FINM 345/Stat 390 Stochastic Calculus, Autumn 2009

Floyd B. Hanson, Visiting Professor

Email: fhanson@uchicago.edu

Master of Science in Financial Mathematics Program
University of Chicago

Lecture 3 (from Singapore)
Diffusion Stochastic Calculus

6:30-9:30 pm, 12 October 2009, Kent 120 in Chicago

7:30-10:30 pm, 12 October 2009 at UBS Stamford

7:30-10:30 am, 13 October 2009 at Spring in Singapore

FINM 345 Stochastic Calculus:

3. Jump-Diffusion Basic Stochastic Chain Rules:

3.1. Diffusion Calculus Basic Chain Rules:

- Recall most **basic dt–precision rule**: $(dW)^2(t) \stackrel{\text{dt}}{=} dt$.
- Higher order zero examples from summary Table 2.1.1: $(dW)^3(t) \stackrel{\text{dt}}{=} 0$, $dtdW(t) \stackrel{\text{dt}}{=} 0$, $(dt)^2 \stackrel{\text{dt}}{=} 0$, etc.
- Preliminary forms for increments of general function $\mathcal{G}(t) \equiv G(W(t), t)$:

$$\Delta G(W(t), t) \equiv G(W(t+\Delta t), t+\Delta t) - G(W(t), t),$$

so with $\Delta W(t) \equiv W(t + \Delta t) - W(t)$,

$$\Delta G(W(t), t) = G(W(t) + \Delta W(t), t+\Delta t) - G(W(t), t).$$

- Similarly for the differential,

$$dG(W(t), t) = G(W(t) + dW(t), t+dt) - G(W(t), t).$$

- **3.1.1. Special Diffusion Examples with $G = G(W(t))$:**
  - **Itô Cubic Integral** using binomial expansion:

$$\begin{aligned}\Delta[W^3](t) &= (W+\Delta W)^3(t) - W^3(t) \\ &\stackrel{\text{bin}}{=} \left(3W^2\Delta W + 3W(\Delta W)^2 + (\Delta W)^3\right)(t),\end{aligned}$$

so in the **dt–precision limit** as $\Delta t \longrightarrow dt \longrightarrow 0$,

$$d\left[W^3\right](t) \stackrel{\text{dt}}{=} (3W^2(t)dW(t) + 3W(t)dt)\ ,$$

where the $3wdt$ is the Itô correction to the deterministic differential $d(w^3) = 3w^2dw$. Solving for $W^2(t)dW(t)$ and Itô integrating,

$$\int_{t_0}^{t} W^2(s)dW(s) \underset{\text{dt}}{\overset{\text{int}}{=}} \frac{1}{3}\left(W^3(t) - W^3(t_0)\right) - \int_{t_0}^{t} W(s)ds,$$

where $\left(\underset{\text{dt}}{\overset{\text{int}}{=}}\right)$ means integrating using dt–precision limits, a formal version of IMS equality $\left(\stackrel{\text{ims}}{=}\right)$.

Note that the Itô integral has only been reduced to an explicit exact term plus a Riemann integral of $W(t)$.

- **Itô General Power Integral** using the full binomial expansion theorem (Online Appendix B.150),

$$\Delta\left[W^{m+1}\right](t) = (W+\Delta W)^{m+1}(t) - W^{m+1}(t)$$
$$\stackrel{\text{bin}}{=} \sum_{i=0}^{m} \binom{m+1}{i} W^i(t)(\Delta W)^{m+1-i}(t),$$

where the passage to the limit as $\Delta t \to dt \to 0^+$ and the dt–precision limits leading to the Itô integral form,

$$\int_0^t W^m(s)dW(s)$$

and its reduction to an exact integral plus Riemann integral has been left as an Exercise.

- **Itô Exponential Integral** : Using the **laws of exponentials (LOE)** and the first few terms of the exponential expansion (B.53), going directly to the formal differential form and skipping the more general increment form to expedite applied stochastic calculations,

$$d\left[e^{W}\right](t) = \left(e^{W+dW} - e^{W}\right)(t) \stackrel{\text{loe}}{=} \left(e^{W}\left(e^{dW}-1\right)\right)(t)$$
$$\stackrel{\text{dt}}{=} \left(e^{W}\left(dW + \frac{1}{2}(dW)^2\right)\right)(t),$$

neglecting differential forms that are zero in dt–precision limit, such as $dW^3(t) \stackrel{\text{dt}}{=} 0$, $dtdW(t) \stackrel{\text{dt}}{=} 0$, $(dt)^2 \stackrel{\text{dt}}{=} 0$ and higher powers.

Using the basic mean square limit differential form, $(dW)^2(t) \stackrel{\mathrm{dt}}{=} dt$, so

$$d\left[e^W\right](t) \stackrel{\mathrm{dt}}{=} \left(e^W\left(dW + \frac{1}{2}dt\right)\right)(t).$$

This is almost like the deterministic differential, $d(e^w) = e^w dw$, but here with an Itô stochastic correction $e^{W(t)}dt/2$. Solving for $e^{W(t)}dW(t)$, the Itô integral of the exponential of $W(t)$ yields the implicit integration

$$\int_{t_0}^{t} e^{W(s)}dW(s) \underset{\mathrm{dt}}{\stackrel{\mathrm{int}}{=}} e^{W(t)} - e^{W(t_0)} - \frac{1}{2}\int_{t_0}^{t} e^{W(s)}ds$$

by the FTSC.

As with the integral of $W^2(t)$, the Itô integral of $e^{W(t)}$ cannot be Itô integrated exactly and must be numerically simulated if needed, e.g., using the Itô partial sums form of the stochastic exponential,

$$S_{i+1} \stackrel{\text{ifa}}{=} \sum_{j=0}^{i} \exp(W_i)\Delta W_i \;\;\&\;\; W_{i+1} \stackrel{\text{ifa}}{=} \sum_{j=0}^{i} \Delta W_j,$$

for $t = t_{i+1} = (i+1)\Delta t$ for $t_0 = 0$ evenly spaced using $\Delta t_i = \Delta t$. The error is

$$E_{i+1} \stackrel{\text{ifa}}{=} S_{i+1} - \left(\exp(W_{i+1}) - 1 - \frac{1}{2}\sum_{j=0}^{i} \exp(W_j)\Delta t\right),$$

between the partial sums S_{i+1} and the difference approximation to the right-hand side, if $t_0 = 0$, $W(t_0) = 0$ & $\exp(W(t_0)) = 1$.

Remember that the cumulative noise W_i should **always be approximated by sums of simulated independent increments** ΔW_j for $j=0:i-1$, else big problems. In the differential of the pure exponential, there is a clue to an exact differential in the Itô mean square sense, since the factor $(dW+dt/2)$ suggests subtracting $t/2$ from $W(t)$. In fact,

$$d\left[e^{W(t)-t/2}\right] \stackrel{\text{dt}}{=} e^{W(t)-t/2}dW(t),$$

so

$$\int_0^t e^{W(s)-s/2}dW(s) \underset{\text{dt}}{\stackrel{\text{int}}{=}} e^{W(t)-t/2}-1,$$

the perfect analog to the the deterministic integral $\int_0^w e^v dv = e^w - 1$, if an Itô shift $t/2$ is used.

- **Chain Rule 3.1. For $G(W(t))$.**
  *Let $G(w)$ be twice continuously differentiable. Then the **differential form** of the **Itô stochastic chain rule for $G(W(t))$** is*

$$dG(W(t)) \overset{\text{dt}}{=} G'(W(t))dW(t) + \frac{1}{2}G''(W(t))dt,$$

  *corresponding to the **integral form** of the **Itô stochastic chain rule for $G(W(t))$**,*

$$G(W(t)) \underset{\text{dt}}{\overset{\text{int}}{=}} G(W(t_0)) + \int_{t_0}^{t} G'(W(s))dW(s) + \frac{1}{2}\int_{t_0}^{t} G''(W(s))ds,$$

  for $0 \leq t_0 \leq t$.

Sketch of Proof: Assuming $G(w)$ is twice continuously differentiable in the argument w, we then see that $G(W(t))$ has the differential:

$$dG(W(t)) = G(W(t)+dW(t)) - G(W(t))$$
$$\stackrel{\mathrm{dt}}{=} G'(W(t))dW(t) + \frac{1}{2}G''(W(t))(dW)^2(t),$$

keeping only terms of dt–precision, neglecting terms such as $dW^3(t)$, $dtdW(t)$ and $(dt)^2$, finally using $(dW)^2(t) \stackrel{\mathrm{dt}}{=} dt$, essentially satisfying the mean square limit. This yields the differential form of the Itô stochastic **chain rule for $G(W(t))$** and the integral form easily follows. □

The last term in the second derivative is the Itô stochastic correction to the deterministic chain rule.

Rewriting Stochastic Chain Rule 3.1 yields the **fundamental theorem of stochastic calculus** according to the Itô:

Corollary 3.1. Form $G'dW$ of Fundamental Theorem of Itô Integral Calculus for Diffusions.

Let $G(w)$ be twice continuously differentiable. Then

$$\int_{t_0}^{t} G'(W(s))dW(s) \overset{\text{int}}{\underset{\text{dt}}{=}} G(W(t)) - G(W(t_0)) - \frac{1}{2}\int_{t_0}^{t} G''(W(s))ds.$$

Remark: Recall the more elementary integral of a differential form of the fundamental theorem of stochastic diffusion calculus, which in fact leads to the exact part of the Itô version, using G **and the FTSC**,

$$\int_{t_0}^{t} dG(W(s)) \overset{\text{ims}}{=} G(W(t)) - G(W(t_0)).$$

- **3.1.2. Diffusion Examples with $G = G(W(t), t)$, having Explicit Time Dependence**:
  - **Chain Rule 3.2. for $G(W(t), t)$ (Itô's Lemma or Formula)**:
    Let **G(w,t)** be twice continuously differentiable in **w** and once continuously differentiable in **t**. Then the differential Itô stochastic **chain rule for $G(W(t), t)$** is

$$dG(W(t), t) \stackrel{\text{dt}}{=} \left(G_t + \frac{1}{2}G_{ww}\right)(W(t), t)dt + G_w(W(t), t)dW(t),$$

corresponding to the integral form of the Itô stochastic **chain rule for $G(W(t), t)$**,

$$G(W(t), t) \stackrel{\text{int}}{\underset{\text{dt}}{=}} G(W(t_0), t_0) + \int_{t_0}^{t} G_w(W(s), s)dW(s) + \int_{t_0}^{t} \left(G_t + \tfrac{1}{2}G_{ww}\right)(W(s), s)ds$$

for $0 \le t_0 \le t$.

Sketch of Proof: Assuming $G(w,t)$ is twice continuously differentiable in the argument w and once continuously differentiable in t, by using a mean square order modification of the Taylor approximation in (B.181), $G(W(t),t)$ has the differential

$$
\begin{aligned}
dG(W(t),t) = {} & G(W(t)+dW(t),t+dt) - G(W(t),t) \\
\stackrel{\text{dt}}{=} {} & G_t(W(t),t)dt + G_w(W(t),t)dW(t) \\
& + \tfrac{1}{2}G_{ww}(W(t),t)(dW)^2(t)
\end{aligned}
$$

where the partial derivatives are denoted with subscripts, i.e.,

$$
G_w(w,t) \equiv \frac{\partial G}{\partial w}(w,t), \quad G_t(w,t) \equiv \frac{\partial G}{\partial t}(w,t),
$$

$$
G_{ww}(w,t) \equiv \frac{\partial^2 G}{\partial w^2}(w,t).
$$

{Beware of silly, confusing pure notation "W_t" for processes W(t), etc.}

Taking the Itô mean square limit with $(dW)^2(t) \stackrel{\mathrm{dt}}{=} dt$ and neglecting the higher order differential forms that are zero in the dt–precision sense, such as $dW^3(t)$, $dtdW(t)$ and $(dt)^2$, yields the differential form of the Itô Chain Rule for $G(W(t),t)$. Again, the last term in the second derivative is the Itô stochastic correction to the deterministic chain rule. Translating the symbolic differential form gives the substantial Itô stochastic integral form. $\square$

- **3.1.3. Remarks on Functions, Values and Partial Derivatives & ∂–Phobias:**
  - For readers without much PDE background and apologies to those with such background, we note that there are certain concepts that are important and there are subtle differences in the function G and its values $G(w, t)$. This is particularly true when there are two or more independent variables, such as the $w = W(t)$ and t in $G(W(t), t)$. This does not arise when there is just one independent variable, such as x in $y = f(x)$. Another complication is that the $W(t)$ is a nondifferentiable function, so **do not form its derivative, but only compute its differential $dW(t)$**, and that is best done formally by the increment form of the differential.

- The symbol G denotes a function specified by a set of rules for its calculation, while $G(w, t)$ is the value of that function with its first argument evaluated at state w and with the second argument at time t. Similarly, $G(W(t), t)$ is the value of G specified at the random variable or state $W(t)$ at time t in place of the realized or dummy variable w. Further, $X(t) = G(W(t), t)$ is the path of the state in time and is nondifferentiable along with $W(t)$, i.e., $X(t)$ is a composite function in time through both arguments of G, implicitly through $W(t)$ and explicitly through the second argument t.

- Using **limits of Newton's quotient for derivatives**, the partial derivatives of $G(w,t)$ are defined, also giving several alternate notations, at (w,t) as

$$G_w(w,t) = \frac{\partial G}{\partial w}(w,t) = \left(\frac{\partial G}{\partial w}\right)\bigg|_{\substack{t \\ \text{fixed}}}(w,t)$$
$$= \lim_{\Delta w \to 0} \frac{G(w+\Delta w,t) - G(w,t)}{\Delta w}$$

and

$$G_t(w,t) = \frac{\partial G}{\partial t}(w,t) = \left(\frac{\partial G}{\partial t}\right)\bigg|_{\substack{w \\ \text{fixed}}}(w,t)$$
$$= \lim_{\Delta t \to 0} \frac{G(w,t+\Delta t) - G(w,t)}{\Delta t},$$

provided the limits exist. **Hence, partial derivatives with one of the variables fixed are based on the definition of ordinary derivatives and are calculated as ordinary derivatives.**

- ∘ The partial derivatives G_w and G_t are defined as rules based upon the target function rule G. For the topics here, when the first argument is a random variable $w = W(t)$, $(\partial G/\partial w)(W(t), t)$ is just G_w evaluated at the first variable $w = W(t)$ after differentiation. We would **never write** $G_{W(t)}$ due to the non-differentiable properties of $W(t)$.

  The partial derivative is calculated first, and then it is evaluated. For example, $G_w(1, 2)$ can be computed if we know G_w and it has a unique value at $(1, 2)$, but $(G(1, 2))_w = 0 = (G(1, 2))_t$, since $G(1, 2)$ has a fixed, constant value, presumably unique, at $(1, 2)$. The order of partial differentiation and partial derivative function evaluation are very important.

$\{$So, Value $\neq$ Function.$\}$

- Another, more relevant, example illustrating the difference in the differential is multiplying by dt to avoid obtaining the singular derivative of $W(t)$, i.e.,

$$dG(W(t),t) \stackrel{\mathrm{dt}}{=} \left(G_t dt + G_w dW(t) + \frac{1}{2} G_{ww} dt\right)(W(t),t),$$

contains the partial derivative of the function G with respect to t evaluated at $(W(t),t)$, $(\partial G/\partial t)(W(t),t)dt$, rather than the partial derivative with respect to t written as the derivative of the value $G(W(t),t)$, $(\partial G/\partial t)(W(t),t)dt$, which makes no sense since it would involve the derivative of the nondifferentiable $W(t)$ in t with probability one (recall Th. 1.1, page L1-p26).

- **Corollary 3.1. Itô Integral of Partial Derivative**:
Let $g(W(t),t)$ satisfy the conditions of the IMS Definition 2.1.1 (page L2-p19) for an Itô stochastic integral and be once continuously differentiable in w. Let $G(w,t)$ be the antiderivative of $g(w,t)$ with respect to w, i.e., $G_w(w,t) = g(w,t)$, and let $G(w,t)$ be twice continuously differentiable in w, but only once in t. If for $0 \le t_0 \le t$, then

$$\int_{t_0}^{t} g(W(s),s)dW(s) \stackrel{\text{int}}{\underset{\text{dt}}{=}} G(W(t),t) - G(W(t_0),t_0) - \int_{t_0}^{t} (G_t + 0.5 * g_w)(W(s),s)ds.$$

Sketch of Proof: The proof follows directly from **Chain Rule 3.2** by rearranging terms, since $G_w = g$ and $G_{ww} = g_w$. $\square$

- **Remark**: Thus, the Itô stochastic diffusion integral of $g(W(t), t)$ can be reduced to an exact integral $G(W(t), t) - G(W(t_0), t_0)$ with respect to w, less a quasi-deterministic Riemann integral over the diffusion shifted drift function $(G_t + 0.5 * g_w)(W(t), t)$. Thus, if the partial differential equation $(G_t + 0.5 * g_w)(w, t) = 0$ is valid with $g_w(w, t) = G_{ww}(w, t)$, then the integral of $g(W(t), t)$ is equal to the exactly integrated part $G(W(t), t) - G(W(t_0), t_0)$ in the Itô mean square sense. **This idea can be the basis for constructing exact stochastic diffusion integrals.**

- **Example — Merton's Analysis of the Black–Scholes Option Pricing Model**:
  At this point in the text, a good, but abused application in finance is Merton's (1973) mathematical justification and generalization of the Black–Scholes (1973 too) financial options pricing model (see the text Ch. 10 *Applications in Financial Engineering*). The text survey elaborates on Merton's model, which has several state dimensions — the bond, the stock and the option. Multi-dimension SDEs are covered in Chapter.

- **3.1.4. Itô Stochastic Natural Exponential Construction** :

  From the differential of $\exp(W(t))$ (page L3-p6) it is seen that the stochastic exponential is not like the deterministic natural exponential, where the derivative is proportional to the original function.

  - **Deterministic Reference:** For example, the natural exponential e^x in the natural base e has the differential property
  $$d\left(e^x\right) = e^x dx,$$
  returning the original function times dx, and has the inverse relationship to the natural logarithm
  $$e^{\ln(x)} = x$$
  for $x > 0$.

However, when the base $b > 0$ and in particular $b \neq e$, then **by the law of exponentials (LOE)**

$$d\left(b^{x}\right) \stackrel{\text{loe}}{=} d\left(e^{x\ln(b)}\right) = b^{x}\ln(b)dx,$$

returning an additional factor $\ln(b)$.

For more generality, consider the deterministic model

$$d\left(e^{ax}\right) = ae^{ax}dx,$$

where the parameter a is a nonzero constant.

- **General Diffusion Exponential SDE**: The corresponding stochastic model **concerns finding** the process $X(t) = G(W(t), t)$ such that

$$\begin{aligned} dX(t) &= dG(W(t), t) \stackrel{\mathrm{dt}}{=} aG(W(t), t)dW(t) \\ &= aX(t)dW(t). \end{aligned}$$

{**Note, the above equation is not a theorem, but a problem.**} The explicit t dependence is needed to avoid correction factors in dt. Applying the appropriate stochastic $G(W, t)$–chain rule to illustrate a technique for inverting the chain rule to get the desired model in terms of the composite function G.

- **Simple Derivation for Pair of G–PDEs**:

We have

$$\begin{aligned} aG(W(t),t)dW(t) &\overset{\mathrm{dt}}{=} dG(W(t),t) \\ &\overset{\mathrm{dt}}{=} \left(G_t(W(t),t) + \tfrac{1}{2}G_{ww}(W(t),t)\right)dt \\ &\quad + G_w(W(t),t)dW(t). \end{aligned}$$

Since the differentials, $dW(t)$ and dt, can be independently varied **functionally** in this equation, the coefficients of $dW(t)$ and dt can be separately set equal to their values on both sides of the equation (dropping the arguments of G for simplicity):

$$G_w = aG \;\;\&\;\; G_t + \frac{1}{2}G_{ww} = 0.$$

{Note also, this is an example of a general technique for finding exact solutions of SDEs.}

- **First PDE Exponential Solution in State w**: The solution of the first partial differential equation (PDE), $G_w = aG$, really an ODE with t held fixed, is

$$G(w,t) = A(t)e^{aw},$$

since $d(e^{-aw})/dw = -ae^{-aw}$ (differentiation is allowable for a regular continuous, i.e., nonstochastic, function) so

$$\left(e^{-aw}G\right)_w = e^{-aw}\left(G_w - aG\right) = 0,$$

which shows that the solution $G = Ae^{aw}$ satisfies the first PDE, $G_w = aG$, by substitution, $e^{-aw} \neq 0$. Here, $A(t)$ is a **function, rather than constant, of integration** since the differential equation is only in w and t is arbitrary, although held fixed in the equation.

Remark:

Given a differentiable function $F(w,t)$, the notation $F_w(w,t) = 0$ is shorthand for the partial deriviative

$$\left(\frac{\partial F}{\partial w}\right)_{\substack{t \\ \text{fixed}}}(w,t) = 0.$$

This means that $F(w,t) = A(t)$ for some function A of t, since t is held fixed in the partial differentiation with respect to w.

- **Second PDE Exponential Solution in Time t:** Upon substituting this current functional form into the second PDE, $G_t + 0.5G_{ww} = 0$, using

$$(A(t)e^{aw})_t = e^{aw}(A(t))_t = A'(t)e^{aw},$$
$$(A(t)e^{aw})_{ww} = A(t)(e^{aw})_{ww} = a^2A(t)e^{aw},$$

then

$$A'(t)e^{aw} + \frac{a^2}{2}A(t)e^{aw} = 0.$$

Canceling out the common nonzero factor e^{aw},

$$A'(t) + \frac{a^2}{2}A(t) = 0,$$

and solving for the function of integration yields

$$A(t) = Ce^{-a^2t/2},$$

where C is a **genuine constant of integration**.

Remark: Note that an ultimate test of a solution of a differential equation solution is the **substitution test**, i.e., substituting the solution back into the equation and verifying that the equation and any conditions are satisfied.

Substitution of $A(t)$ back into its ODE to verify the solution,

$$A'(t) + \frac{a^2}{2}A(t) = Ce^{-a^2t/2} \cdot \left(-\frac{a^2}{2} + \frac{a^2}{2}\right) = 0.$$

{**Differential Equation Substitution Rule: It does not matter how you obtained a solution, because if you can substitute it back into the equation and show that it satisfies the equation then that is all you have to do. Of course, it is useful to know how to solve DEs too.**}

- **Reassembling Parts into $X(t)$**: By reassembling the parts of the solution, we obtain the Itô general stochastic form of the natural exponential (exponential in the natural base e),

$$X(t) = G(W(t), t) = Ce^{aW(t)-a^2t/2},$$

systematically deriving and generalizing what previously was a guess for the stochastic natural-like exponential $e^{W(t)-t/2}$. The extra exponential term $(-a^2t/2)$ is the special Itô correction that forces something like the deterministic linear growth model $dy(t) = ay(t)dt$ for the stochastic exponential growth in the diffusion $W(t)$ when $dX(t) = aX(t)dW(t)$.

Since $W(0^+) = 0$ with probability one, $X(0^+) = G(0, 0^+) = C$, with probability one, is the initial value of the state $X(t)$, while a is a rate of growth. The basic moments of the state trajectory can be calculated by using the density $\phi_{W(t)}(w)$ for $W(t)$ (page L1-p23).

◦ **Completing the Square Technique**:
Some of the details are given to illustrate the use of the **completing the square** technique when computing exponential moments with respect to normal distributions. An illustration of the completing the square technique is presented for the expectation of an exponential whose exponent is linear (or affine) in $W(t)$, i.e., $\exp(a(t)W(t)+b(t))$ for fixed t.

Lemma 4.1. Completing the Square for $\mathrm{E}[K(t)\exp(a(t)W(t)+b(t))]$:
Let $a(t)\neq 0$, $b(t)$ and $K(t)\neq 0$ be bounded deterministic functions of t. Then

$$\mathrm{E}\left[K(t)e^{a(t)W(t)+b(t)}\right] = K(t)e^{a^2(t)t/2+b(t)}.$$

Proof: Since the Wiener process density,

$$\phi_{W(t)}(w) = \frac{1}{\sqrt{2\pi t}} e^{-w^2/(2t)},$$

$-\infty < w < +\infty$, $t > 0$, from page L1-p23, is essentially a function of the sampled dummy variable w and t is only a parameter that we can hold fixed during the integration, the deterministic functions of time are treated as constants. By the **laws of exponents**, the exponent of the density and the exponent of the argument of the expectation with the dummy variable substitution $W(t) = w$ are added together to obtain a complete square of all quadratic w terms,

$$\begin{aligned} & -w^2/(2t) + a(t)w + b(t) \\ & \qquad = -(w - a(t)t)^2/(2t) + a^2(t)t/2 + b(t). \end{aligned} \tag{3.1}$$

Thus,

$$\mathrm{E}\left[K(t)e^{a(t)W(t)+b(t)}\right] = K(t)\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{-(w-a(t)t)^2/(2t)} \cdot e^{+a^2(t)t/2+b(t)}dw$$

$$\stackrel{\text{loe}}{=} K(t)e^{a^2(t)t/2+b(t)}\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{-v^2/(2t)}dv$$

$$= K(t)e^{a^2(t)t/2+b(t)}\mathrm{E}[1],$$

$$= K(t)e^{a^2(t)t/2+b(t)},$$

where the fixed part of the integral with exponent $(a^2(t)t/2+b(t))$ has been separated out and the change of variables $v=w-a(t)t$ with $dv=dw$, t being fixed, in the integral has been used to transform the completed square part of the expectation integral into conservation of probability $\mathrm{E}[1]=1$ for the standard Wiener process.

$\square$

Example — Mean State $\mathrm{E}[X(t)]$: Using Lemma 4.1,

$$\mathrm{E}\left[Ce^{aW(t)-a^2t/2}\right] = C = X(0^+),$$

so the mean trajectory is a constant at the initial level $X(0^+)$. However, the state variance, again using **Lemma 4.1** but with $a(t)$ replaced by $2a$ following application of the variance-expectation identity (B.186), $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}^2[X]$, to use the expectation result above, is

$$\begin{aligned}
\mathrm{Var}\left[Ce^{aW(t)-a^2t/2}\right] &= \mathrm{E}\left[\left(Ce^{aW(t)-a^2t/2}\right)^2\right] \\
&\quad -\mathrm{E}^2\left[Ce^{aW(t)-a^2t/2}\right] \\
&= C^2\mathrm{E}\left[e^{2aW(t)-a^2t}\right] - C^2 \\
&= C^2\left(e^{a^2t} - 1\right).
\end{aligned}$$

Examining the standard deviation, or square root of the variance,

$$\sigma_{X(t)} = \sqrt{\mathrm{Var}[X(t)]} = C\sqrt{e^{a^2t} - 1} \sim Ce^{a^2t/2}$$

as $t \to \infty$, it is seen that the RMS of stochastic fluctuations grows exponentially with exponent $a^2t/2$ starting initially at $\sigma_{X(0^+)} = 0^+$.

Figure 3.1 is an illustration of the simulation of the integral of this natural exponential in the special case

$$\begin{aligned} I[\mathbf{gdW}](t) &= \int_0^t g(W(s), s)dW(s) \\ &= \int_0^t e^{W(s)-s/2}dW(s) \\ &\stackrel{\text{ims}}{=} e^{W(t)-t/2} - 1, \end{aligned}$$

i.e., when $a = 1 = C$.

Also plotted in **Figure 3.1** is the diffusion process $W(t)$ for comparison and the error,

$$E_{i+1} = S_{i+1} - I_{i+1},$$

between the simulation of the integral by Itô finite difference partial sums

$$S_{i+1} = \sum_{j=0}^{i} g_j \Delta W_j,$$

and the simulation of the exact mean square integral value for $I[\mathrm{gdW}](t)$,

$$I_{i+1} = g_{i+1} - 1,$$

for $i = 0 : n$, where the integrand is

$$g_i = \exp(W_i - t_i/2)$$

with $W_i = \sum_{j=0}^{i-1} \Delta W_j$ and $t_i = i\,\Delta t$ for $i = 0 : n+1$.

Observe that the integral initially tracks the W_i simulated noise, but eventually diverges from it. Also, the error slowly degrades as time t_i becomes long (not shown), in this case for $n = 10,000$ (note that this is an approximate sample size since random sample size is $n + 1 = 10,001$ random increments) and $t = 2.0$. The MATLAB code for the exactly integrable $g(W(t), t)$ in the Itô mean square diffusion integral sense is given in Program C.12, called **`intgwtdw.m`** in Online Appendix C.

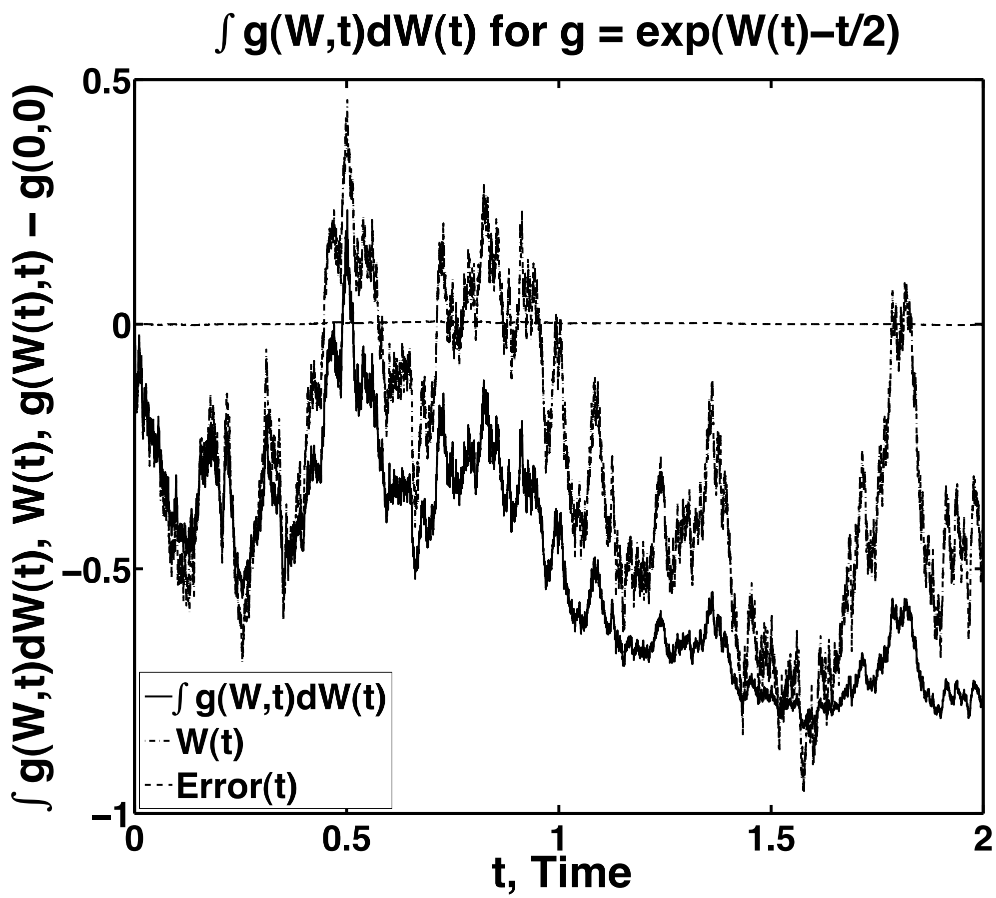

Figure 3.1: Simulated Itô IFA for stochastic diffusion integral $I_n[\mathbf{gdW}](t_{i+1}) = \sum_{j=0}^{i} g_j \Delta W_j$ for $i = 0 : n$, using MATLAB **randn** with sample size $n = 10,000$ on $0 \leq t \leq 2.0$. Presented are the simulated Itô partial sums S_{i+1}, the simulated noise W_{i+1} and the error E_{i+1} relative to the exact integral, $I^{(\mathrm{ims})}[\mathbf{gdW}](t_{i+1}) \overset{\mathrm{ims}}{=} \exp(W_{i+1} - t_{i+1}/2) - 1$, in the Itô mean square sense.

- ***$\sum_i g\Delta W_i$ book MATLAB code example (edited)***:

```
function intgwtdw
% Fig. 4.1 Book code example for int[g(w,t)dw], [t0,t]
%   by RNG Simulation (3/2007):
%   Generation is by summing g(W(i),t(i))dW(i)
%      for i=0:n, but converted base 0 to base 1:
%      matlab[G(W(i),T(i))DW(i);i=1:N+1]  ...
%             = math[g(W(i),t(i))dW(i);i=0:n].
%   Sample g(w,t) = exp(w-t/2), exact g(w,t)-1, [0,t].
clc % clear variables,
clf % clear figures
fprintf('\nfunction intgwtdw OutPut:')
nfig = 0; % figure counter.
TF= 2.0;T0 = 0;N = 20000; NI = N+1;dt = (TF-T0)/NI;
%                          time grid:  Fixed Delta{t}.
sqrtdt = sqrt(dt); % Set std. Wiener inc. time scale.
% Begin Sample Path Calculation:
kstate = 1;
```

```
randn('state',kstate);   % set state to repeat.
DW = sqrtdt*randn(1,NI); % random  vector of N+1
T = zeros(1,NI+1); T(1) = 0;   % set T initially.
W = zeros(1,NI+1);   % Set W(1) for base 1 vector.
S = zeros(1,NI+1);   % Set integral sum initially.
gv = zeros(1,NI+1); gv(1) = g(W(1),T(1)); % Initial;
Err = zeros(1,NI+1); % Set Error initially.
for i = 1:NI % Sim. Sample paths by Inc. Accum.:
    T(i+1) = T(i) + dt;
    W(i+1) = W(i) + DW(i);
    gv(i+1) = g(W(i+1),T(i+1));
    S(i+1) = S(i) + gv(i)*DW(i);%see subfun. below.
    Err(i+1) = S(i+1) - (gv(i+1) -gv(1)); %  g -> gv
end
T(1,NI+1) = TF;   % Correct for  rounding errors.
% Begin Plot:
nfig = nfig + 1;
fprintf('\n\nFigure(%i): int[g](t) vs t Sims.\n',nfig)
```

```
figure(nfig);
scrsize = get(0,'ScreenSize'); % for target screen
ss = [5.0,4.0,3.5]; % figure spacing factors
plot(T,S,'k-',T,W,'k-.',T,Err,'k--','LineWidth',2);
title('\int g(W,t)dW(t) for g = exp(W(t)-t/2)'...
    ,'FontWeight','Bold','Fontsize',44);
ylabel('\int g(W,t)dW(t), W(t), g(W(t),t) - g(0,0)'...
    ,'FontWeight','Bold','Fontsize',44);
xlabel('t, Time'...
    ,'FontWeight','Bold','Fontsize',44);
hlegend=legend('\int g(W,t)dW(t)','W(t)','Error(t)'...
    ,'Location','SouthWest');
set(hlegend,'Fontsize',36,'FontWeight','Bold');
set(gca,'Fontsize',36,'FontWeight','Bold','linewidth',3);
set(gcf,'Color','White','Position' ...
   ,[scrsize(3)/ss(nfig) 60 scrsize(3)*0.6 scrsize(4)*0.8]);
% End Main
%
```

```
function gv = g(W,T)
% Example g(W(t),t) = exp(W(t) - t/2);
% Exact integral = g(W(t),t) - 1;
gv = exp(W - T/2);
% End intgwtdw Code
```

In **Figure 3.2** **(Corrected from text 4.2, which was incorrectly a copy of 4.3)**, the chain rule formulation of the Itô diffusion integral of the simple exponential $g(X(t), t) = \exp(W(t))$ of the Example, on **(L3-p5)**, is compared to the Itô partial sums $S_{i+1} = \sum_{j=0}^{i} g_j \Delta W_j = \sum_{j=0}^{i} \exp(W_j) \Delta W_j$. Unlike the stochastic natural exponential $\exp(W(t) - t/2)$, the simple exponential is not exactly integrable in the Itô mean square sense since the stochastic chain rule introduces a quasi-deterministic, regular-type integral for the diffusion term

$$-0.5 G_w(w, t) = -0.5 g(w, t) = -0.5 \exp(w).$$

The partially integrated chain rule form is thus

$$I_{i+1} = \exp(W_i) - 1 - 0.5 * \sum_{j=0}^{i} \exp(W_j)\Delta t$$

with $G_t(w,t)=0$. In the figure the error between the two approximations of the integral is $E_{i+1} = S_{i+1} - I_{i+1}$ and the underlying diffusive noise is $W(t)$. The error is very small for a sample size of $n=10,000$. The integration significantly dampens the fluctuations in the original noise $W(t)$. The MATLAB code for this figure is given in **Program C.13**, called **`intgxtdw.m`** in Online Appendix C. Compare this code **`intgxtdw.m`** with the prior special case code **`intgwtdw.m`** when $X(t) = W(t)$.

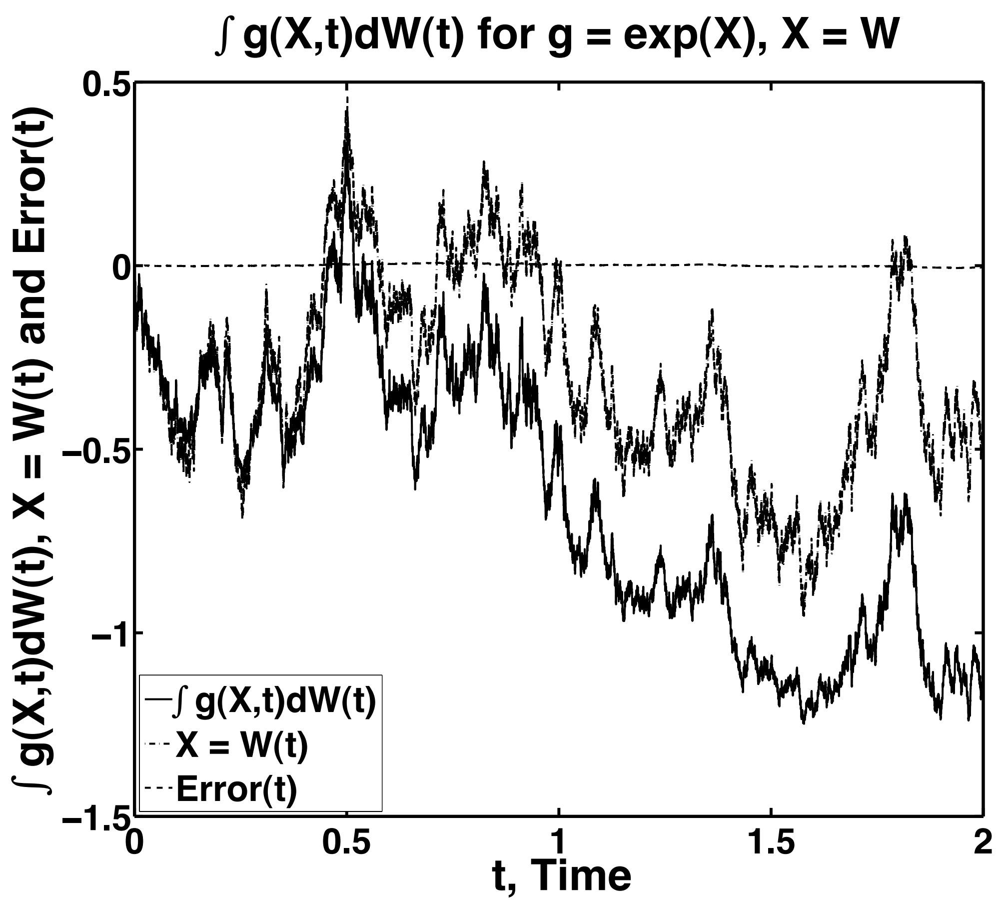

Figure 3.2: Simulated Itô IFA to the diffusion integral $I_n[g](t_{i+1}) = \sum_{j=0}^{i} g_j \Delta W_j = \sum_{j=0}^{i} \exp(W_j) \Delta W_j$ for $i = 0 : n$, using MATLAB **randn** with sample size $n+1 = 10,001$ on $0 \leq tv \leq 2.0$. Presented are the simulated Itô partial sums S_{i+1}, the simulated noise W_{i+1} and the error E_{i+1} relative to the stochastic chain rule partially integrated form I_{i+1} given on **(L3-p46)**.

- $\sum_i \exp(W)\Delta W_i$ ***MATLAB code example (edited)***:

```
function intgxtdw
% Fig. 4.2 Book code for int[g(x(t),t)dw], [t0,t]
%   Generation is by summing g(X(i),t(i))dW(i)   for
% matlab[G(X(i),T(i))DW(i);i=1:N+1]
% = math[g(X(i),t(i))dW(i);i=0:n].
% Int[gdW](t)=G(W,t)-G(0,0)-Int[(g_t+0.5*g_w)(w,t)dt];
% G_w(w,t) = g(w,t), G_{ww}(w,t) = g_w(w,t).
% Here  g(x,t) = exp(x) and x = w.
clc % clear variables
clf % clear figures
fprintf('\nfunction intgxtdw OutPut:')
nfig = 0; % figure counter.
TF= 2.0;T0 = 0;N =(TF-T0)*10000;NI=N+1;dt =(TF-T0)/NI;
sqrtdt = sqrt(dt); % Set standard Wiener increment.
% Begin Sample Path Calculation:
kstate = 1;
randn('state',kstate);  % set state if repeated.
```

```
dW = sqrtdt*randn(1,NI); % Generate samples for dW(t)
t = zeros(1,NI+1); t(1) = T0;  % set T initially.
W = zeros(1,NI+1);  % Set W(1) in place of W(0) = 0
X = zeros(1,NI+1);  % Set integral sum initially.
gv = zeros(1,NI+1); gv(1) = g(X(1),t(1)); %initialize
sdw = zeros(1,NI+1); sdt = zeros(1,NI+1); %initialize
ev = zeros(1,NI+1); % Set Error initially.
for i = 1:NI % Simulated Sample paths by Inc. Accum.
    t(i+1) = i*dt;
    W(i+1) = W(i) + dW(i);
    X(i+1) = W(i+1); % Set State for this g Example.
    gv(i+1) = g(X(i+1),t(i+1));
    sdw(i+1) = sdw(i)+gv(i)*dW(i);% g subfunction.
    sdt(i+1) = sdt(i)-gthgw(X(i+1),t(i+1))*dt;% ''
    ev(i+1) = sdw(i+1)-exact(X(i+1),t(i+1))-sdt(i+1);
    % CAUTION: For given g only!
end
t(NI+1) = TF;  % Correct for rounding errors.
```

```
% Begin Plot:
nfig = nfig + 1;
fprintf('\n\nFigure(%i): int[g](t) vs t Sims\n',nfig)
figure(nfig)
scrsize = get(0,'ScreenSize'); % for target screen
ss = [5.0,4.0,3.5]; % figure spacing factors
plot(t,sdw,'k-',t,W,'k-.',t,ev,'k--','LineWidth',2);
title('\int g(X,t)dW(t) for g = exp(X), X = W'...
    ,'FontWeight','Bold','Fontsize',44);
ylabel('\int g(X,t)dW(t), X = W(t) and Error(t)'...
    ,'FontWeight','Bold','Fontsize',44);
xlabel('t, Time'...
    ,'FontWeight','Bold','Fontsize',44);
hlegend=legend('\int g(X,t)dW(t)','X = W(t)','Error(t)'...
    ,'Location','SouthWest');
set(hlegend,'Fontsize',36,'FontWeight','Bold');;
set(gca,'Fontsize',36,'FontWeight','Bold','linewidth',3);
set(gcf,'Color','White','Position' ...
```

```
   ,[scrsize(3)/ss(nfig) 60 scrsize(3)*0.6 scrsize(4)*0.8]);
% End Main
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function gv = g(x,t) % ignore warning on unused t.
% Sample g(X(t),t) only, e.g.,
%1% gv = exp(x-t/2); % x = w.
%2% gv = exp(x); % x = w.
%3% gv = x; % x = w.
gv = exp(x);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function exactv = exact(x,t)
% Sample g(X(t),t) exact integrals only, e.g.,
%1% exactv = exp(x-t/2) - 1; % i.e., x=w, G=exp(w-t/2).
%2% exactv = exp(x) - 1; % i.e., x=w, G=exp(w).
%3% exactv = 0.5*(x^2-t);  % i.e., x=w, G=0.5*(w^2-t).
exactv = exp(x) - 1; % i.e.,  x=w, G=exp(w).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
```

```
function gthgwv = gthgw(x,t)
% Reg. Correction Int. of (G_t+0.5*G_{ww}), G_w=g
%1% gthgwv = 0; %i.e.,g=exp(x-t/2)=G,G_t=-0.5*G,G_{ww}=G
%2% gthgwv =0.5*exp(x);%i.e.,G=exp(w),G_t=0,G_{ww}=exp(w)
%3% gthgw 0; %i.e.,g=x=w,G=0.5*(w^2-t),G_t=-0.5,G_{ww}=1
gthgwv = 0.5*exp(x); % i.e.,G=exp(w),G_t=0,G_{ww}=exp(w)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% End intexpwtdw Code
```

- **3.2. Transformations of Linear Diffusion Equations**:

Consider the diffusion SDE, linear in the state process $X(t)$, with time-dependent coefficients

$$dX(t) = X(t)\left(\mu(t)dt + \sigma(t)dW(t)\right), \tag{3.2}$$

where the initial condition is $X(t_0) = x_0 > 0$ with probability one, $\mu(t)$ is called the **drift** or deterministic coefficient and $\sigma(t)$ is called the **volatility** or standard deviation of the diffusion term. The **diffusion coefficient** is usually defined as $\mathcal{D} = \sigma^2(t)/2$, so $\sigma(t) = \sqrt{2\mathcal{D}}$. The linear form of (3.2) is sometimes called the **multiplicative noise** case, **such as used in finance**, the state $X(t)$ multiplies the stochastic terms, and the word **noise** refers to the randomness or stochastic properties here.

In the deterministic case, transforming the state variable into its logarithm makes the right-hand side independent of the transformed state variable, so let

$$Y(t) = F(X(t)) \equiv \ln(X(t)).$$

Since we have F depending on $X(t)$ rather than $W(t)$, we go back to the basic treatment of the change as an increment and expand the increment to second order,

$$\begin{aligned} dY(t) &= \log(X(t) + dX(t)) - \log(X(t)) \\ &\stackrel{\mathrm{dt}}{=} \frac{1}{X(t)} dX(t) - \frac{1}{2X^2(t)} (dX)^2(t) \\ &\stackrel{\mathrm{dt}}{=} (\mu(t)dt + \sigma(t)dW(t)) - 0.5\sigma^2(t)(dW)^2(t) \\ &= (\mu(t) - 0.5\sigma^2(t))dt + \sigma(t)dW(t), \end{aligned} \tag{3.3}$$

where we again used $(dW)^2(t) \stackrel{\mathrm{dt}}{=} dt$ and dropped terms zero in dt–precision.

Use has been made of the following partial derivatives when $F(x) = \ln(x)$,

$$F_t(x) \equiv 0, \; F_x(x) = 1/x, \; F_{xx}(x) = -1/x^2.$$

The final line in (3.3), with only differentials dt & $dW(t)$, is also called **additive noise** since it just adds to the state value and can be immediately integrated, as opposed to the multiplicative noise in the original SDE in (3.2). In the above derivation, the Itô stochastic correction on the drift $\mu(t)$ is the negative of the **diffusion coefficient** $\sigma^2(t)/2$. The final right-hand side (3.3) defines a differential simple Gaussian process (B.24), with the diffusion specified ($\pm$) by the **infinitesimal mean** $\mathrm{E}[dY(t)] = (\mu(t) - 0.5\sigma^2(t))dt$ and **infinitesimal variance** of $\mathrm{Var}[dY(t)] = \sigma^2(t)dt$.

So, the **infinitesimal mean** here is defined as

$$\mathrm{E}[dY(t)]$$

and the **infinitesimal variance** is defined as

$$\mathrm{Var}[dY(t)],$$

both defining a **diffusion**, given some technical conditions, in each case neglecting orders smaller than $\mathrm{ord}(dt)$ (*means exact order* dt*, so "*$< \mathrm{ord}(dt)$*" means the same as "*$= \mathrm{o}(dt)$*" and "*$\leq \mathrm{ord}(dt)$*" means the same as "*$= \mathrm{O}(dt)$*"*) .

An alternate method of deriving (3.3) is to use the Itô stochastic chain rule for $G(W(t))$, but with $W(t)$ replaced by $X(t)$, subsequently expanding the differentials $dX(t)$ and $(dX)^2(t)$, then replacing them by the SDE in (3.2) and neglecting any terms that are zero in the dt–precision of modeling.

Since the right-hand side of (3.3) does not depend on the state $Y(t)$, we can immediately integrate for $Y(t)$ given the coefficient functions leading to

$$Y(t) = y_0 + \int_{t_0}^{t} (\mu(s) - 0.5\sigma^2(s))ds + \int_{t_0}^{t} \sigma(s)dW(s), \tag{3.4}$$

where $y_0 = \ln(x_0)$.

Exponentiation leads to the formal solution for the original state,

$$X(t) = x_0 \exp\left(\int_{t_0}^{t} (\mu(s) - 0.5\sigma^2(s))ds + \int_{t_0}^{t} \sigma(s)dW(s) \right). \tag{3.5}$$

- **3.2.1. Linear Diffusion SDEs with Constant Coefficients**: If the SDE has **constant coefficients**, $\mu(t) = \mu_0$ and $\sigma(t) = \sigma_0$, while letting $t_0 = 0$, then the solution is simpler:

$$X(t) = x_0 \exp\left((\mu_0 - 0.5\sigma_0^2)t + \sigma_0 W(t)\right). \quad (3.6)$$

Note that if $X(0^+) = x_0$ is initially positive as declared, then the solution $X(t)$ will never become negative by the property of the exponential for real arguments and the transformation $Y(t) = \ln(X(t))$ is proper with $X(t) > 0$. The state $X(t)$ positivity feature is very important in biological and financial applications.

In the **additive noise** case, borrowing the exponent form in (3.3), the relation between the new and old values of Y is computed by adding the noise

$$Y(t+\Delta t) = Y(t) + (\mu_0 - 0.5\sigma_0^2)\Delta t + \sigma_0 \Delta W(t) \tag{3.7}$$

or recursively in the time-step Δt_i from t_i to t_{i+1} and then summing the recursion,

$$\begin{aligned} Y_{i+1} &= Y_i + (\mu_0 - 0.5\sigma_0^2)\Delta t_i + \sigma_0 \Delta W_i \\ &= y_0 + \sum_{j=0}^{i} \left((\mu_0 - 0.5\sigma_0^2)\Delta t_j + \sigma_0 \Delta W_j\right). \end{aligned}$$

So taking the expectation,

$$\mathbf{E}[Y_{i+1}] = y_0 + \sum_{j=0}^{i} \left(\mu_0 - 0.5\sigma_0^2\right) \Delta t_j$$
$$= y_0 + (\mu_0 - 0.5\sigma_0^2) \sum_{j=0}^{i} \Delta t_j .$$

This result should be compared to the corresponding deterministic additive or arithmetic recursion with constant a,

$$z_{i+1} = z_i + a, \quad \Longrightarrow \quad z_{i+1} = z_0 + (i+1) \cdot a,$$

so the corresponding additive parameter form is

$$\mathbf{E}[Y_{i+1}] = y_0 + (i+1)(\mu_0 - 0.5\sigma_0^2)\overline{\Delta t_i}^{(am)},$$

where

$$\overline{\Delta t_i}^{(am)} = \frac{1}{i+1} \sum_{j=0}^{i} \Delta t_j$$

is the **arithmetic mean (AM)** of the first $(i+1)$ time-steps.

As the **multiplicative noise** property can be seen by rewriting (3.6) as a single step,

$$X(t+\Delta t)=X(t)\exp\left((\mu_0-0.5\sigma_0^2)\Delta t+\sigma_0\Delta W(t)\right), \quad (3.8)$$

so the new noise exponential contribution from $\Delta W(t)$ multiplies the current value of the solution $X(t)$ to produce the new value $X(t+\Delta t)$. The corresponding recursive form in the time-step Δt_i from t_i to t_{i+1}, followed by a summing of the recursion, yields

$$\begin{aligned} X_{i+1} &= X_i \exp\left((\mu_0-0.5\sigma_0^2)\Delta t_i+\sigma_0\Delta W_i\right) \\ &= x_0 \exp\left(\sum_{j=0}^{i}\left((\mu_0-0.5\sigma_0^2)\Delta t_j+\sigma_0\Delta W_j\right)\right) \\ &= x_0 \prod_{j=0}^{i} \exp\left((\mu_0-0.5\sigma_0^2)\Delta t_j+\sigma_0\Delta W_j\right), \end{aligned}$$

using the laws of exponents to turn the exponential of a sum into a product of exponentials. **We consider a meaning for the Geometric name in Geometric Brownian Motion (GBM)** . . . .

Thus, taking the expectation and using the **completing the squares Lemma 4.1**

$$\mathrm{E}[X_{i+1}] = x_0 \prod_{j=0}^{i} \exp\left(\mu_0 \Delta t_j\right). \qquad (3.9)$$

This result should be compared to the corresponding deterministic multiplicative recursion or geometric progression with constant r, $x_{i+1} = r x_i = x_0 r^{i+1}$, so the corresponding multiplicative parameter form is

$$\mathrm{E}[X_{i+1}] = x_0 \left(\overline{\xi_i}^{(gm)}\right)^{i+1},$$

where

$$\overline{\xi_i}^{(gm)} = \left(\prod_{j=0}^{i} e^{\mu_0 \Delta t_j}\right)^{\frac{1}{i+1}}$$

is the **geometric mean (GM)** of the first $(i+1)$ growth steps $\xi_{\mathrm{j}} = e^{\mu_0 \Delta t_j}$ for $j = 0 : i$.

Applications include stochastic population growth, where $X(t)$ is the population size, $\mu(t)$ is an intrinsic growth rate (rate of growth in the absence of stochastic or other effects in the environment) and the $\sigma(t)X(t)dW(t)$ denotes the **stochastic effects**. The term $\sigma(t)X(t)dW(t)$ is called **demographic stochasticity** since it looks like a stochastic perturbation from $\mu(t)$, i.e., $\mu dt \longrightarrow \mu dt + \sigma dW(t)$. Similarly, perturbations of nonlinear saturation terms are called **environmental stochasticity**. In biology, multiplicative or geometric noise is also called **density independent noise**, since $dX(t)/X(t)$ is independent of $X(t)$.

Another application is financial engineering, where $X(t)$ is the investment return, $\mu(t)$ is the mean appreciation rate and $\sigma(t)$ is the investment volatility. In stochastic finance, the process $X(t)$ is called **geometric Brownian motion (GBM)** due to the linear scaling on the right-hand side for the $dX(t)$ and, in particular, due to the stochastic noise being multiplied by the state process $X(t)$, i.e., the multiplicative noise. In finance, one of the earliest stochastic stock models was from the thesis of **Bachelier (1900)**, in which **additive noise** was used, but this work did not attract much attention until **after Black and Scholes (Spring 1973) jointly with the mathematical justification of Merton (Spring 1973)**, and others began using **multiplicative noise stock and options models**.

Multiplicative models are more appropriate in finance as well as in biology, since **random effects are more likely to compound (as in compound interest) rather than add**.

{Note for the linear models, we have

Additive Noise $\Longleftrightarrow$ Arithmetic Mean

&

Multiplicative Noise $\Longleftrightarrow$ Geometric Mean

&

Multiplicative Wiener Noise $\Longleftrightarrow$ Geometric Brownian Motion }

See also Chapter 10 on **financial engineering applications**.

For the constant coefficient case of the linear stochastic diffusion SDE, the solution can be shown to have a log-normal distribution.

- **Theorem 3.2. Solution of the Constant Coefficient, Linear Stochastic Diffusion SDE is Log-Normally Distributed**: Let $X(t)$ satisfy

$$dX(t) = X(t)\left(\mu_0 dt + \sigma_0 dW(t)\right), \tag{3.10}$$

$X(0) = x_0 > 0$ with probability one, where μ_0 and $\sigma_0 > 0$ are constants. Then, the distribution of $X(t)$,

$$\Phi_{X(t)}(x) = \Phi_n\left(\ln(x); \mu_n(t), (\sigma_n)^2(t)\right), \tag{3.11}$$

where Φ_n is the normal distribution defined in (B.18),

$$\mu_n(t) = \ln(x_0) + \left(\mu_0 - 0.5\sigma_0^2\right)t \ \& \ (\sigma_n)^2(t) = \sigma_0^2 t.$$

Proof: Using the **probability inversion** Lemma B19, the distribution for the solution $X(t)$ in (3.6) can be derived by reducing the distribution for $X(t)$ to the known one for Wiener process $W(t)$ by inverting $X(t)$ i n favor of $W(t)$. It is important here that $x_0 > 0$, $\sigma_0 > 0$ and that the natural logarithm $\ln(x)$ is an increasing function preserving the direction of an inequality.

$$
\begin{aligned}
\Phi_{X(t)}(x) &= \mathrm{Prob}[X(t) \le x] \\
&= \mathrm{Prob}\left[x_0 \exp\left((\mu_0 - 0.5\sigma_0^2)t + \sigma_0 W(t)\right) \le x\right] \\
&= \mathrm{Prob}\left[\left(\mu_0 - 0.5\sigma_0^2\right) t + \sigma_0 W(t) \le \ln(x/x_0)\right] \\
&= \mathrm{Prob}\left[W(t) \le \left(\ln(x/x_0) - (\mu_0 - 0.5\sigma_0^2)t\right)/\sigma_0\right] \\
&= \Phi_{W(t)}\left(\left(\ln(x/x_0) - (\mu_0 - 0.5\sigma_0^2)t\right)/\sigma_0; 0, t\right) \\
&= \Phi_n\left(\ln(x); \ln(x_0) + \left(\mu_0 - 0.5\sigma_0^2\right) t, \sigma_0^2 t\right).
\end{aligned}
$$

{**Note that monotonicity is required, e.g.,**

$$\ln'(x) = 1/x > 0.\}$$

The last step follows from the **conversion identity** from the standard Wiener distribution $\Phi_{W(t)}$ in (B.22) to normal distribution Φ_n, given for Φ_n in Exercise 9 on p. B.70. Thus, the probability distribution of the solution $X(t)$ is the general **lognormal distribution** of Online Subsection B.1.6, where the exponent has the normal distribution mean

$$\mu_n(t) = \ln(x_0) + \left(\mu_0 - 0.5\sigma_0^2\right) t$$

and normal variance

$$\sigma_n^2(t) = \sigma_0^2 t,$$

i.e., the logarithm of the solution $X(t)$ has a general normal distribution, where the lognormal moment formulas are given in Properties B.20. $\square$

The probability density of $\boldsymbol{X(t)}$ is found using the regular calculus chain rule by differentiating the distribution to yield

$$\phi_{X(t)}(x) = x^{-1}\phi_n\left(\ln(x); \mu_n(t), \sigma_n^2(t)\right). \qquad (3.12)$$

Although the differentiation of the $\ln(x)$ distribution argument leads to an algebraic pole in $\phi_{X(t)}(x)$, $\phi_{X(t)}(0^+) \equiv 0$, which is in fact the limit as $x \longrightarrow 0^+$. The leading part of the exponentially small normal distribution term $\exp(-\ln^2(x)/(2\sigma_0^2 t))$ dominates the simple algebraic pole $1/x = \exp(-\ln(x))$ as $x \longrightarrow 0^+$ with the larger logarithmic exponent in magnitude.

- **3.3. Functions of General Diffusion States & Time, $F(X(t),t)$:**

The derivation for the special chain rule for the linear SDE logarithm transformation suggests that a more general chain rule for $F(X(t),t)$ will be needed.

Rule 3.2. Chain Rule for Diffusion $F(X(t),t)$:
Let $Y(t) = F(X(t),t)$, such that function $F(x,t)$ is twice continuously differentiable in x and once in t. Let the $X(t)$ process satisfy the diffusion SDE,

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t), \quad (3.13)$$

$X(0) = x_0$ w.p.o., while $f(X(t),t)$ and $g(X(t),t)$ satisfy the IMS integrability conditions with the $W(t)$ argument replaced by the $X(t)$ arguments of f and g.

Then

$$\begin{aligned} dY(t) &= dF(X(t),t) \\ &\stackrel{\mathrm{dt}}{=} \left(F_t + fF_x + \frac{1}{2}g^2F_{xx}\right)(X(t),t)dt \\ &\quad + (gF_x)\,(X(t),t)dW(t), \end{aligned}$$

where wholesale arguments have been used for the coefficient functions multiplying dt and $dW(t)$, respectively.

Sketch of Proof: Formally, using the increment form of the differential,

$$\begin{aligned} dY(t) &= Y(t+dt) - Y(t) \\ &= F(X(t+dt), t+dt) - F(X(t), t) \\ &= F(X(t) + dX(t), t+dt) - F(X(t), t). \end{aligned}$$

Next, mean square approximations are used with their implied precision-dt,

$$\begin{aligned} dY(t) &\overset{\text{dt}}{=} \left(F_t dt + F_x dX(t) + \frac{1}{2} F_{xx} (dX)^2(t)\right)(X(t), t) \\ &\overset{\text{dt}}{=} \left(F_t dt + F_x \cdot (f dt + g dW(t)) + \frac{1}{2} F_{xx} g^2 dt\right)(X(t), t) \\ &= \left(\left(F_t + f F_x + \frac{1}{2} g^2 F_{xx}\right) dt + g F_x dW(t)\right)(X(t), t) \end{aligned}$$

where the dependence of the coefficients and their derivatives is denoted by a *wholesale* $(X(t), t)$ at the end of each line, the diffusion SDE (3.13) has been substituted for $dX(t)$ and its square, the latter being truncated by the basic diffusion rule $(dW)^2(t) \overset{\text{dt}}{=} dt$ and other rules to neglect zero terms to dt–precision, such as $(dW)^3(t)$, $dtdW(t)$ and $(dt)^2$, given in the useful Table 2.1. $\square$

$\otimes$ **End of Lecture 3 Notes.**