FINM345/STAT390 Stochastic Calculus – Hanson – Autumn 2009

Lecture 1 Homework: Stochastic Jump and Diffusion Processes

(Due by Lecture 2 in Chalk FINM345 Digital Dropbox)

You must show your work, code and/or worksheet for full credit.
There are 10 points per question if correct answer.

1. **Gaussian Process from Zero Mean – Unit Variance Wiener Process:** Let $\{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n, t_0 = 0; t_{n+1} = T\}$ be a variably spaced partition of the time interval $[0, T]$ with $\Delta t_i > 0$. Show the following properties and justify them by giving a reason for every step, such as a property of the process or a property of expectations:

   (a) Let $G(t) = \mu t + \sigma W(t)$ and $\Delta G(t_i) \equiv G(t_i + \Delta t_i) - G(t_i)$ with μ and $\sigma > 0$ constants, then show that
   $$\mathrm{E}[\Delta G(t_i)] = \mu \Delta t_i,$$
   $$\mathrm{Var}[\Delta G(t_i)] = \sigma^2 \Delta t_i,$$
   and
   $$\mathrm{Cov}[\Delta G(t_i), \Delta G(t_j)] = \sigma^2 \Delta t_i \, \delta_{i,j}$$
   for $i, j = 0 : n$, where $\delta_{i,j}$ is the Kronecker delta.

   (b) Simulate the trajectories of the processes $G(t)$ with a professional graph, using 4 samples, with constant parameter values μ =2.448e-4, σ =1.121e-2 and $\Delta t = 1/252$, and $T = 2$ is the total time interval, i.e., simulate the integral of $dG(t) = \mu dt + \sigma dW(t)$.

2. **Simple Poisson Process with Non-Unit Amplitude:** Let $\{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n, t_0 = 0; t_{n+1} = T\}$ be a variably spaced partition of the time interval $[0, T]$ with $\Delta t_i > 0$. Show the following properties and justify them by giving a reason for every step, such as a property of the process or a property of expectations:

   (a) Let $H(t) = \nu P(t)$ and $\Delta H(t_i) \equiv H(t_i + \Delta t_i) - H(t_i)$ with $\lambda > 0$ and $\nu > -1$ constants, then show that
   $$\mathrm{E}[\Delta H(t_i)] = \nu \lambda \Delta t_i,$$
   $$\mathrm{Var}[\Delta H(t_i)] = \nu^2 \lambda \Delta t_i,$$
   and
   $$\mathrm{Cov}[\Delta H(t_i), \Delta H(t_j)] = \nu^2 \lambda \Delta t_i \delta_{i,j}$$
   for $i, j = 0 : n$.

   (b) Simulate the trajectories of the process $H(t)$ with a professional graph, using 4 samples, with constant parameter values λ =5.241, ν =7.45e-3, Δt =1/252, and $T = 2$ years is the total time interval, i.e., simulate the integral of $dH(t) = \nu dP(t)$.

3. **Wiener and Poisson Tables Partial Justification:**

   (a) Derive the $m = 3 : 4$ entries in Table 1 on page L1-p29 for $\mathrm{E}[|\Delta W(t)|^m]$.

   (b) Derive the $m = 3 : 4$ entries in Table 1 on page L1-p58 for $\mathrm{E}[(\Delta P(t))^m]$ and $\mathrm{E}[(\Delta P(t) - \lambda \Delta t)^m]$.

4. Integral of Wiener Differential Squared:

(a) Simulate the integrals of $dX(t) = (dW(t))^2$ using four (4) samples of sample size $n = 1000$ each and time interval $T = 2$ units, plotting $X(t)$ versus t on a professional graph.

(b) Do the same for sample size $n = 10000$ to simulate convergence. simulated convergence.

(c) Verify the nondifferentiability of $W(t)$ by showing that $\mathrm{E}[\Delta W(t)/\Delta t] = 0$, but $\mathrm{Var}[\Delta W(t)/\Delta t] = 1/\Delta t \to +\infty$ as $\Delta t \to 0^+$.

5. Integral of Product of Time and Wiener Differentials:

(a) Simulate the integrals of $dY(t) = dt \cdot dW(t)$ using four (4) samples of sample size $n = 1000$ each and time interval $T = 2$ units, plotting $Y(t)$ versus t on a professional graph.

(b) Do the same for sample size $n = 10000$ to simulate convergence.