FinM 345/Stat 390 Stochastic Calculus, Autumn 2009

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Lecture 9 (from Chicago)

Stochastic-Volatility, Jump-Diffusion Option Pricing

6:30-9:30 pm, 23 November 2009 at Kent 120 in Chicago

7:30-10:30 pm, 23 November 2009 at UBS Stamford

7:30-10:30 am, 24 November 2009 at Spring in Singapore

FinM 345 Stochastic Calculus:

9. Stochastic-Volatility, Jump-Diffusion (SVJD) Option Pricing:

• 9.1. SVJD European Option Pricing:

- ***9.1.1 Joint Stock and Variance SDE Dynamics:***

Along with jump effects in equity returns, a Northwestern University financial econometric group (Torben Anderson, with Benzoni and Lund, 2002) found that stochastic volatility was important too in estimating parameter for returns, and this is confirmed by other financial econometric investigators, like Bates (1996) in exchange rates and Bakshi et al. (1997) in options pricing. The SVJD European option pricing solution, with log-uniform jump-amplitudes, discussed here is adapted from a paper of *Yan and Hanson (2006)*.

A much used model for stochastic volatility (mostly posed as a stochastic variance $\mathbf{V(t)}$ model) is the square root diffusion model, here written for the asset or stock in risk-neutral form,

$$dS^{(\mathrm{rn})}(t)/S^{(\mathrm{rn})}(t) = (r_0 - \lambda_0 \overline{\nu})dt + \sqrt{V(t)}dW_s(t) + d\mathrm{CP}(t,Q), \tag{9.1}$$

with initial condition $S(0) = S_0 > 0$, risk-neutral rate r_0, jump rate $\lambda_0 = \mathrm{E}[P(t)]/t$, mean jump-amplitude $\overline{\nu} = \mathrm{E}[\nu(Q)]$, stochastic variance $V(t)$, stock stochastic diffusion $W_s(t)$, compound Poisson jump process $\mathrm{CP}(t,Q) = \sum_{j=1}^{P(t)} \nu(Q_j)$, IID mark is uniform $\phi_Q(q) = U(q;(a,b)/(b-a)$ (U is the unit step function), where $Q = \ln(1+\nu(Q))$ and $a < 0 < b$. So $\overline{\nu} = (e^b - e^a)/(b-a) - 1$.

The so-called **stochastic volatility** $\left(\mathrm{SV} = \sqrt{V(t)}\right)$ SDE shares the square root diffusion term,

$$dV(t) = \kappa_v(t)(\theta_v(t) - V(t))dt + \sigma_v(t)\sqrt{V(t)}dW_v(t), \tag{9.2}$$

with $V(0) = V_0 > 0$, **mean-reversion rate** $\kappa_v(t) > 0$, **mean-reversion level** $\theta_v(t)$, **volatility of volatility** (i.e., of variance, also called "**vol of vol**") $\sigma_v(t)$, and variance stochastic diffusion $W_v(t)$, having the joint correlation coefficient with the stock price,

$$\rho_v(t) = \mathrm{Corr}[W_s(t), W_v(t)] = \frac{\mathrm{Cov}[dW_s(t), dW_v(t)]}{\sqrt{\mathrm{Var}[dW_s(t)]\mathrm{Var}[dW_v(t)]}}.$$

The square root diffusions like (9.2) go back to **Feller** (1951) who worked out a lot of the details of this kind of model and other diffusion properties parallel to the work of Kolmogorov, both working with the corresponding PDEs rather than the SDEs that are now "***in fashion***".

However, (9.2) is commonly called the **Heston SV model** and also the **CIR model** for Cox, Ingersoll and Ross (1985), working on interest rate models.

The square-root Heston model allow for inclusion of systematic volatility risk and can generate analytically tractable method of option pricing without sacrificing accuracy.

This SV model is very singular to transformations and the Itô chain rule due to the square root term leading to problems unless the variance is bounded away from zero, unlike the diffusion or jump-diffusion where any singularity due the log-transformation $Y(t)=\ln(S(t))$ is automatically removed. This Heston model discussion is adapted from a SVJD optimal portfolio paper appendix of ***Hanson (2008)***.

* **Mean Reversion Level:** The mean-reversion level stems from the observation that excess or deficiency of volatility at some $V_0 > 0$ for $t_0 = 0$, say, should decay back to some reasonable level, $\theta_v \neq V_0$, so consider the **deterministic SV equation** for $V^{(\mathrm{det})}(t)$,

$$dV^{(\mathrm{det})}(t) = \kappa_v(\theta_v - V^{(\mathrm{det})}(t))dt,$$

let $Z(t) = \ln(|V^{(\mathrm{det})}(t) - \theta_v|)$ and $\{\kappa_v, \theta_v\}$ be constants, so $dZ(t) = dV^{(\mathrm{det})}(t)/(V^{(\mathrm{det})}(t) - \theta_v) = -\kappa_v dt$, $Z(t) = Z(0) - \kappa_v t = \ln(|V_0 - \theta_v|) - \kappa_v t$, and $|V^{(\mathrm{det})}(t) - \theta_v| = \exp(Z(t)) = |V_0 - \theta_v| \exp(-\kappa_v t)$, representing exponential decay of the absolute excess from V_0 to θ_v at rate κ_v, the decay or growth of $V^{(\mathrm{det})}(t)$ toward θ_v depends on the sign, $\mathrm{sgn}_v \equiv \mathrm{sgn}(V_0 - \theta_v)$.

* **Nonnegativity of $V(t)$ and Consistency with respect to Itô's Diffusion Approximation:** Feller (1951) settled the nonnegativity long ago for square root diffusions using very elaborate Laplace transform techniques on the corresponding Kolmogorov forward equation, finding a noncentral chi-squared distribution for the solution. Feller classified the boundary conditions finding in the time-independent form notation here, positivity is assured if $1 < 2\kappa_v\theta_v/\sigma_v^2$ with zero boundary conditions in value and flux, while if $0 < 2\kappa_v\theta_v/\sigma_v^2 < 1$ then only non negativity can be assured. This is important in finance because the volatility often occurs as a divisor as in the Black-Scholes option pricing formula and and Merton's optimal portfolio problem.

Feller's approach is far too difficult to present here, but we can use the methods of the course to show that there is a perfect square solution form.

Letting $Y(t) = F(V(t), t)$ and using **Itô's lemma** to get the transformed SDE,

$$\begin{aligned} dY(t) &= F_t(V(t), t)dt + F_v(V(t), t)dV(t) \\ &\quad + 0.5F_{vv}(V(t), t)\sigma_v^2(t)V(t)dt, \end{aligned} \tag{9.3}$$

to dt-precision. Then a simpler form is sought with volatility-independent noise term, i.e.,

$$dY(t) = \left(\mu_y^{(0)}(t) + \mu_y^{(1)}(t)\Big/\sqrt{V(t)}\right)dt + \sigma_y(t)dW_v(t), \tag{9.4}$$

with $Y(0) = F(V_0, 0)$, where $\mu_y^{(0)}(t)$, $\mu_y^{(1)}(t)$ and $\sigma_y(t)$ are time-dependent coefficients to be determined. Due to the appearance of the singular term $1/\sqrt{V(t)}$ as $V(t) \to 0^+$, the variance has to be bounded away from zero, $V(t) \geq \varepsilon_v > 0$.

Equating the coefficients of $dW_v(t)$ terms between (9.3) and (9.4), given $V(t)=v\geq 0$, leads to

$$F_v(v,t)=\left(\frac{\sigma_y}{\sigma_v}\right)(t)\frac{1}{\sqrt{v}}, \tag{9.5}$$

and then partially integrating (9.5) yields

$$F(v,t)=2\left(\frac{\sigma_y}{\sigma_v}\right)(t)\sqrt{v}+c_1(t), \tag{9.6}$$

which is the desired transformation with a function of integration $c_1(t)$.

Additional differentiations of (9.5) produce

$$F_t(v,t)=2\left(\frac{\sigma_y}{\sigma_v}\right)'(t)\sqrt{v}+c_1'(t) \quad \& \quad F_{vv}(v,t)=-\frac{1}{2}\left(\frac{\sigma_y}{\sigma_v}\right)(t)v^{-3/2}.$$

Terms of order $v^0dt=dt$ imply that $c_1'(t)=\mu_y^{(0)}(t)$, but this equates two unknown coefficients, so we set $\mu_y^{(0)}(t)=0$ for simplicity.

Equating terms of order $\sqrt{v}dt$ and integrating imply

$$\left(2\left(\frac{\sigma_y}{\sigma_v}\right)' - \kappa_v\left(\frac{\sigma_y}{\sigma_v}\right)\right)(t) = 0$$

imply

$$\left(\frac{\sigma_y}{\sigma_v}\right)(t) = \left(\frac{\sigma_y}{\sigma_v}\right)(0)e^{\overline{\kappa}_v(t)/2},$$

where

$$\overline{\kappa}_v(t) \equiv \int_0^t \kappa_v(y)dy.$$

For convenience, we set $\sigma_y(0) = \sigma_v(0)$. For order $v^{-1/2}dt$, we obtain

$$\mu_y^{(1)}(t) = e^{\overline{\kappa}_v(t)/2}(\kappa_v\theta_v - 0.25\sigma_v^2)(t),$$

completing the coefficient determination.

Assembling these results we form the solution as follows,

$$Y(t) = 2e^{\overline{\kappa}_v(t)/2}\sqrt{V(t)}.$$

Inverting for $V(t)$ yields the **desired nonnegativity result**:

$$V(t) = 0.25e^{-\overline{\kappa}_v(t)}Y^2(t) \geq 0, \tag{9.7}$$

due to the perfect square form, where

$$Y(t)/2 = \sqrt{V_0} + I_g(t) \tag{9.8}$$

and where

$$I_g(t) = 0.5\int_0^t e^{\overline{\kappa}_v(s)/2}\left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V}}\right)(s)ds + (\sigma_v dW_v)(s)\right). \tag{9.9}$$

This is an implicit form that is also singular unless the variance $V(t)$ is bounded away from zero, $V(t) \geq \varepsilon_v > 0$. Ideally, it would be desiraable that the reciprocal volatility $1/\sqrt{V(t)}$ is integrable in t as $V(t) \to 0^+$, so the singularity will be ignorable in theory, but after all $V(t)$ is only implicitly defined as a solution in (9.7).

However, it is necessary to check the consistency of the Itô diffusion approximation in (9.3) because of the competing limits as the time-increment $\Delta t \to 0^+$ as a proxy for dt-precision and variance singularity as $V(t) \to 0^+$, i.e., $\varepsilon_v \to 0^+$. The partial derivatives of $F(v,t)$ given in (9.5) and following equations imply that the satisfy the power relation $\partial^k F/\partial v^k = \beta_k(t) v^{-(2k-1)/2}$ for some $\beta_k(t)$ while the mean estimate of of the dominant diffusion term factor is $\mathrm{E}[|\sigma_v v \Delta W_v|^k] = \alpha_k v^k (\Delta t)^{k/2}$, so that the products of these terms are an estimate of the corresponding significant terms in the Taylor,

$$\frac{\partial^k F}{\partial v^k} \mathrm{E}[(\sigma_v \Delta W_v)^k] = \gamma_k \frac{\Delta t}{\sqrt{v}} \left(\frac{\Delta t}{v}\right)^{(k-2)/2},$$

separated into the order $\Delta t/\sqrt{v}$ of the diffusion $(k=2)$ term and the factor relative to it. Hence, for all of the term higher order than $k=2$, we need $\Delta t/v \ll 1$, i.e., $\Delta t \ll \varepsilon_v \ll 1$.

*** Consistent Singular Limit Formulation for Theory and Computation:** Since as $V(t) \longrightarrow 0^+$, the singular integral with (9.7) needs a proper method of integration specified.

First (9.7)-(9.9) is reformulated as a recursion using some algebra for the next time increment Δt and the method of integration is specified for each subsequent time step, i.e.,

$$V(t+\Delta t) = e^{-\Delta\overline{\kappa}_v(t)} \lim_{\varepsilon_v \to 0^+} \left(\sqrt{V^{(\varepsilon_v)}(t)} + e^{-\overline{\kappa}_v(t)/2} \Delta I_g^{(\varepsilon_v)}(t) \right)^2, \tag{9.10}$$

where $V^{(\varepsilon_v)}(t) = \max(V(t), \varepsilon_v)$ with $\varepsilon_v > 0$ such that $\Delta t \ll \varepsilon_v \ll 1$ as $\Delta t \longrightarrow 0^+$ to ensure that the time-step goes to zero faster than the cutoff singular denominator, where

$$\Delta\overline{\kappa}_v(t) \equiv \int_t^{t+\Delta t} \kappa_v(s) ds \longrightarrow \kappa_v(t)\Delta t$$

as $\Delta t \to 0^+$.

Similarly, a scaled increment of integral $I_g^{(\varepsilon_v)}$ is defined by

$$e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t) \equiv 0.5\int_t^{t+\Delta t} e^{(\overline{\kappa}_v(s)-\overline{\kappa}_v(t)/2} \\ 0.5\left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V^{(\varepsilon_v)}}}\right)(s)ds + (\sigma_v dW_v)(s)\right) \\ \to 0.5\left(\left(\frac{\kappa_v\theta_v - \frac{1}{4}\sigma_v^2}{\sqrt{V}}\right)(t)\Delta t + (\sigma_v\Delta W_v)(t)\right), \tag{9.11}$$

as $\Delta t \to 0^+$ & $\varepsilon_v \to 0^+$ such that $\Delta t \ll \varepsilon_v \ll 1$ to ensure that the time-step goes to zero faster than the cutoff singular denominator, for Itô diffusion approximation (9.3) consistency and the numerical consistency of the solution (9.10). We have also been using a modification of the ***method of ignoring the singularity*** of Davis and Rabinowitz (1965).

In fact, a Taylor expansion for small $\Delta t \ll 1$ and $\varepsilon \ll 1$ such that $\Delta t \ll \varepsilon_v \ll 1$ confirms that (9.10)-(9.11) recovers the Heston (1993) model, proving solution consistency. Thus, the square in (9.10) formally justifies the nonnegativity of the variance and the volatility of the Heston model, for a proper computational nonnegativity-preserving procedure.

However, for the general validity of applications of the chain rule and simulations, the Δt-variance limit

$$\Delta t \ll \varepsilon_v \ll 1 \tag{9.12}$$

required for (9.10)-(9.11) implies that the ***non-negative variance condition*** $V(t) \geq 0$ ***is questionable in both theory and simulation***.

* **A Nonsingular, Explicit, Exact Solution:** In any event, the singular term in (9.7)-(9.9) vanishes in the special parameter case, such that

$$\kappa_v(t)\theta_v(t) = \frac{1}{4}\sigma_v^2(t), \quad \forall\, t.$$

Hence, we obtain a nonnegative, nonsingular exact solution

$$V(t) = e^{-\overline{\kappa}_v(t)}\left(\sqrt{V_0} + 0.5\int_0^t e^{\overline{\kappa}_v(s)/2}(\sigma_v dW_v)(s)\right)^2, \qquad (9.13)$$

with the numerical form corresponding to (9.10)-(9.11) simplifies to,

$$V(t+\Delta t) = e^{-\Delta\overline{\kappa}_v(t)}\left(\sqrt{V(t)}\right.$$
$$\left. + 0.5 e^{-\overline{\kappa}_v(t)/2}\int_t^{t+\Delta t} e^{\overline{\kappa}_v(s)/2}(\sigma_v dW_v)(s)\right)^2. \qquad (9.14)$$

Similarly, the chain rule for the integrating factor form $\exp(\overline{\kappa}_v(t))V(t)$ with constant $\theta_v(t) = \theta_0$ for the stochastic volatility SDE (9.2) leads to a somewhat simpler integrated form,

$$V(t) = \max\left(V^{(\mathrm{det})}(t) + \int_0^t e^{\overline{\kappa}_v(s) - \overline{\kappa}_v(t)}\left(\sigma_v\sqrt{V}dW_v\right)(s), 0\right), \quad (9.15)$$

using the maximum with respect to zero to remove spurious numerical simulations in absence of a perfect square form. In (9.15),

$$V^{(\mathrm{det})}(t) = V_0 e^{-\overline{\kappa}_v(t)} + \theta_0\left(1 - e^{-\overline{\kappa}_v(t)}\right)$$

is the deterministic part of $V(t)$ for constant θ_0 . Note that there is only a linear change of dependent variable according to the stochastic chain rule [Hanson (2007)] using the transformation $Y(t) = \exp(\overline{\kappa}_v(t))V(t)$.

So the deterministic part is easily separated out from the square-root dependence and replaces the mean-reverting drift term. The $V^{(\mathrm{det})}(t)$ will be positive for positive parameters. However, as *Lord et al. (2007)* point out, a sufficiently accurate simulation scheme and a large number of simulation nodes are required so that the right-hand side of SV SDE (9.2) generates nonnegative values. Nonnegative values using the stochastic Euler simulation have been verified for Heston's (1993) constant risk-neutralized parameter values ($\kappa_v = 2.00$, $\theta_v = 0.01$, $\sigma_v = 0.10$) as long as the scaled number of nodes per unit time $N/(\kappa_v T) > 100$.

Hence, since the variance by definition for real processes cannot be negative, practical considerations suggest replacing occurrences of $V(t)$ by $\max(V(t), \varepsilon_v)$, where $\Delta t \ll \varepsilon_v \ll 1$ for numerically consistent variance values by keeping by keeping the small variance cut-off larger than the small computational time step.

For more information on stochastic volatility models see the texts Fouque, Papanicolaou and Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, 2000 or Gatheral, *The Volatility of Surface*, 2006 or Lewis, *Option Valuation under Stochastic Volatility*, 2000.

* Risk-Neutral SVJD European Call Option Pricing:

The risk-neutral interest rate is specified by the risk-neutral expectation, $\mathrm{E}^{(\mathrm{rn})}[S(T)|S(t_0)] = S(t_0)\exp(r_0(T-t_0))$ for risk-free spot rate r_0 and some reference time $t_0 \geq 0$ with risk-neutral drift $\mu^{(\mathrm{rn})} = r_0 - \lambda_0 \overline{\nu}$.

Let C denote the price at time t of a European style call option on $S(t)$ with strike price K and expiration time T. Using the fact that the terminal payoff of a European call option on the underlying stock S with strike price K is $\max(S(T) - K, 0)$ and assuming the short-term interest rate r_0 is constant over the lifetime of the option.

The price of the European call at time t equals the **discounted, conditional expected payoff**,

$$\begin{aligned} C(s,v,t;K,T) &= e^{-r_0(T-t)}\mathrm{E}^{(\mathrm{rn})}[\max(S(T)-K,0) \\ &\qquad |S(t)=s,V(t)=v] \\ &= e^{-r_0(T-t)}\left(\int_K^\infty y\phi_{S^{(\mathrm{rn})}(T)}(y|s,v)dy\right. \\ &\qquad \left. -K\int_K^\infty \phi_{S^{(\mathrm{rn})}(T)}(y|s,v)dy\right) \\ &= sP_1(s,v,t;K,T) \\ &\qquad -Ke^{-r_0(T-t)}P_2(s,v,t;K,T), \end{aligned} \tag{9.16}$$

where $\mathbf{E}^{(\mathrm{rn})}$ is the expectation with respect to the risk-neutral conditional probability density,

$\phi_{S^{(\mathrm{rn})}(T)}(y|S(t)=s,V(t)=v)$, given $(S(t),V(t))$.

The **risk-neutral tail probability that $S(T) > K$** is

$$P_1(s,v,t;K,T) = e^{-r_0(T-t)}\int_K^\infty y\phi_{S^{(\mathrm{rn})}(T)}(y|s,v)dy/S(t)$$
$$= \int_K^\infty y\phi_{S^{(\mathrm{rn})}(T)}(y|s,v)dy \qquad (9.17)$$
$$/\mathrm{E}^{(\mathrm{rn})}[S(T)|S(t)=s,V(t)=v],$$

by the risk-neutral property, since the integrand is nonnegative and the integral over $[0,\infty)$ is one.

The **risk-neutral in-the-money (ITM) tail probability**,

$$P_2(S(t),V(t),t;K,T) = \mathrm{Prob}[S(T) > K|S(t)=s,V(t)=v], \qquad (9.18)$$

is the complementary risk-neutral distribution function. The European option evaluation problem is to evaluate $P_1 = P_1^{(\mathrm{rn})}$ and $P_2 = P_2^{(\mathrm{rn})}$ under the distribution assumptions embedded in the risk-neutral probabilities (measures).

The difficulty is that the cumulative distribution function for most distributions is infeasible [Bates (1996)]. We use some of techniques of Bates (1996), Heston (1993 and Bakshi et al. (1997).

The usual change of variable is made from the stock price $S^{(\mathrm{rn})}(t)$ to to the risk-neutral stock log-return $L(t) \equiv \ln(S^{(\mathrm{rn})}(t))$. By the Itô's chain rule, the log-return process satisfies the SDE

$$dL(t) = (r_0 - \lambda_0 \overline{\nu} - V(t)/2)dt + \sqrt{V(t)}dW_s(t) + \sum_{j=1}^{dP(t)} Q_j \tag{9.19}$$

convert the call price to log-return variables,

$$\widehat{C}(L(t), V(t), t; \kappa_c, T) \equiv C(S(t), V(t), t; K, T), \tag{9.20}$$

i.e.,

$$\begin{aligned}\widehat{C}(\ell, v, t; \kappa_c, T) = e^{-r_0(T-t)}\mathrm{E}^{(\mathrm{rn})}[\max(e^{L(T)} - e^{\kappa_c}, 0) \\ |L(t) = \ell, V(t) = v]\end{aligned} \tag{9.21}$$

where the log-strike-price is $\kappa_c \equiv \ln(K)$ or $K = \exp(\kappa_c)$.

* **Risk-Neutral PIDE Derivation:** As in Merton's (1973) BS justification paper, we need to convert our SDE formulation to a numerically and analytically more desirable PDE. So using a Merton-like Itô-expansion of $\widehat{C}(\ell,v,t)=\widehat{C}(L(t),V(t),t;\kappa_c,T)$ with respect to the processes $L(t)$ and $V(t)$ along with t, except the jump-chain rule is included, to change from an system of SDEs to a single PDE, called a **PIDE (partial integro-differential equation)**:

$$\begin{aligned}
0 &= \frac{\partial \widehat{C}}{\partial t}+\mathcal{A}\left[\widehat{C}\right](\ell,v,t) \\
&\equiv \frac{\partial \widehat{C}}{\partial t}+\left(r_0-\lambda_0\overline{\nu}-\frac{1}{2}v\right)\frac{\partial \widehat{C}}{\partial \ell} \\
&\quad +\kappa_v(\theta_v-v)\frac{\partial \widehat{C}}{\partial v}+\frac{1}{2}v\frac{\partial^2 \widehat{C}}{\partial \ell^2}+\rho_v\sigma_v v\frac{\partial^2 \widehat{C}}{\partial \ell \partial v}+\frac{1}{2}\sigma_v^2 v\frac{\partial^2 \widehat{C}}{\partial v^2} \\
&\quad -r_0\widehat{C}+\lambda_0\int_{-\infty}^{\infty}\left(\widehat{C}(\ell+q,v,t)-\widehat{C}(\ell,v,t)\right)\phi_Q(q)dq.
\end{aligned} \tag{9.22}$$

Technically, the resulting formula is called **Dynkin's Theorem or Formula** (see [Hanson (2007), Chapt. 7]). which says, in our case, that (9.21) is the solution to the SVJD option price backward **PIDE (9.22)**.

The risk neutral call formula (9.16) can be written in the current state variables,

$$\widehat{C}(\ell, v, t; \kappa_c, T) = e^{\ell}\widehat{P}_1(\ell, v, t; \kappa_c, T) - e^{\kappa_c - r(T-t)}\widehat{P}_2(\ell, v, t; \kappa_c, T).$$

Inserting this into (9.22) and separating assumed independent terms $\widehat{P}_1$ and $\widehat{P}_2$, produces two PIDEs for the risk-neutralized probabilities $\widehat{P}_i(\ell, v, t; \kappa_c, T)$ for $i = 1:2$,

$$\begin{aligned} 0 &= \frac{\partial \widehat{P}_1}{\partial t} + \mathcal{A}_1[\widehat{P}_1](\ell, v, t; \kappa_c, T) \\ &\equiv \frac{\partial \widehat{P}_1}{\partial t} + \mathcal{A}[\widehat{P}_1](\ell, v, t; \kappa_c, T) + v\frac{\partial \widehat{P}_1}{\partial \ell} + \rho_v \sigma_v v \frac{\partial \widehat{P}_1}{\partial v} \\ &\quad + (r_0 - \lambda_v \overline{\nu})\,\widehat{P}_1 + \lambda_v \int_{-\infty}^{\infty} (e^q - 1)\widehat{P}_1(\ell + q, v, t)\phi_Q(q)dq. \end{aligned} \tag{9.23}$$

This is subject to the boundary condition at the expiration time $t = T$:

$$\widehat{P}_1(\ell, v, T; \kappa_c, T) = \mathbb{I}_{\ell > \kappa_c}, \tag{9.24}$$

where $\mathbb{I}_{\ell > \kappa_c}$ is the indicator function for the set $\{\ell > \kappa_c\}$. Similarly,

$$\begin{aligned} 0 &= \frac{\partial \widehat{P}_2}{\partial t} + \mathcal{A}_2[\widehat{P}_2](\ell, v, t; \kappa_c, T) \\ &\equiv \frac{\partial \widehat{P}_2}{\partial t} + \mathcal{A}[\widehat{P}_2](\ell, v, t; \kappa_c, T) + r_0 \widehat{P}_2, \end{aligned} \tag{9.25}$$

subject to the boundary condition at the expiration time $t = T$:

$$\widehat{P}_2(\ell, v, T; \kappa_c, T) = \mathbb{I}_{\ell > \kappa_c}. \tag{9.26}$$

* Dynkin's Formula for Multi-State Jump-Diffusions:

Given an n-state system $\vec{X}(t) = [X_i(t)]_{n\times 1}$ such that

$$dX_i = \mu_i dt + \sigma_i dW_i(t) + \nu_i dP_i(t),$$

with state-time $(\vec{X}(t), t)$ coefficients, ν_i also depending on the ith mark Q_i, an integrable "payoff" function $U(\vec{x})$, "time-discount factor" $\psi(t)$ and a "pricing" functional

$$u(\vec{x}, t) \equiv \psi(t) \cdot \mathrm{E}\left[U(\vec{X}(T)) \,|\vec{X}(t) = \vec{x}\right],$$

then $u(\vec{x}, t)$ satisfies the following PIDE:

$$\begin{aligned} 0 = {} & u_t(\vec{x}, t) + \psi'(t) u(\vec{x}, t)/\psi(t) \\ & + \sum_{i=1}^{n} \mu_i u_{x_i}(\vec{x}, t) + 0.5 \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{i,j}\sigma_i\sigma_j u_{x_i,x_j}(\vec{x}, t) \\ & + \sum_{i=1}^{n} \lambda_i \int_{\mathcal{Q}_i} (u(\vec{x} + \nu_i(\vec{x}, t, q_i)\vec{e}_i, t) - u(\vec{x}, t))\phi_{Q_i}(q_i) dq_i, \end{aligned} \tag{9.27}$$

where $\rho_{i,j}$ is the correlation coefficient between diffusions, λ_i is the jump-rate for independent Poissons and e_i is the ith state unit vector, only one jump being likely at any instant.

* Characteristic Function Formulation for Solution:

Character functions are Fourier or complex variable transforms for functions, e.g., processes here, that are related to moment generating functions and are used in the definition of a kind of jump-diffusion called Lévy processes.

- The ***characteristic functions*** (constant $i=\sqrt{-1}$) are

$$f_j(\ell,v,t;y,T) \equiv -\int_{-\infty}^{\infty} e^{iy\kappa} d\widehat{P}_j(\ell,v,t;\kappa,T), \quad (9.28)$$

- Satisfying the ***same PIDEs*** as the $\widehat{P}_j(\ell,v,t;\kappa,T)$:

$$\frac{\partial f_j}{\partial t} + \mathcal{A}_j[f_j](\ell,v,t;\kappa,T) = 0, \quad (9.29)$$

where $\mathcal{A}_j$ represents the corresponding full backward operators in (9.23) and (9.25) with boundary conditions, $f_j(\ell,v,T;y,T) = +e^{iy\ell}$, respectively for $j = 1:2$.

* Solution Details:

- For the Fourier transforms f_j for $j=1\!:\!2$,

$$h_j(\tau)=\frac{(\eta_j^2-\Delta_j^2)(e^{\Delta_j\tau}-1)}{\sigma_v^2(\eta_j+\Delta_j-(\eta_j-\Delta_j)e^{\Delta_j\tau})}; \qquad (9.30)$$

$$\begin{aligned} g_j(\tau) &= ((r_0-\lambda_v\overline{\nu})iy-\lambda_0\overline{\nu}\delta_{j,1}-r_0\delta_{j,2})\tau \\ &\quad +\lambda_0\tau\int_{-\infty}^{\infty}(e^{(iy+\delta_{j,1})q}-1)\phi_Q(q)dq \qquad (9.31) \\ &\quad -\frac{\kappa_v\theta_v}{\sigma_v^2}\left(2\ln\left(1-\frac{(\Delta_j+\eta_j)(1-e^{-\Delta_j\tau})}{2\Delta_j}\right)+(\Delta_j+\eta_j)\tau\right), \end{aligned}$$

where

$$\eta_j=\rho\sigma(iy+\delta_{j,1})-k \;\;\&\;\; \Delta_j=\sqrt{\eta_j^2-\sigma^2 iy(iy\pm 1)};$$

$$\int_{-\infty}^{\infty}(e^{(iy+1)q}-1)\phi_Q(q)dq=\frac{e^{(iy+1)b}-e^{(iy+1)a}}{(b-a)(iy+1)}-1.$$

- The ***tail probabilities*** P_j for $j = 1:2$ are

$$P_j(S(t), V(t), t; K, T) = \frac{1}{2} \tag{9.32}$$
$$+ \frac{1}{\pi}\int_{0^+}^{+\infty} \mathrm{Re}\left[\frac{e^{-iy\ln(K)} f_j(\ln(S(t)), V(t), t; y, T)}{iy}\right] dy,$$

by complex integration on equivalent contours yielding a residue of $1/2$ and a principal value integral in the limit to the left of the apparent singularity at $y = 0^+$, since the integrand is bounded in the singular limit.

- **Put Option by Put-Call Parity:** Since we are dealing with European options with fixed exercise time, the parity is still valid, so the European put option price at fixed (K, T) is

$$P(S(t), V(t), t) = C(S(t), V(t), t) + Ke^{-r_0(T-t)} - S(t), \tag{9.33}$$

easily calculated once the call option price is known.

* **Computing Inverse Fourier Integrals:** The inverse Fourier integral (9.32) can be computed by means of standard procedures of numerical integration with some precautions. Two methods are compared: the discrete Fourier transform (DFT) with Gaussian Quadrature sub-integral refinement for accuracy and the other is the fast Fourier transform (FFT) for speed of computation.

◦ **Discrete Fourier Transform (DFT) Approximations:** Since the integrand of (9.32) has a bounded limit as $y \longrightarrow 0^+$, is otherwise smooth and decays very fast, it is rewritten in the general approximate form for DFT,

$$\begin{aligned} I[F](\kappa) &\equiv \int_0^\infty F(y;\kappa)dy \\ &\simeq \sum_{j=1}^{N} I_j(\kappa) = \sum_{j=1}^{N} \int_{(j-1)h}^{jh} F(y;\kappa)dy, \end{aligned} \tag{9.34}$$

for sufficiently large N.

Such integrals are the basis of the discrete Fourier transform, where h is a fixed gross step size depending on some **integral cutoff** $R_y = \max[y] \simeq N * h$. The sub-integrals on $((j-1)h, jh)$ in (9.34) for $j = 1 : N$ are computed by means of **ten-point Gauss-Legendre** formula for refined accuracy need for oscillatory integrands and for the fact that it is an **open quadrature formula that avoids any non-smooth behavior as** $y \to 0^+$. The number of steps N is not static, but ultimately determined by a local stopping criterion: the integration loop is stopped if the ratio of the contribution of the last strip to the total integration becomes smaller than 0.5e-7. By trials, h=5 is a good choice that we can get sufficiently fast convergence and good precision.

- **Using Fast Fourier Transform (FFT):**

(*After Carr and Madan (1999)*)

- **Initial call option price ($d\widehat{P}_2 = \widehat{p}^{(\mathrm{rn})} d\kappa_c$):**

$$\widehat{C}(\ell, v, t; \kappa_c, T) = -\int_{\kappa_c}^{\infty} e^{-r(T-t)}(e^{\ell} - e^{k}) d\widehat{P}_2(\ell, v, t; k, T). \quad (9.35)$$

- **Modified call option price to remove the singularity:**

$$\widehat{C}^{(\mathrm{mod})}(\ell, v, t; \kappa_c, T, \alpha) = e^{\alpha\kappa_c}\widehat{C}(\ell, v, t; \kappa_c T). \quad (9.36)$$

- **Fourier transform of $\widehat{C}^{(\mathrm{mod})}(\ell, v, t; \kappa_c, T, \alpha)$:**

$$\Psi(\ell, v, t; y, T, \alpha) = \int_{-\infty}^{\infty} e^{iy\kappa}\widehat{C}^{(\mathrm{mod})}(\ell, v, t; \kappa, T, \alpha) d\kappa. \quad (9.37)$$

- Thus,

$$\widehat{C}(\ell, v, t; \kappa_c, T) = \frac{e^{-\alpha\kappa_c}}{\pi}\int_{0}^{\infty} e^{-iy\kappa}\Psi(\ell, v, t; y, T, \alpha) dy. \quad (9.38)$$

where, by putting (9.35)-(9.37) together,

$$\Psi(\ell, v, t; y, T, \alpha) = -e^{-r(T-t)} \int_{-\infty}^{\infty} e^{(\alpha+iy)\kappa} \int_{\kappa}^{\infty} (e^{\ell} - e^{k}) \cdot d\widehat{P}_2(\ell, v, t; k, T) d\kappa$$
$$= \frac{e^{-r(T-t)} f_2(y - (\alpha+1)i)}{\alpha^2 + \alpha - y^2 + i(2\alpha+1)y}; \qquad (9.39)$$

- Transfer the Fourier integral into discrete Fourier transform (DFT) and incorporate Simpson's rule (Carr and Madan (1999)) to increase accuracy of the ***FFT application for Fourier inverses:***

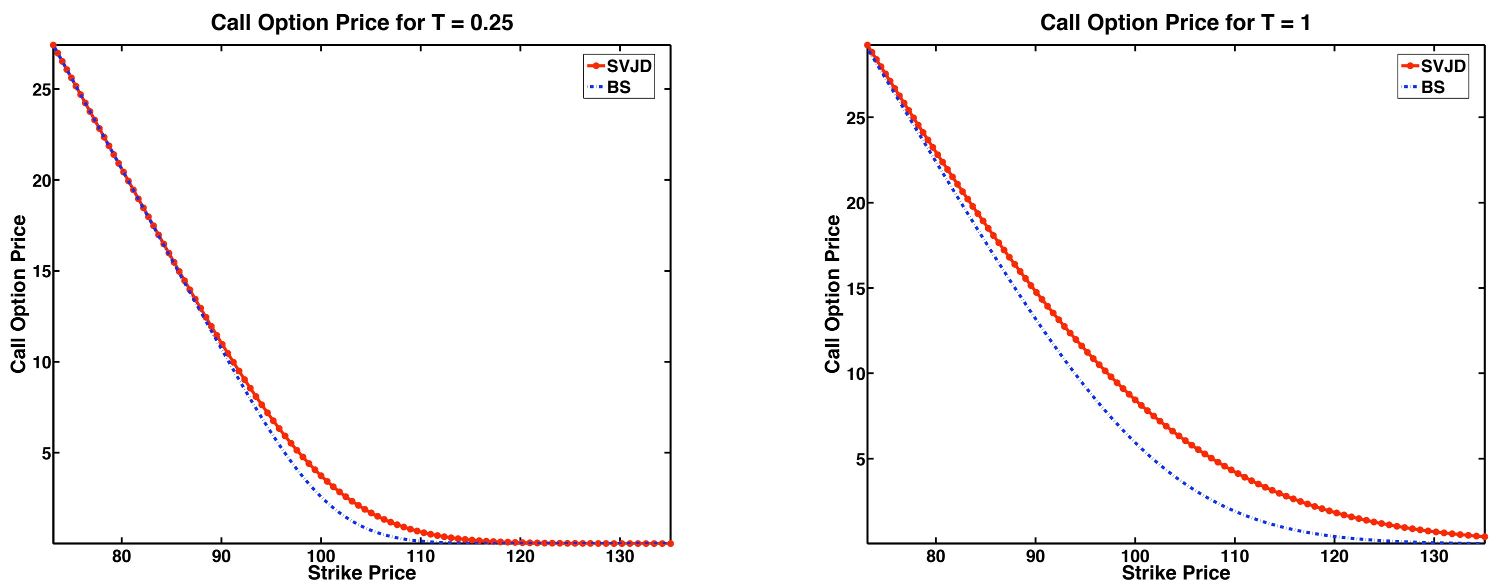
$$C(S(t), V(t), t; \kappa, T) = \frac{e^{-\alpha\kappa}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}jk} e^{iy_j(L-\ln(S))} \Psi(y_j) \cdot \frac{\Delta y}{3}[3 + (-1)^{(j+1)} - \delta_{j,1}], \qquad (9.40)$$

where $\alpha = 2.0$ and for the Simpson's rule (a Carr accuracy innovation?) $\Delta y = 0.25$ are used.

◦ Numerical Results for Call and Put Options

- **Two numerical algorithms give the same results within accuracy standard.** The FFT method can compute different levels strike price near at-the-money (ATM) in 5 seconds. The standard integration method can give out the results for one specific strike price in about 0.5 seconds. The implementations are using MATLAB 6.5 and on the PC with 2.4GHz CPU.
- The **option prices from the stochastic-volatility jump-diffusion (SVJD) model are compared with those of Black-Scholes model:**
  Parameters: $r_0 = 3\%$, $S_0 = \$100$, $V_0 = 0.012$, $\rho_v = -0.622$, $\theta_v = 0.53$, $\kappa_v = 0.012$, $\sigma_v = 7\%$, $\lambda_0 = 64$, $a = -0.028$, $b = 0.026$ (various sources) .

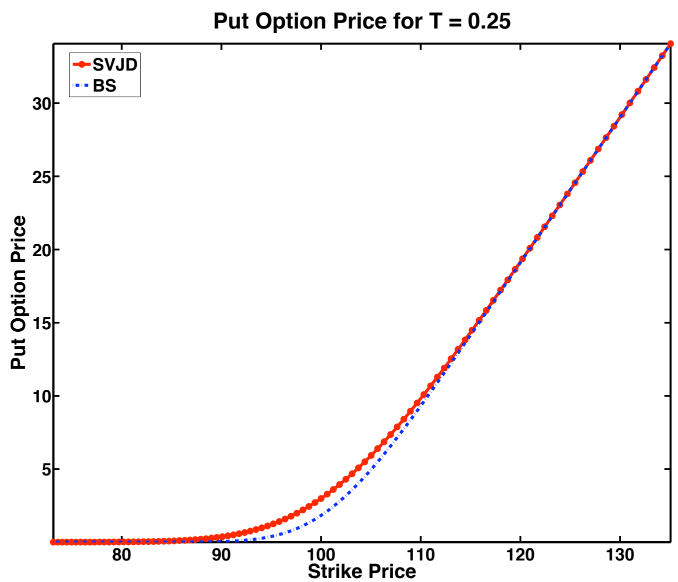
○ Call Option Prices:



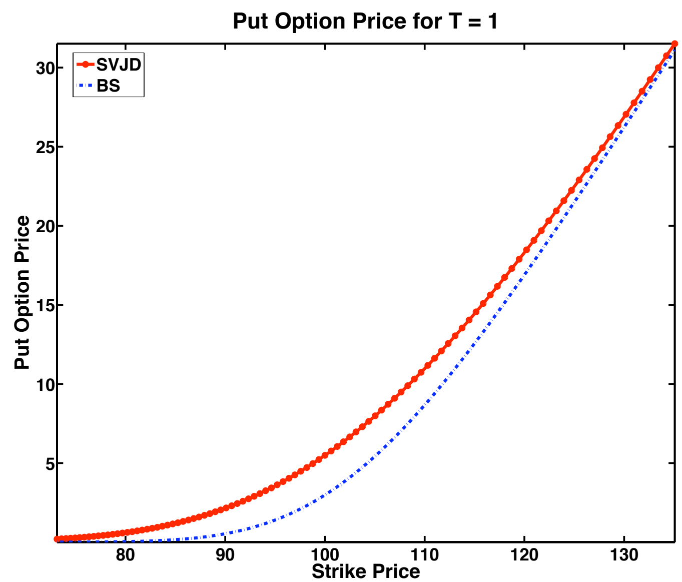
(a) Call prices for $T = 0.25$. (b) Call prices for $T = 1.0$.

Figure 9.1: Call option prices for the SVJD model compared to the corresponding pure diffusion Black-Scholes values for $T = \{0.25, 1.0\}$.

◦ Put Option Prices:



(a) Put prices for $T = 0.25$.



(b) Put prices for $T = 1.0$.

Figure 9.2: Put option prices for the SVJD model by ***put-call parity*** compared to the corresponding pure diffusion Black-Scholes values for $T = \{0.25, 1.0\}$.

○ **Conclusions on SVJD Option Pricing [Yan-Hanson (2006)]:**

- ***Proposed an alternative stochastic-volatility, jump-diffusion (SVJD) model***. The stochastic variance has mean-reversion with square-root noise and the jump-amplitude has log-uniform distribution.
- ***Characteristic functions of the log-terminal stock price*** and the conditional risk neutral probability are ***derived***.
- The option prices are expressed in terms of characteristic functions in ***formally closed form***.
- Accurate and fast computing algorithms are compared, using a ***10-point Gauss, discrete Fourier transform (DFT)*** and an ***FFT***, but FFT is fast for many $(\boldsymbol{K}, \boldsymbol{T})$ values, but DFT is better for selected discrete $(\boldsymbol{K}, \boldsymbol{T})$.
- The ***Black-Scholes prices are higher than SVJD model option prices***, especially for longer $\boldsymbol{T}$ and near-ATM $\boldsymbol{K}$.

Summary of Lecture 9?

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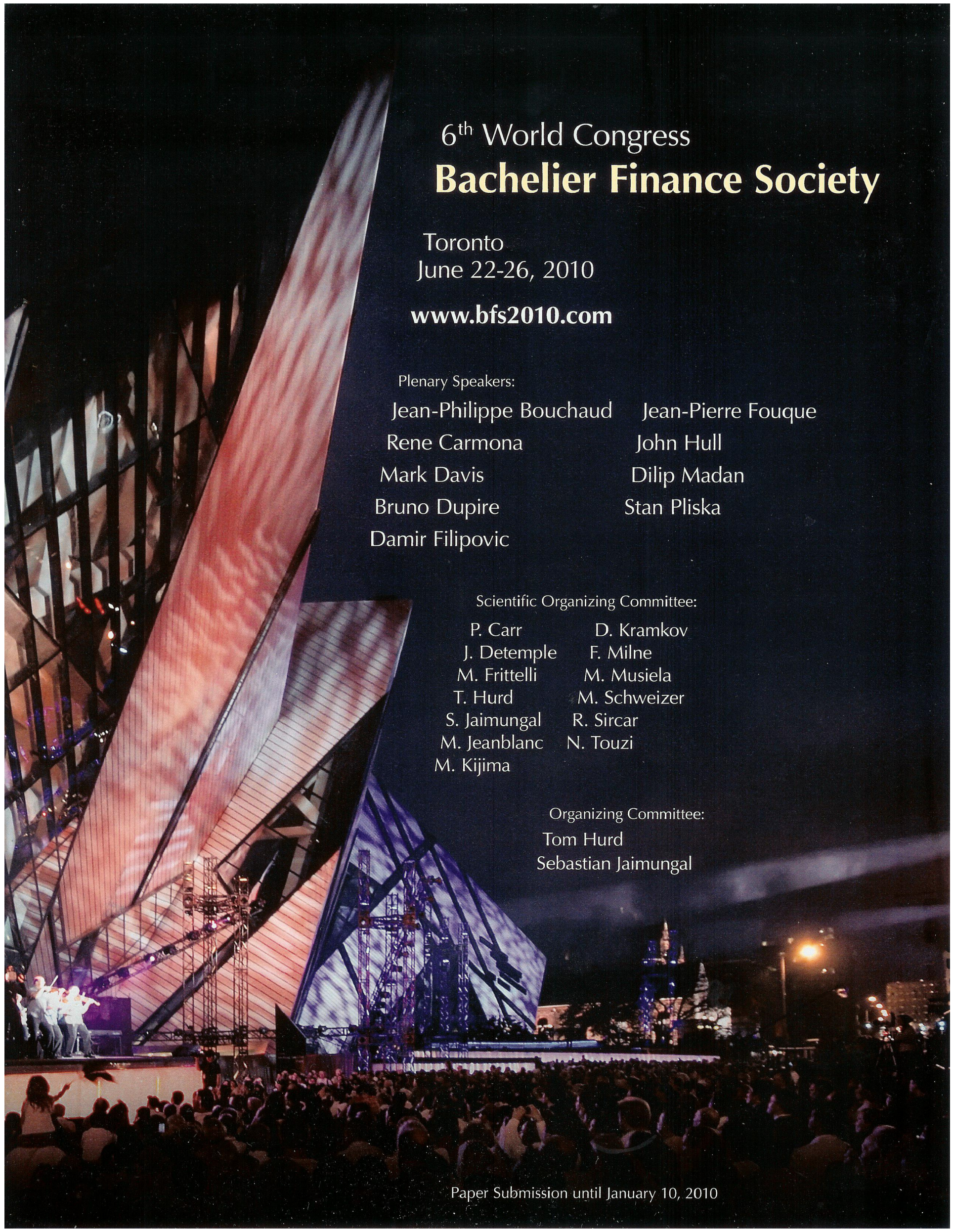
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