/2+\lambda\overline{\nu})\tau+\mathcal{S}_k}\int_{-d_{2,k}}^{\infty} dz\, f_Z^{(\mathrm{n})}(z;0,1)e^{\sigma\sqrt{\tau}z} - e^{-r\tau}K\int_{-d_{2,k}}^{\infty} dz\, f_Z^{(\mathrm{n})}(z;0,1)\Big].$$

Either using the complete the square technique or letting $y = z - \sigma\sqrt{\tau}$, along with $\exp(\sigma Z)$ normal integral formula, then

$$C^{(\mathrm{rn})}\left(s,t;K,T,r,\vec{\theta}\right) = \sum_{k=0}^{\infty} p_k(\lambda\tau)\mathrm{E}_{\mathcal{S}_k}\Big[s\, e^{\mathcal{S}_k - \lambda\bar{\nu}\tau} F_Z^{(\mathrm{n})}(d_{1,k};0,1) \\ - e^{-r\tau} K F_Z^{(\mathrm{n})}(d_{2,k};0,1)\Big]\,,$$

where we have used the identities, $-d_{2,k} \equiv -d_{1,k} + \sigma\sqrt{\tau}$ and $\int_{-d}^{\infty} = \int_{-\infty}^{d}$ for even integrable integrands. Finally, we form the compound-jump-diffusion risk-neutral European call option price as a compound-Poisson mixture of Black-Scholes call option prices,

$$C^{(\mathrm{rn})}\left(s,t;K,T,r,\vec{\theta}\right) = \sum_{k=0}^{\infty} p_k(\lambda\tau)\mathrm{E}_{\mathcal{S}_k}\big[\\ C^{(\mathrm{bs})}\left(se^{\mathcal{S}_k - \lambda\bar{\nu}\tau}, t; K, t+\tau, r, \sigma\right)\big]\,,$$

where, for example, in the case of a single uniform jump amplitude model, $\vec{\theta} = [\sigma, \lambda, a, b]^\top$. The formula reduces to the Black-Scholes if $\lambda = 0$ and $\vec{\theta} = [\sigma]$.

Due to the complex nature of this call option price formula with Poisson and IID RV expectations, perhaps Monte Carlo simulations[a] would be most practical, especially since a Poisson simulation of the number of jumps would keep the Poisson sum finite and the Poisson sum and the IID RV expectation could be combined. For instance, following Zhu and Hanson ('05), we can replace the Poisson sum and the sum $\mathcal{S}_k$ in the single uniform distribution case, with sample IID Poisson variates P_i for $i = 1\!:\!n$ and standard RVs $U_{i,j}$ for $j = 1\!:\!P_i$, on **(a,b)** by the estimate

$$\widehat{\mathcal{S}}_i = \sum_{j=1}^{P_i} Q_{i,j} = \sum_{j=1}^{P_i} (a + (b-a)U_{i,j}) = aP_i + (b-a)\sum_{j=1}^{P_i} U_{i,j}.$$

Then for the Poisson sample size n, the Monte Carlo estimate of the the

[a]Zhu and Hanson ('05) give elaborate Monte Carlo procedures with variance reduction techniques for European options in a jump-diffusion model with uniformly distributed jump-amplitudes, showing that jump-diffusion options are worth more than Black-Scholes diffusion options, in the paper at *http://www.math.uic.edu/ hanson/pub/CDC2005/cdc05zhweb.pdf*.

call option price, starting at $t = 0$, is simply,

$$\widehat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C^{(\mathrm{bs})}\Big(S_0 e^{\widehat{\mathcal{S}}_i - \lambda\overline{\nu}T}, 0; K, T, r, \sigma\Big) \equiv \frac{1}{n}\sum_{i=1}^{n} C_i^{(\mathrm{bs})},$$

where $C_i^{(\mathrm{bs})}$ is IID compound Poisson variate along with $\widehat{\mathcal{S}}_i$, so $\widehat{C}_n \to C^{(\mathrm{rn})}(S_0, 0; K, T, r, \sigma)$ as $n \to \infty$ with probability one, with standard deviation,

$$\sigma_{\widehat{C}_n} = \frac{1}{\sqrt{n}}\sqrt{\mathrm{Var}[C_i^{(\mathrm{bs})}]} \simeq \frac{1}{\sqrt{n}}\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(C_i^{(\mathrm{bs})} - \widehat{C}_n)^2},$$

where in the last term the unbiased sample variance estimate was used.

There is more to the Monte Carlo application than these basic estimates, i.e., there are variance and bias reduction techniques to improve performance and accuracy.

- ***7.3 Nonparametric, Multivariate Kernel Regression:***[a]

Kernel smoothers are useful for smoothly fitting data to a curve, particularly when the user wants to do some continuous operations on the curve, like plotting and finding optima. Whereas, splines fit smooth curves by numerical interpolation by matching values and derivatives at data points using a low degree polynomial (cubics are often used, fitting up to second derivatives) interpolation, the kernel smoothers are related to the kernel density estimators, except that kernel smoothing regression gives an estimation of an expectation the response scalar variable, the y), relative to the explanatory m-vector, the $\vec{x}$.

For ***independent or explanatory vector***, the distribution is represented by a normalized kernel, $K_{\vec{X}}$, with common scaled bandwidth b_x, the estimated smooth function for a sample of n independent observations, $\{\vec{x}_i, y_i | i = 1\!:\!n\}$, has the form

$$y \approx \phi(\vec{x}; b_x) = \sum_{i=1}^{n} y_i K_{\vec{X}}\left(\frac{\vec{x} - \vec{x}_i}{b_x}\right) \Bigg/ \sum_{j=1}^{n} K_{\vec{X}}\left(\frac{\vec{x} - \vec{x}_j}{b_x}\right),$$

[a]This and other sections comes from Carmona ('04) Chapter 4, but the kernel smoothing part is not recommended for students.

where the kernel $K_{\vec{X}}(\vec{\xi})$ is some model proper (i.e., integrates to one on the domain) density like normal, uniform or triangular and is used with a ***standardized argument*** $\vec{\xi}$ centered about some data point $\vec{x}_i$ and normalized with the scale of the bandwidth b_x for better computational properties. Standardized variables reduces the effects of floating point truncation errors. The normal kernel is often used because of supporting theory. Also, due to centered arguments, usually the kernel is assumed to symmetric, i.e., $K(-\vec{\xi}) = K(\vec{\xi})$.

Actually, the smoothed function $\phi(\vec{x})$ is basically a simulation of the conditional expectation of a dependent or response variable y conditioned on the independent or explanatory variable $\vec{x}$, since

$$\begin{aligned} \mathrm{E}[Y|\vec{X} = \vec{x}] &= \int_{-\infty}^{+\infty} y f_{Y|\vec{X}}(y|\vec{X} = \vec{x}) dy \\ &= \int_{-\infty}^{+\infty} y f_{\vec{X},Y}(\vec{x}, y) dy / f_{\vec{X}}(\vec{x}) \end{aligned}$$

by a ***"Bayes' rule" for densities***, $f_{Y|\vec{X}}(y|\vec{X} = \vec{x}) = f_{\vec{X},Y}(\vec{x}, y)/f_{\vec{X}}(\vec{x})$, following from the definition of conditional probability (Hanson, ('04), p. B26).

For motivation, consider the ***univariate kernel estimation*** of $f_X(x)$,

$$f_X(x) \simeq \widehat{f}_X(x) = \frac{1}{nb_x}\sum_{i=1}^{n} K_X\left(\frac{x - x_i}{b_x}\right),$$

and that the joint density has the estimate, assuming the that the joint kernel is separable, i.e., $K_{X,Y}(\xi,\eta) = K_X(\xi)\cdot K_Y(\eta)$, and that

$$f_{X,Y}(x,y) \simeq \widehat{f}_{X,Y}(x,y) = \frac{1}{nb_xb_y}\sum_{i=1}^{n} K_{X,Y}\left(\frac{x - x_i}{b_x}, \frac{y - y_i}{b_y}\right)$$

then

$$\widehat{f}_{X,Y}(x,y) = \frac{1}{nb_xb_y}\sum_{i=1}^{n} K_X\left(\frac{x - x_i}{b_x}\right) K_Y\left(\frac{y - y_i}{b_y}\right)$$

and

$$\begin{aligned} f_X(x)\mathrm{E}[Y|X = x] &= \int_{-\infty}^{+\infty} y f_{X,Y}(x,y)dy \\ &\simeq \frac{1}{nb_xb_y}\sum_{i=1}^{n} K_X\left(\frac{x - x_i}{b_x}\right)\int_{-\infty}^{+\infty} yK_Y\left(\frac{y - y_i}{b_y}\right)dy \\ &= \frac{1}{nb_xb_y}\sum_{i=1}^{n} y_iK_X\left(\frac{x - x_i}{b_x}\right), \end{aligned}$$

since, in the y-integral, letting $\eta = (y - y_i)/b_y$,

$$\int_{-\infty}^{+\infty} y K_Y\left(\frac{y - y_i}{b_y}\right) dy = b_y\left(y_i + \int_{-\infty}^{+\infty} \eta K_Y(\eta) dy\right) = b_y y_i,$$

by the fact that K_Y is also a symmetric and proper density like K_X.
Finally by reassembling our formulas, we have the desired motivational result,

$$\mathrm{E}[Y|X = x] \simeq \sum_{i=1}^{n} y_i K_X\left(\frac{x - x_i}{b_x}\right) \Bigg/ \sum_{j=1}^{n} K_X\left(\frac{x - x_j}{b_x}\right),$$

the denominator sum coming from using another approximation $f_X(x) \simeq \widehat{f}_X(x)$ consistent with the joint density estimated approximation.

The multivariate case is more complicated, or rather tedious, due to dimensional complexity, even if the kernel is separable in $\vec{x}$, i.e.,

$$K_{\vec{X}}\left(\frac{\vec{x} - \vec{x}_i}{b_x}\right) = \prod_{j=1}^{m} K_{X_j}\left(\frac{x_j - x_{i,j}}{b_x}\right).$$

- ***7.4 Kernel Regression for Implied Volatility Computations:***

In the implied volatility inverse problem, here starting at $t = 0$,

$$\widetilde{C}_0^{(\mathrm{bs})}\left(M_{k,j}, T_j; \sigma_{k,j}^{(\mathrm{iv})}\right) = C^{(\mathrm{mkt})}(K_k, T_j),$$

where the moneyness variable $M_{k,j} = M_k^{(0)} = S_0/K_k$ for $k = 1\!:\!n_k$ helps to reduce the problem dimensionality. We also assume here the risk-free rate is taken from the current U.S. Federal Reserve Target Rate, so is fixed in this computation. The market European call (or put) option price can be obtained from the CBOE Delayed Market Quotes, Quote Table Download page[a] using an appropriate market symbol[b]. The first two items listed in the first column of the quote table will be the 2- digit ***year*** and the ***exercise month*** followed by the strike price.

[a] *http://www.cboe.com/delayedquote/QuoteTableDownload.aspx* (See description on how change comma-delimited format to Excel, if wanted.

[b] E.G., for S&P 500 Index option SPX (Caution: some companies also use that symbol and you have to avoid getting the index itself, rather than the option.) the product specification is at *http://www.cboe.com/products/indexopts/spx_spec.aspx*.

The expiration date[a], in case of the SPX option is the Saturday after the third Friday of the month (e.g., if ***09 Feb***[b] is the expiration month then 21 Feb. is the expiration date.). Prices are listed in CBOE points and come with a current multiplier of ***$100***, so for example 200.00 points is $20,000.00. SPX is a European style option as is XEO is an option on the S&P100 Index, while OEX options on that index are American, early exercise, style. Index options are different from stock options in many ways. Mileage, i.e., specifications will vary for other options.

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[a]See Hull ('06, 6th Edition) for very practical description the CBOE quote table of hard to find information, pp. 187 & 316-317 (recommended).

[b] A fragment of the top left corner of the SPX quote table looks like

SPX (S&P 500 INDEX) Feb 15 2009 @ 13:45 ET	826.84	-8.35		
Calls	LastSale	Net	Bid	Ask
09 Feb 200.00 (SPV BD-E)	635.10	0.0	619.50	622.40

so a market call estimate would $\boldsymbol{C^{(\mathrm{mkt})}(20000, 4/252)}$, counting 4 trading days due to the market Monday Holiday. Note for the SPX, there are no weekly options (ticker: JX[A,B,D,E]) and weeklys are different from monthlys and long-term versions.

SPX (S&P 500 INDEX)	826.84	-8.35				
Feb 15 2009 @ 13:45 ET						
Calls	Last Sale	Net	Bid	Ask	Vol	Open Int
09 Feb 200.00 (SPV BD-E)	635.10	0.0	619.50	622.40	0	1
09 Feb 300.00 (SPV BT-E)	0.0	0.0	519.60	522.30	0	0
09 Feb 325.00 (SPV BE-E)	0.0	0.0	494.50	497.40	0	0
09 Feb 350.00 (SPV BJ-E)	0.0	0.0	469.60	472.40	0	0
09 Feb 375.00 (SPV BO-E)	0.0	0.0	444.60	447.40	0	0
09 Feb 400.00 (SZU BT-E)	421.05	0.0	419.60	422.40	0	70
09 Feb 425.00 (SZU BE-E)	403.45	0.0	394.60	397.40	0	98
09 Feb 450.00 (SZU BJ-E)	377.10	0.0	369.60	372.40	0	100
09 Feb 475.00 (SZU BO-E)	0.0	0.0	344.60	347.40	0	0
09 Feb 490.00 (SZU BR-E)	0.0	0.0	329.50	332.20	0	0
09 Feb 500.00 (SYU BT-E)	326.00	0.0	319.60	322.40	0	2875
09 Feb 525.00 (SYU BE-E)	330.30	0.0	294.60	297.40	0	98
09 Feb 550.00 (SYU BJ-E)	265.00	0.0	269.60	272.40	0	130
09 Feb 560.00 (SYU BL-E)	269.00	0.0	259.60	262.40	0	50
09 Feb 575.00 (SYU BP-E)	356.00	0.0	244.60	247.50	0	75
09 Feb 580.00 (SYU BY-E)	0.0	0.0	239.60	242.50	0	0
09 Feb 590.00 (SYU BR-E)	0.0	0.0	229.60	232.50	0	0
09 Feb 600.00 (SYG BT-E)	231.05	0.0	219.60	222.50	0	1959
09 Feb 610.00 (SYG BB-E)	221.15	0.0	209.60	212.50	0	2
09 Feb 620.00 (SYG BD-E)	0.0	0.0	199.60	202.50	0	0
09 Feb 625.00 (SYG BE-E)	202.90	0.0	194.60	197.60	0	53
09 Feb 630.00 (SYG BF-E)	0.0	0.0	189.70	192.60	0	0
09 Feb 635.00 (SYG BG-E)	0.0	0.0	184.70	187.60	0	0
09 Feb 640.00 (SYG BH-E)	0.0	0.0	179.70	182.60	0	0
09 Feb 645.00 (SYG BI-E)	0.0	0.0	174.70	177.60	0	0
09 Feb 650.00 (SYG BJ-E)	182.50	+9.30	169.70	172.70	5	146
09 Feb 655.00 (SYG BK-E)	0.0	0.0	164.70	167.70	0	0
09 Feb 660.00 (SYG BL-E)	167.00	+12.50	159.80	162.60	3	13
09 Feb 665.00 (SYG BM-E)	165.45	0.0	154.80	157.70	0	20
09 Feb 670.00 (SYG BN-E)	0.0	0.0	149.80	152.80	0	0
09 Feb 675.00 (SYG BO-E)	154.60	0.0	144.80	147.70	0	95
09 Feb 680.00 (SYG BP-E)	0.0	0.0	140.00	142.80	0	0
09 Feb 685.00 (SYG BQ-E)	0.0	0.0	135.00	137.80	0	0

Figure 1: CBOE Quote Table for S&P 500 Index Options with only call option columns (put columns supressed) from ***page 1*** of Delayed Quote Download page from February 15, 2009.

09 Feb 1050.00 (SPQ BJ-E)	0.0	-0.05	0.0	0.05	2	20430
09 Feb 1055.00 (SPQ BK-E)	0.10	0.0	0.0	0.10	0	1293
09 Feb 1060.00 (SPQ BL-E)	0.05	0.0	0.0	0.10	1	827
09 Feb 1065.00 (SPQ BM-E)	0.10	0.0	0.0	0.10	0	257
09 Feb 1070.00 (SPQ BN-E)	0.05	-0.05	0.0	0.10	1	2488
09 Feb 1075.00 (SPQ BO-E)	0.05	0.0	0.0	0.05	0	6145
09 Feb 1080.00 (SPQ BP-E)	0.10	0.0	0.0	0.05	0	1405
09 Feb 1085.00 (SPQ BQ-E)	0.05	0.0	0.0	0.10	0	513
09 Feb 1090.00 (SPQ BR-E)	0.05	0.0	0.0	0.05	0	347
09 Feb 1095.00 (SPQ BS-E)	0.05	0.0	0.0	0.05	0	185
09 Feb 1100.00 (SPT BT-E)	0.05	0.0	0.0	0.05	0	19166
09 Feb 1105.00 (SPT BA-E)	0.05	0.0	0.0	0.05	0	50
09 Feb 1110.00 (SPT BB-E)	0.05	0.0	0.0	0.05	0	337
09 Feb 1115.00 (SPT BC-E)	0.05	0.0	0.0	0.05	0	20
09 Feb 1120.00 (SPT BD-E)	0.05	0.0	0.0	0.05	0	2013
09 Feb 1125.00 (SPT BE-E)	0.05	0.0	0.0	0.05	0	8318
09 Feb 1130.00 (SPT BF-E)	0.05	0.0	0.0	0.05	0	3050
09 Feb 1140.00 (SPT BH-E)	0.05	0.0	0.0	0.05	0	1396
09 Feb 1150.00 (SPT BJ-E)	0.05	0.0	0.0	0.05	0	19733
09 Feb 1160.00 (SPT BL-E)	0.05	0.0	0.0	0.05	0	48
09 Feb 1175.00 (SPT BO-E)	0.05	0.0	0.0	0.05	0	1729
09 Feb 1180.00 (SPT BP-E)	0.25	0.0	0.0	0.05	0	43
09 Feb 1200.00 (SZP BT-E)	0.05	0.0	0.0	0.05	0	482
09 Feb 1225.00 (SZP BE-E)	0.05	0.0	0.0	0.05	0	939
09 Feb 1250.00 (SZP BJ-E)	0.50	0.0	0.0	0.05	0	21
09 Feb 1275.00 (SZP BO-E)	1.20	0.0	0.0	0.05	0	30
09 Feb 1300.00 (SXY BT-E)	0.20	0.0	0.0	0.05	0	18003
09 Feb 1325.00 (SXY BE-E)	0.65	0.0	0.0	0.05	0	504
09 Feb 1350.00 (SXY BJ-E)	0.0	0.0	0.0	0.05	0	0
09 Feb 1375.00 (SXY BO-E)	0.0	0.0	0.0	0.05	0	0
09 Feb 1400.00 (SXZ BT-E)	0.40	0.0	0.0	0.05	0	25
09 Feb 1450.00 (SXZ BJ-E)	0.0	0.0	0.0	0.05	0	0
09 Feb 1500.00 (SXM BT-E)	0.05	0.0	0.0	0.05	0	2802
09 Mar 200.00 (SPV CD-E)	707.60	0.0	618.00	620.80	0	360
09 Mar 300.00 (SPV CT-E)	0.0	0.0	518.10	520.90	0	0
09 Mar 325.00 (SPV CE-E)	0.0	0.0	493.10	495.90	0	0

Figure 2: CBOE Quote Table for S&P 500 Index Options with only call option columns from *page 4* of Delayed Quote Download page from February 15, 2009.

09 Mar 1450.00 (SLQ CJ-E)	1.60	0.0	0.0	1.00	0	253
09 Mar 1500.00 (SQP CT-E)	1.25	0.0	0.0	1.00	0	1
09 Mar 1550.00 (SQP CJ-E)	0.0	0.0	0.0	1.00	0	0
09 Apr 200.00 (SPV DD-E)	0.0	0.0	615.80	620.30	0	0
09 Apr 300.00 (SPV DT-E)	0.0	0.0	516.10	520.60	0	0
09 Apr 350.00 (SPV DJ-E)	0.0	0.0	466.30	470.80	0	0
09 Apr 375.00 (SPV DO-E)	0.0	0.0	441.50	446.00	0	0
09 Apr 400.00 (SZU DT-E)	0.0	0.0	416.70	421.20	0	0
09 Apr 425.00 (SZU DE-E)	0.0	0.0	392.10	396.60	0	0
09 Apr 450.00 (SZU DJ-E)	0.0	0.0	367.30	371.80	0	0
09 Apr 480.00 (SZU DP-E)	0.0	0.0	337.90	342.40	0	0
09 Apr 490.00 (SZU DR-E)	0.0	0.0	328.10	332.60	0	0
09 Apr 500.00 (SYU DT-E)	334.15	0.0	318.30	322.80	0	7
09 Apr 510.00 (SYU DB-E)	0.0	0.0	308.60	313.10	0	0
09 Apr 515.00 (SYU DU-E)	0.0	0.0	303.70	308.20	0	0
09 Apr 520.00 (SYU DD-E)	0.0	0.0	298.90	303.40	0	0
09 Apr 525.00 (SYU DE-E)	0.0	0.0	294.10	298.60	0	0
09 Apr 530.00 (SYU DF-E)	0.0	0.0	289.20	293.70	0	0
09 Apr 540.00 (SYU DH-E)	0.0	0.0	279.60	284.10	0	0
09 Apr 550.00 (SYU DJ-E)	0.0	0.0	270.00	274.50	0	0
09 Apr 560.00 (SYU DL-E)	0.0	0.0	260.40	264.90	0	0
09 Apr 570.00 (SYU DN-E)	0.0	0.0	250.90	255.40	0	0
09 Apr 575.00 (SYU DP-E)	0.0	0.0	246.20	250.70	0	0
09 Apr 580.00 (SYU DY-E)	0.0	0.0	241.50	245.90	0	0
09 Apr 585.00 (SYU DQ-E)	0.0	0.0	236.80	241.20	0	0
09 Apr 590.00 (SYU DR-E)	0.0	0.0	232.10	236.50	0	0
09 Apr 600.00 (SYG DT-E)	0.0	0.0	222.70	227.10	0	0
09 Apr 610.00 (SYG DB-E)	0.0	0.0	213.40	217.80	0	0
09 Apr 620.00 (SYG DD-E)	0.0	0.0	204.20	208.60	0	0
09 Apr 625.00 (SYG DE-E)	0.0	0.0	199.60	204.00	0	0
09 Apr 630.00 (SYG DF-E)	0.0	0.0	195.00	199.40	0	0
09 Apr 640.00 (SYG DH-E)	0.0	0.0	186.00	190.40	0	0
09 Apr 650.00 (SYG DJ-E)	264.50	0.0	177.00	181.40	0	50
09 Apr 660.00 (SYG DL-E)	0.0	0.0	168.10	172.50	0	0
09 Apr 670.00 (SYG DN-E)	0.0	0.0	159.30	163.70	0	0
09 Apr 675.00 (SYG DO-E)	242.00	0.0	155.00	159.40	0	50

Figure 3: CBOE Quote Table for S&P 500 Index Options with only call option columns from ***page 10*** of Delayed Quote Download page from February 15, 2009.

10 Dec 2500.00 (SYZ LU-E)	0.05	0.0	0.05	0.95	0	9617
11 Dec 250.00 (SZJ LE-E)	543.70	0.0	544.00	549.80	0	13
11 Dec 280.00 (SZJ LP-E)	519.00	0.0	519.20	524.80	0	102
11 Dec 300.00 (SZJ LF-E)	497.80	0.0	503.50	509.00	0	82
11 Dec 350.00 (SZJ LG-E)	0.0	0.0	463.80	469.70	0	0
11 Dec 400.00 (SZJ LB-E)	423.40	0.0	426.20	432.00	0	32
11 Dec 450.00 (SZJ LI-E)	388.80	0.0	390.60	395.90	0	5
11 Dec 500.00 (SZJ LC-E)	360.00	0.0	355.90	361.50	0	3
11 Dec 550.00 (SZJ LK-E)	331.60	0.0	323.20	328.70	0	5
11 Dec 600.00 (SZJ LR-E)	352.50	0.0	291.90	297.90	0	5
11 Dec 650.00 (SZJ LM-E)	324.50	0.0	263.10	268.80	0	5
11 Dec 700.00 (SZJ LA-E)	245.00	0.0	234.90	240.90	0	50
11 Dec 800.00 (SZJ LL-E)	197.50	0.0	185.00	191.00	0	865
11 Dec 850.00 (SZJ LJ-E)	172.55	-2.45	162.20	168.00	3	4210
11 Dec 900.00 (SZJ LT-E)	148.00	+8.50	141.40	147.20	10	2647
11 Dec 950.00 (SZJ LS-E)	128.00	0.0	122.30	128.20	0	456
11 Dec 1000.00 (SZT LR-E)	131.35	0.0	105.00	111.00	0	353
11 Dec 1100.00 (SZT LT-E)	81.00	-7.00	76.00	81.90	300	1088
11 Dec 1200.00 (SZT LU-E)	56.60	0.0	53.40	59.10	0	1970
11 Dec 1300.00 (SZT LW-E)	40.00	0.0	36.20	41.90	0	811
11 Dec 1400.00 (SZT LA-E)	32.00	0.0	23.60	29.30	0	95
11 Dec 1500.00 (SZV LT-E)	20.70	0.0	15.40	19.60	0	1151
11 Dec 1600.00 (SZV LO-E)	15.00	0.0	9.90	12.90	0	5750
11 Dec 1700.00 (SZV LA-E)	17.70	0.0	5.70	8.70	0	25
11 Dec 1800.00 (SZV LD-E)	0.0	0.0	3.00	5.80	0	0
11 Dec 1900.00 (SZV LI-E)	0.0	0.0	1.40	4.00	0	0
11 Dec 2000.00 (SZV LE-E)	2.50	0.0	0.55	2.60	0	4260

Figure 4: CBOE Quote Table for S&P 500 Index Options with only call option columns from *page 23* of Delayed Quote Download page from February 15, 2009 (Long term LEAP options, up to 3 years).

- ***7.7 Implied Volatility Algorithm with Kernel Regression and Numerical Inversion:***

Returning to the implied volatility computations, here is our pseudo-algorithm:

1. Select an option to study and download the quote table from the CBOE, or other exchange that allow public domain downloads, from the delayed quote page.

2. Select a few exercise times T_j, some in weeks and others in months (do not forget to convert to years, since the FRB risk-free rates are in years and that dominates the units. Also, short exercise times are more likely to produce implied volatility smiles (like a minimum curve), while long times produce smirks (like a maximum curve). If you have a volatility surface in mind, then you will need more than a few exercise times.

3. Next select a number of strike values K_k, enough data to produce a respectable implied volatility curve of the $\sigma^{(\mathrm{iv})}_{k,j}$ (do not forget to account for quote table scaling) versus moneyness $M^{(0)}_k = S_0/K_k$, where S_0 is the current price of the underlying asset which should be on the top of your quote table (CBOE rules).

4. Compute the model call (or put) option price data using a grid of volatility values σ_i for $i = 1\!:\!n$ that produce a realistic range of option prices $C_{i,k,j} \equiv C(K_k, T_j; \sigma_i)$ with your set of contract parameters (K_k, T_j) or equivalently $(M^{(0)}_k, T_j)$ given S_0 using the Black-Scholes model (p. 5; could be a test case) or the compound-jump-diffusion model with a simple Monte Carlo simulation (p. 17), filling in with some estimate of jump-parameters.

5. Now form the function forming the estimated market option price curve for each value of the moneyness $M_k^{(0)}$ with fixed T_j and other parameters using the kernel smoothing technique, written here in scalar variable form,

$$y \approx \phi(x; b_x) = \sum_{i=1}^{n} y_i K_X\left(\frac{x - x_i}{b_x}\right) \Bigg/ \sum_{\ell=1}^{n} K_X\left(\frac{x - x_\ell}{b_x}\right),$$

with Gaussian kernel and bandwidth from code or estimated, where $x_i = \sigma_i$ here (in general could be a vector, if of just several dimensions) and corresponding response data $y_i = C_i$, with other parameters suppressed However, in particular, the contract set $\left(M_k^{(0)}, T_j\right)$ need to keep track of for each (k, j)-kernel smoothing operation. See the MATLAB public domain kernel smoothing regression (KSR) code **ksr**, potentially a vector-argument kernel so could use $\vec{x}_{i;k,j} = \left[\sigma_i, M_k^{(0)}, T_j\right]'$ with output response $y_{i;k,j}$, described later. However, other regression methods such as maximum likelihood could be used or other smoothing methods such as spline interpolation.

6. Then, for the implied volatility step is basically a nonlinear zero-finding problem: find an x^* such that

$$\phi(x^*, \cdot) = C^{(\mathrm{mkt})}_{k,j},$$

for each fixed k and j where $C^{(\mathrm{mkt})}_{k,j} \equiv C^{(\mathrm{mkt})}(K_k, T_j)$, or find an $\vec{x}^*_{k,j}$ such that

$$\phi(\vec{x}^*_{k,j}, \cdot) = C^{(\mathrm{mkt})}_{k,j},$$

which in principle could determine a volatility surface. There are many basic methods that could be used here, such as the classic univariate zero finder **`fzero`** for scalar function of a scalar variable that is in MATLAB or Newton's methods or any of its quasi-variants.

7. When the roots for each (k, j) data pair for $k = 1\!:\!n_k$ and for $j = 1\!:\!n_j$ are assembled, then implied volatility curves versus moneyness and parameterized by exercise time can be plotted. Otherwise, the implied or local volatility surface can be plotted against both moneyness and exercise or maturity time is three-dimensional graphs with **`surf or mesh`**.

- ***7.7 Kernel Smoothing Regression (KSR):***[a]

1. Syntax: `function r = ksr(x, y, b, n)`, computes the Gaussian kernel regression of `y` versus `x` and outputs the structure `r`.
2. Input: The `x` is the explanatory data `n`-vector, `y` is the response data `n`-vector, `b` is a specified bandwidth of the kernel (if the user wants `ksr` to compute an optimal bandwidth then use `r = ksr(x, y)` form, and `n` is specified data length but should not be needed. %
3. Output: The `r` is a structure, such that `r.h` is the computed bandwidth and `r.n` is the number of samples and `r.f(r.x) = y(x) +e` is the form of the regression computed, all when the short form is used. The regression is plotted for the forms `r = ksr(x, y)` adn e `r = ksr(x, y, b)`.

See the MATLAB Central Exchange for more documentation and code (not tested).

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[a] Kernel Smoothing Regression by Yi Cao, 2008, *http://www.mathworks.com/matlabcentral/fileexchange/19195*.

- ***7.8 Root or Zero Finder (`fzero`):***

1. Syntax:

   `[x,fval,exitflag,output]=fzero(@f,x0,options);`

   solves the zero or root problem for a scalar valued function `f` of a single scalar argument `x`, for an x^* such that $f(x^*) = 0$ given a start $x0$ and objective function `f` appearing as the first argument as the pointer or handle `@f` usually pointing to a subfunction within the main function m-file.

2. Additional parameter can be passed to the (sub-)function `f` using a global statement in called and calling functions, as with `fminsearch` of Lecture 5, in fact, the syntax is much like that of `fminsearch`, except of the multivariate properties.

3. The output arguments have essentially the same descriptions as those in `fminsearch` and all but `x` are optional.

See D. Higham ('04) Chapters 14 and 20 for other methods for implied volatility, including Monte Carlo. ***For instance,*** $f(x) = \phi(x) - C_{k,j}^{(\mathrm{mkt})}$ ***and*** $x0 = \sigma_k$.

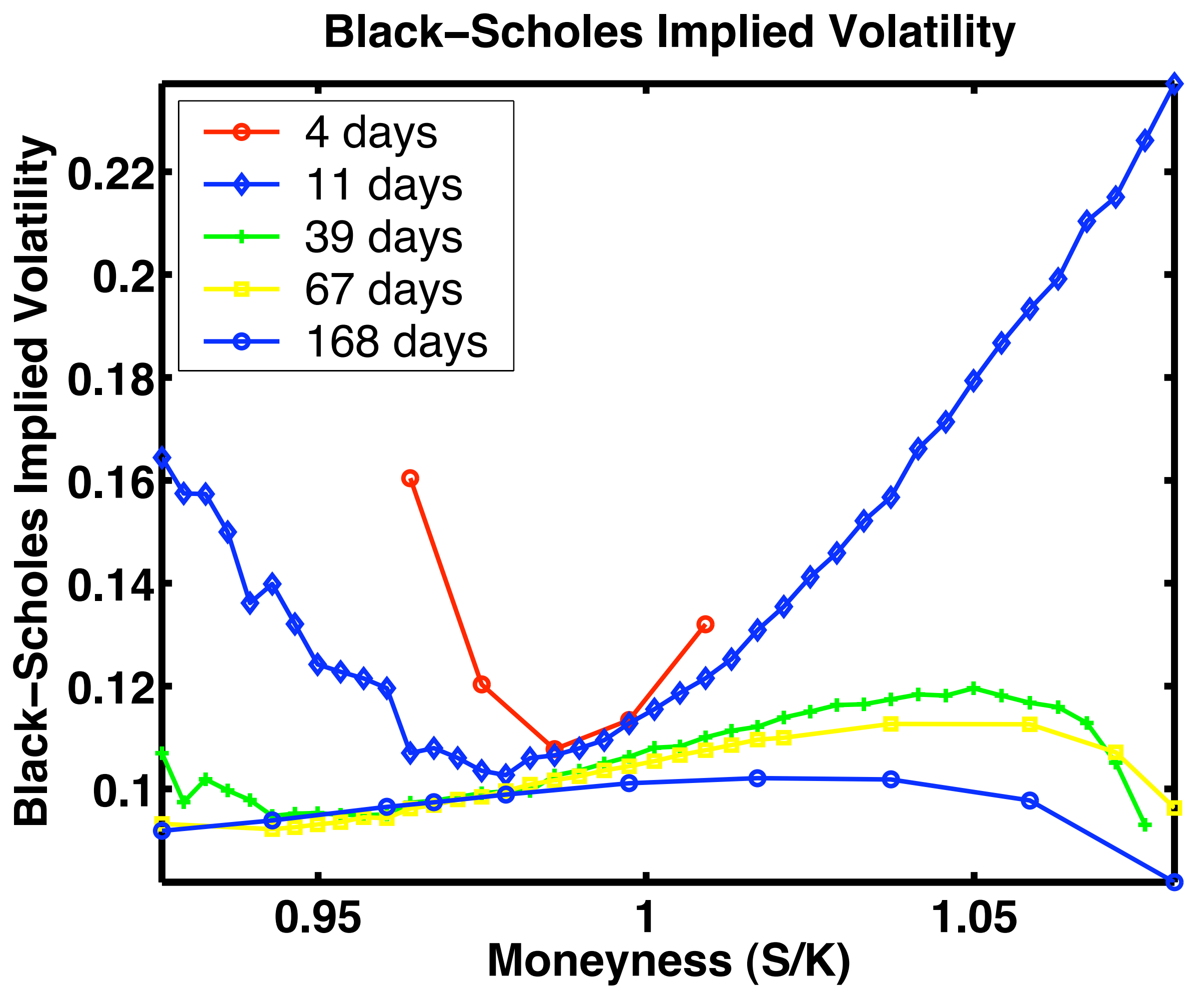
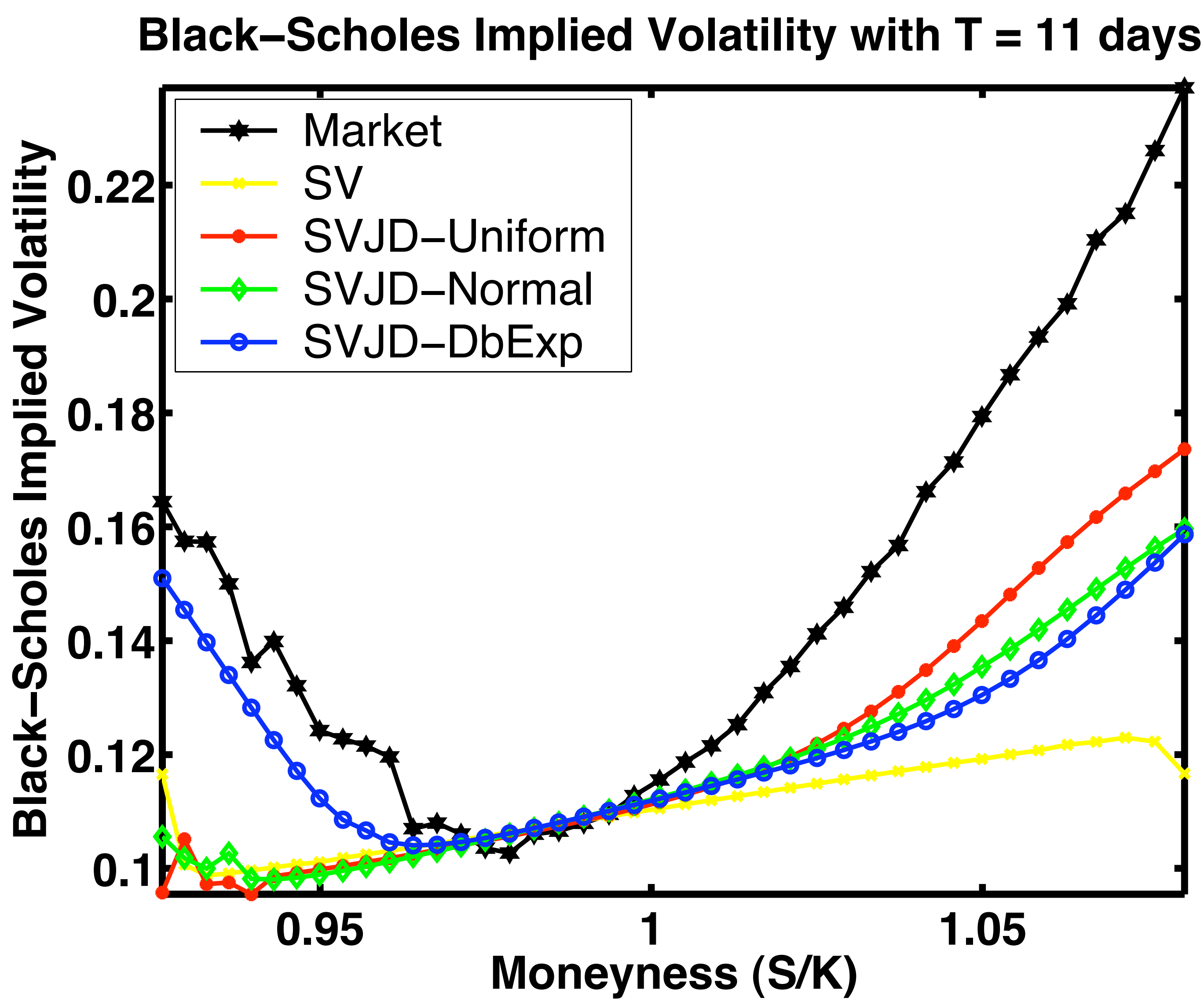
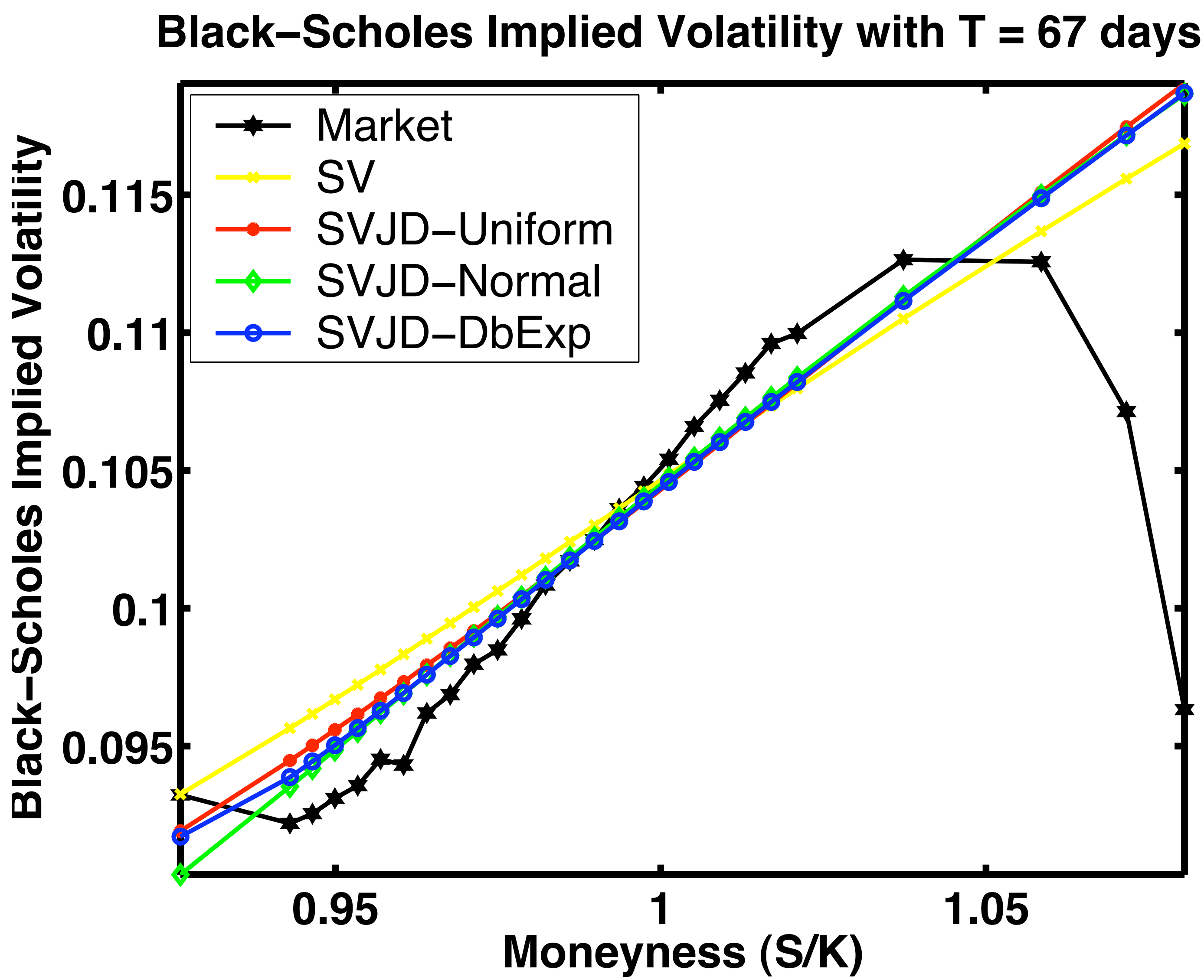
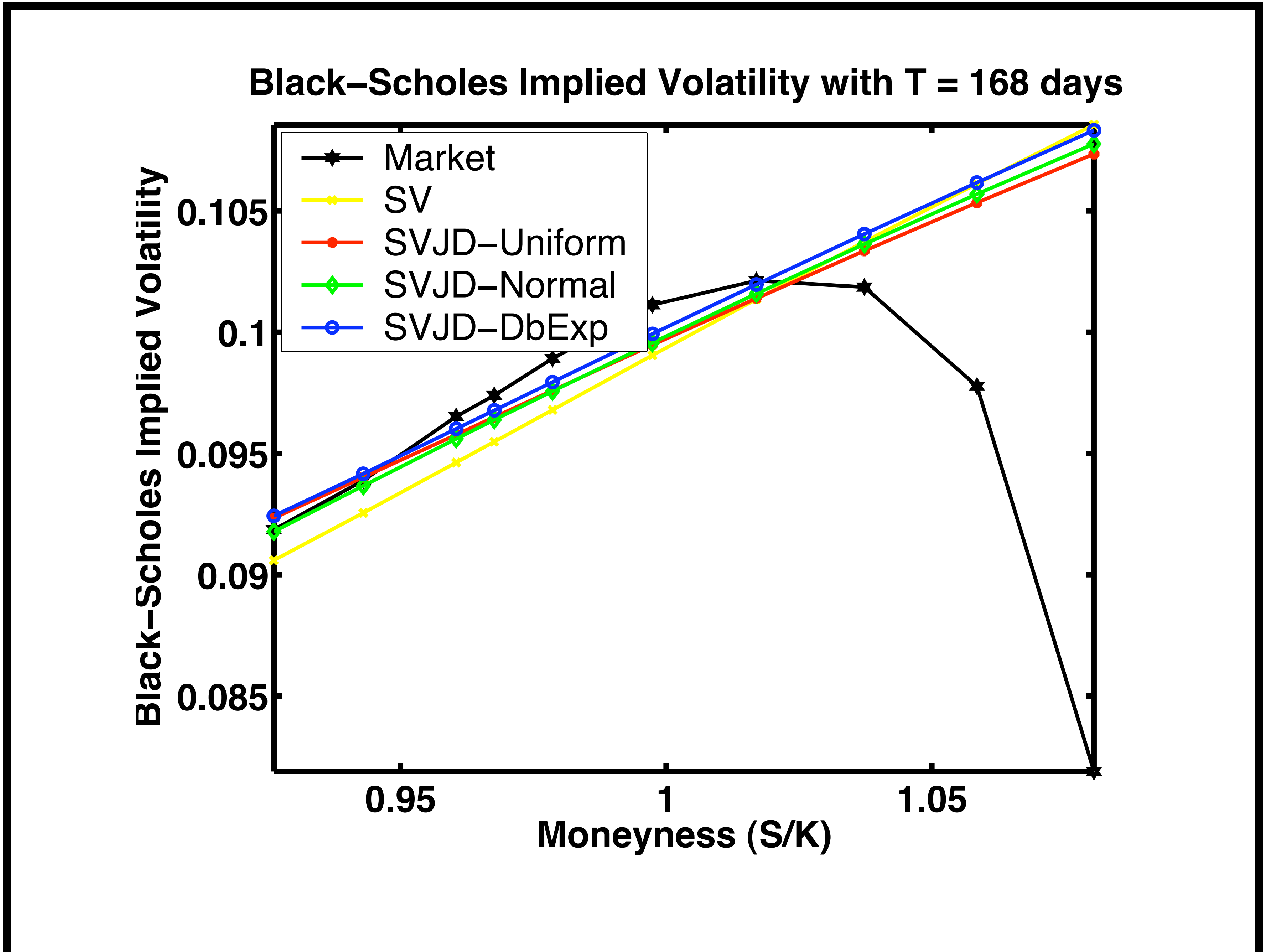


Figure 5: Black-Scholes implied volatility by SPX European call options of 5 different maturities, using a maximum likelihood to minimize the mean square error (MSE) between market observations and BS predictions. See also Figs. 6, 7 & 8. Option prices were quoted on April 10, 2006 (G. Yan, PhD Thesis, 2006).



Black–Scholes Implied Volatility with T = 11 days
Market
SV
SVJD–Uniform
SVJD–Normal
SVJD–DbExp
Black–Scholes Implied Volatility
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.95
1
1.05
Moneyness (S/K)


Black–Scholes Implied Volatility with T = 67 days
Market
SV
SVJD–Uniform
SVJD–Normal
SVJD–DbExp
Black–Scholes Implied Volatility
0.115
0.11
0.105
0.1
0.095
0.95
1
1.05
Moneyness (S/K)



Black–Scholes Implied Volatility with T = 168 days
Market
SV
SVJD–Uniform
SVJD–Normal
SVJD–DbExp
Black–Scholes Implied Volatility
0.105
0.1
0.095
0.09
0.085
0.95
1
1.05
Moneyness (S/K)