FINM345/STAT390 Stochastic Calculus – Hanson – Autumn 2009

Lecture 2 Homework: Stochastic Jump and Diffusion Processes

(Due by Lecture 3 in Chalk FINM345 Digital Dropbox)

You must show your work, code and/or worksheet for full credit.
There are 10 points per question if correct answer.

Corrections are in ***Red***, 10/06/2009.

1. (a) Show that when $0 \le s \le t$ that

$$\mathrm{E}\left[W^3(t) \;\middle|\; W(s)\right] = W^3(s) + 3(t-s)W(s)\,,$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Use this result to derive the primary martingale property for Markov processes:

$$\mathrm{E}\left[W^3(t) - 3tW(t) \;\middle|\; W(s)\right] = W^3(s) - 3sW(s).$$

{*Remark: The general technique is to seek the expectation of mth power in the separable form,*

$$\mathrm{E}\left[M_W^{(m)}(W(t),t) \;\middle|\; W(s)\right] = M_W^{(m)}(W(s),s)\,,$$

where

$$M_W^{(m)}(W(t),t) = W^m(t) + \sum_{k=0}^{m-1} \alpha_k(t) W^k(t)\,,$$

satisfied for the sequence of functions $\{\alpha_0(t), \ldots, \alpha_{m-1}(t)\}$, that can be recursively solved using the separable form $\alpha_k(t)$ in the order $k = 0 : m-1$; or just use the binomial theorem. Obviously, $m = 3$ here.}

2. (a) Verify that when $0 \le s \le t$ and constant jump rate $\lambda_0 > 0$ that

$$\mathrm{E}\left[P^2(t) \;\middle|\; P(s)\right] = P^2(s) + 2\lambda_0(t-s)P(s) + \lambda_0(t-s)(1+\lambda_0(t-s))\,,$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Find the time polynomials $\alpha_0(t)$ and $\alpha_1(t)$ such that

$$M_P^{(2)}(t) = P^2(t) + \alpha_1(t)P(t) + \alpha_0(t)$$

is a Martingale. Assume $\alpha_k(0) = 0$ for $k = 0 : 1$.
{*Remarks: The primary martingale property is that $\mathrm{E}[X(t)|X(s)] = X(s)$ for some process $X(t)$ and in this case $X(t) = f(P(t))$, but there are also additional technical conditions to define a martingale form. Also, by a simple form of the principle of separation of variables, if $f(t) = g(s)$ for arbitrary values of t and s, then $f(t) = C = g(s)$ where C is a constant.*}

3. Show the limit in the IFA limit for

$$I\left[(dt)^{\alpha}\right](t) \equiv \int_0^t (ds)^{\alpha} dW(s) \stackrel{\text{ims}}{=} 0\,,$$

provided $\alpha > 0$ and is real (i.e., not necessarily an integer). {*Hint: See text, Lemma 2.22 for the case* $\alpha = 1$, *noting that it says IMS, but means the IFA limit only as in the "proof".*}

4. Instead of the IFA, considered the θ-rule of stochastic in integration. $0 < \theta \leq 1$, for the integral,

$$I[WdW](t) = \int_0^t W(s)dW(s) \underset{\text{rule}}{\overset{\theta}{\simeq}} I_n^{(\theta)}[WdW] \equiv \sum_{i=0}^{n} W_{i+\theta}\Delta W_i,$$

where $W_{i+\theta} = W(t_{i+\theta})$ and $\boldsymbol{t_{i+\theta} \equiv t_i + \theta\Delta t_i}$. Note that $W_{i+\theta}$ and ΔW_i and not independent, since they correspond to overlapping intervals, but can be decomposed into partial increments $\Delta_\theta[W_i] \equiv W_{i+\theta} - W_i$ and $\Delta_{1-\theta}[W_i] \equiv W_{i+1} - W_{i+\theta}$ to separate the independent parts from an dependent one.

(a) Assuming that $\mathrm{E}[I[WdW](t)] \underset{\text{rule}}{\overset{\theta}{=}} \lim_{n\to\infty} \mathrm{E}[I_n^{(\theta)}[WdW]]$ use the above decomposition technique to show that

$$\mathbf{E}[\boldsymbol{I_n^{(\theta)}}[\boldsymbol{WdW}]] = \boldsymbol{\theta} \sum_{\boldsymbol{i=0}}^{\boldsymbol{n}} \boldsymbol{\Delta t_i} \quad \& \quad \boldsymbol{E[I[WdW](t)]} \underset{\textbf{rule}}{\overset{\boldsymbol{\theta}}{=}} \boldsymbol{\theta t}.$$

(b) Discuss how the result of part (a) is a counterexample to IFA Th. 2.4 on ***page L2-p33*** and how special the IFA case is among other integration rules.

(c) The result in part (a) suggests that the mean square result is

$$I[WdW](t) \underset{\text{ms}}{\overset{\theta}{=}} 0.5W^2 + \beta(\theta)t,$$

where $\beta(\theta)$ is a correction in θ. Show that $\beta(\theta) = \theta - 0.5$ **for** $I[WdW](t)$ **in the mean, not by the very difficult mean square limit**.

(*Comment: The case* $\theta = 0.5$ *is the midpoint rule and the (Stratonovich) stochastic calculus integrals are usually the same as in regular calculus. The mean square convergence is very difficult so the Itô calculus is used to derive answers and that is why you have not been asked to do it analytically.*)

5. Computationally confirm the mean square limit for the non-Itô (Stratonovich) stochastic integral $I[WdW](t)$ with $\theta = 0.5$ of problem 4 by demonstrating that the midpoint-approximating sum $I_n^{(0.5)}[WdW]$ gives a conjectured integral answer $0.5W^2 + \beta(0.5)$ for sufficiently large n, using two samples with sizes $n = 1000$ and $n = 10000$. Plotting the approximation $I_n[W](t)$ and the error $E_n[W](t) = I_n[W](t) - (0.5W^2 + \beta(0.5))$ versus t for $t \in [0, 2]$, using two graphs, one for each n with both approximation and error on the same graph. List the standard deviation $\mathtt{std}_n$ of the error of each n. Compute the rough estimate of the convergence rate $\hat{\alpha}$ assuming that $\mathtt{std}_n \simeq C/n^{\alpha}$ for constant C. (*Comment: You may modify the Wiener code `wiener09fig1.m` on pages L1-p33 to L1-p35 for the current integrand approximation with* $\theta = 0.5$ *and error.*)

6. Show that the **power rules for Poisson jump integration** can be written as the recursions:

$$\int_0^t P^m(s)dP(s) = \frac{1}{m+1}\left(P^{m+1}(t) - \sum_{k=2}^{m+1}\binom{m+1}{k}\int_0^t P^{m+1-k}(s)dP(s)\right)$$

using the jump form of the stochastic chain rule and the binomial theorem. Illustrate the application of the formulae for $P(t)$ to confirm the results for $m = 0 : 3$ in Table 2.2.1.

7. Show that

$$\int_0^t e^{aP(s)}dP(s) = \left\{\begin{array}{cc} \dfrac{e^{aP(t)}-1}{e^a-1}, & e^a \neq 1 \\ P(t), & e^a = 1 \end{array}\right\},$$

for real constant a, showing that they give the same answers in two ways:

(a) Using the pure Poisson sum form of the theorem for $\sum_{k=0}^{P(t)-1} h(k)$ for function h and the geometric series partial sum.

(b) Using the Zero-One Jump Law and the Fundamental Theorem of Jump Calculus applied to $d\exp(aP(t))$ to evaluate the integral.