FINM345/STAT390 Stochastic Calculus – Hanson – Autumn 2009

Take-Home Final Examination:

Due by 6:30pm CST Monday 07 December 2009 (7:30pm EST at UBS; 7:30am Tuesday 08 Dec. in Singapore) in Chalk FINM345 Assignment Submenu

- **You must show your work, code and/or worksheet for full credit. The work must also be your own and points will be deducted for likely copies.**
- **There are multiple points per question and grades depend on both the correct answer and the quality as well as quantity of the justification.**
- **Points will be deducted for late submission by a point per hour per problem to begin, not to exceed one half the total of earned points.**
- **The exam is open Lecture notes and course textbook, but any other references used must be cited in a scholarly fashion.**
- **In your exam submission, include a copy or reasonable facsimile of this signed statement:**

On my honor this take-home exam is my own work, except for any citation to resources that I have used.

Signed: ______________________________. **(10 points)**

Corrections or emphasis are in ***Red*** as are comments, December 5, 2009

0. Innovations beyond what the problem asks for. (variable points)

1. JD SDE Transformations:

{*Comment: Note that for sufficiently small Δt, and dt, you can use the zero-one jump law, but for general values you need the jump form below.*}

(a) Given

$$dS(t) = S(t)\cdot(\mu(t)dt + \sigma(t)dW(t)) + \sum_{j=1}^{dP(t)} S(T_j^-)\nu(T_j^-, Q_j), \quad (1)$$

show that

$$d\left(\frac{1}{S(t)}\right) = \left(\frac{1}{S(t)}\right)\cdot(f(S(t),t)dt + g(S(t),t)dW(t)) + \sum_{j=1}^{dP(t)} h(S(T_j^-), T_j^-, \nu(T_j^-, Q_j)), \quad (1.5)$$

by finding the functions $f(s,t)$, $g(s,t)$ and $h(s,t,\nu(q))$ explicitly. (10 points)

(b) Given

$$dY(t) = \mu(Y(t),t)dt + \sigma(Y(t),t)dW(t) + \sum_{j=1}^{dP(t)} \nu(Y(T_j^-), T_j^-, Q_j), \quad (2)$$

show that

$$d(\exp(Y(t))) = F(Y(t),t)dt + G(Y(t),t)dW(t) + \sum_{j=1}^{dP(t)} H(Y(T_j^-), T_j^-, \nu(Y(T_j^-), T_j^-, Q_j)), \quad (3)$$

by finding the functions $F(y,t)$, $G(y,t)$ and $H(y,t,\nu(y,t,q))$ explicitly. (15 points)

2. Stochastic Calculus Example from Forward Contracts*:

Consider the price of a forward contract for energy at time t expiring at T, satisfying

$$dF(t,T) = F(t,T)\sigma(t,T)dW(t). \tag{4}$$

(a) Show that

$$F(t,T) = F(0,T)\exp\left(\int_0^t \big(\sigma(s,T)dW(s) - 0.5\sigma^2(s,T)ds\big)\right), \tag{5}$$

by stochastic calculus. (15 points)

(b) If the spot price is $S(t) = F(t,t)$ at t and if $Y(t) = \ln(S(t))$, show that

$$dY(t) = \frac{\partial \ln(F)}{\partial T}(0,t) \cdot dt + \sigma(t,t)dW(t) - 0.5\sigma^2(t,t)dt + dt \cdot \int_0^t \frac{\partial \sigma}{\partial T}(s,t) \cdot (dW(s) - \sigma(s,t)ds), \tag{6}$$

where the partial with respect to T is the partial with respect to the second argument of F or σ. (20 points)

(c) Show that

$$dS(t) = S(t)\big(dY(t) + 0.5\sigma^2(t,t)dt\big). \tag{7}$$

(10 points)

{**Background note: You do not have to know anything about forward contracts for this problem, but ONLY about stochastic calculus. This problem and problem 1(a) arose out of the course or from book questions. Also, from calculus or advanced calculus/analysis, you will need to know how to handle the derivative of a doubly time-dependent integral like $\int_0^t f(s,t)ds$ with respect to t or be able to derive it in dt-precision.*}

{An extended hint: For sufficiently small Δt, or just using dt with dt-precision (={dt}), you want to consider the increment
$\int_0^{t+dt} f(s,t+dt)ds - \int_0^t f(s,t)ds =\{dt\} \quad \left(f(t,t) + \int_0^t f_T(s,t)ds\right) \cdot dt$,
where $f_T(t,T) = (\partial f/\partial T)(t,T)$. and the integral with $dW(s)$ can be handled similarly since the $dW(s)$ is not expanded with dt.}

3. Stock-Variance Covariance: Given the risk-neutral, option pricing, stock-variance stochastic dynamical system,

$$dS(t) = S(t)((r_0 - \lambda_0 \overline{\nu})dt + \sqrt{V(t)}dW_s(t)) + \sum_{j=1}^{dP(t)} S(T_j^-)\nu(Q_j), \ S(0) = S_0 > 0, \quad (8)$$

and

$$dV(t) = \kappa_0(\theta_0 - V(t))dt + \sigma_0\sqrt{V(t)}dW_v(t), \ V(0) = V_0 \geq \varepsilon_0, \quad (9)$$

such that $V(t+dt) = \max(V(t) + dV(t), \varepsilon_0)$, $\mathrm{E}[dP(t)] = \lambda_0 dt$ and $\mathrm{Cov}[dW_s(t), dW_v(t)] = \rho_0 dt$.

(a) Show that, if Δt sufficiently small and ignoring the bound $v \geq \varepsilon_0$, then

$$\mathrm{Cov}[S(t+\Delta t), V(t+\Delta t)|S(t) = s, V(t) = v] \simeq \rho_0\sigma_0 s v \Delta t. \quad (10)$$

(10 points)

(b) Produce and plot simulations of the system $\{S(t), V(t)\}$ versus t for the following parameter values [L10-p27 or Yan-Hanson (ACC2007), from OEX 10 April 2006, mostly],

Table 1: Estimated JD-Uniform Parameters

JD Parameter	r_0	a	b	λ_0	S_0	T	N
JD Values	0.015	-0.140	0.011	0.549	100	2	5000

Table 2: Estimated SV Parameters

SV Parameter	κ_0	θ_0	σ_0	ρ_0	V_0	ε_0	T	N
SV Values	10.62	0.0136	0.175	-0.547	0.0083	**0.005**	2	5000

where N is the number of time intervals in $[0, T]$. (20 points)

(c) Calculate the unbiased sample covariance,

$$\mathcal{S}_N^{(s,v)} = \frac{1}{N}\sum_{i=0}^{N}(S_i - \overline{S}_N)(V_i - \overline{V}_N), \quad (11)$$

from the simulation data $\{S_i, V_i | i = 1 : N\}$ and sample means $\{\overline{S}_N, \overline{V}_N\}$, plus initial data $\{S_0, V_0\}$. Also, represent $\mathcal{S}_N^{(s,v)}$ as a correlation coefficient { *Hint: It is calculated like the continuous time version given in class with the normalization by function of the two sample variances.*}. (10 points)

(d) Calculate the simple moving average covariances, also unbiased, from the same simulation data, i.e.,

$$\mathcal{S}_{j,k,N}^{(s,v)} = \frac{1}{n}\sum_{i=(j-1)n}^{jn}(S_i - \overline{S}_{j,k,N})(V_i - \overline{V}_{j,k,N}), \quad (12)$$

for $j = 1 : k$ for $k = 20$ windows, where $n = N/k$ must be an integer and $\{\overline{S}_{j,k,N}, \overline{V}_{j,k,N}\}$ are the corresponding moving average means, e.g., $\overline{S}_{j,k,N} = \frac{1}{n+1}\sum_{i=(j-1)n}^{jn} S_i$. Also, plot the $\mathcal{S}_{j,k,N}^{(s,v)}$ of each window as correlation coefficients versus t using the midpoint in time for each window. (20 points)

(e) Discuss the significance of the results. (5 points)

4. Very Heuristic Model of American Option Smooth Contact to Put Payoff Function

Consider the first order put option price model

$$P_Q(s) = s^a(\alpha + \beta(s - S^*)) \tag{13}$$

that can mimic the basic properties in the Continuation region, $S > S*$, pre-optimal exercise at S^*, given on L10-p9 for $S \to (S^*)^+$ for P_Q and P'_Q, noting that the extreme limit as $S \to \infty$ is satisfied if $a < 0$ is assumed and implied by the heuristic quadratic approximation on L10-p13.

(a) Find the parameters $\{\alpha, \beta\}$ that satisfy the near exercise limits as $S \to (S^*)^+$ for P_Q and P'_Q, writing $P_Q(s)$ in terms of s and parameters $\{a, S^*, K\}$ only.
(10 points).

(b) Now suppose you **do not know** $S*$ and want to find it using Eq. (13), as a black box or source of data, but to do that let $K = \$100$, $S^* = 0.85K$, $a = -2$ and starting iterate $S_0 = 1.3K$.

For a simple and short way to approximate S^*, combine the two exercise conditions as the sum of quadratics (both terms must be zero at the minimum of $G(s)$),

$$G(s) = (P_Q(s) - (K - s))^2 + (P'_Q(s) + 1)^2. \tag{14}$$

Next, produce a plot of $G(s)$ versus s for $s = (2 * K - S_0) : 0.1 * K : S_0$. Then report the minimum, G_1, and second smallest, G_2, of the discrete G-values. Also give their locations, s_i for $i = 1 : 2$, and give the weighted average location, $s^*_{1,2}$, of the two as a simple approximation of S^*, with the weights proportional to opposites values (G_j with s_i for $j \neq i$) and relative to the sum of the G-values **{i.e.,** $s^*_{1,2} = (G_2 s_1 + G_1 s_2)/(G_1 + G_2)$**}**. Compare your approximation $s^*_{1,2}$ with the preassumed value $S^* = 0.85K$ in terms of a relative error.

{*Comment: If $P_Q(S)$ were the trajectory of the risk-neutral put price instead, then you could use a MATLAB optimization like the basic, derivative-free* `fminsearch` *using avfinal stopping tolerance such as* $|S_{i^*} - S_{i^*-1}| \leq 0.01$ *or the Optimization Toolbox* `lsqnonlin`*, if that toolbox is available.*} (20 points)

(c) Discuss the results, bearing in mind that trying to find a smooth contact is much harder than finding the intersection between two functions. (5 points).

5. **RGW (Roll, Geske, Whaley) Approximations for American Call Option Prices with Early Exercise Only Optimal on the Final Discrete Dividend:** Let $S_0 = \$100$, $r_0 = 2.0\%$ p.a., $\sigma_0 = 25\%$, $K = 80 : 5 : 120$ dollars and $T = 6$ months, with a known final dividend amount $D1 = 1 : 0.5 : 3$ in US dollars per share at date $T1 = 5$ months. There are two prefinal dividends of $D0 = 1$ (alternate better value is $D0 = 0.25$) dollar per share at months $T0 = 2$ and 4.

   (a) Compute the RGW early-exercise American call prices for strike prices and dividends. You can use Sivakumar's rogewhaley.m *Roll, Geske, Whaley (Single Dividend)* code at Global-Derivatives ***GD MATLAB Code List*** and information at ***American Pricing Models: RGW section***. You can also write modifications of the code or write your own code. All needed functions called are contained in the RGW package, but you still need to write the proper driver code and necessary modifications. (10 points)

   (b) Plot the RGW approximation to American call prices versus the moneyness S_0/K with $D1$-values as the parameter for each respective curve using different symbols or other distinct markings. (10 points)

   (c) Plot the critical stock price S^*, in the code S_{star} for calls, versus K for $D1 = 0 = D0$ (alternate better values are $D0 = 0.25$ and $D1 = 1.5$). (10 points)

   (d) Discuss the effects of the **dividend payout**. (5 points)

   {*Hint: watch your units carefully.*}

6. BAW (Barone-Adesi, Whaley) Modified Quadratic Approximation for American Put (or Call?) Option Pricing with Constant Yield Dividend:

{*Note: The dividend yield problem is a continuous rate dividend, UNLIKE the previous discrete dividend problem, but is used because it can be easily included along with the spot rate (See the BAW code), although it is not as realistic as the discrete case.*}

Let $S_0 = \$100$, $r_0 = 2.0\%$ p.a., $\sigma_0 = 25\%$ and $T = 6$ months with a single known dividend yield D. Strike prices of interest are $K = 80 : 5 : 120$ dollars and dividend yields are $D = \{0, 1 : 0.25 : 1.5\}$ percent.

(a) Compute and plot the American put option prices versus moneyness (the natural variable for the approximation), S_0/K, using the given K values for the fixed S_0, with individual curves parameterized by 4 dividend values $D = 0, 1, 1.25, 1.5$. You can use another of Kevin's codes, *Barone-Adesi, Whaley (Quadratic Approximation)* code at Global-Derivatives ***GD MATLAB Code List*** and information at ***American Pricing Models: BAW section***. You can also use modifications of the code or write your own code. All needed called-functions are contained in the package, but you still need to write the encompassing driver code and necessary modifications.

(20 points)

(b) Similarly, compute and plot the American call option prices versus moneyness, S_0/K, using the given K values for the fixed S_0, with individual curves parameterized by 4 dividend values $D = 0, 1, 1.25, 1.5$.

(20 points)

(c) Plot the critical stock price S^*, in the code S_p for puts and S_c for calls, versus K for $D = 1.5$ (replacing $D = 0$, else a problem with the call early exercise and S_c), each on separate graphs. (15 points)

(d) Compute the discrepancy in the put-call parity relation, modified for the dividend yield for American options, when $D = 1.5$ and plot versus moneyness using $K = 80 : 5 : 120$ and the given S_0. {Hint: Put-Call Parity for dividend yield, from Hull Eq. (14.3) 6th Ed., is $C + K * \exp(-r_0 T) = P + S_0 * \exp(-DT)$.}

(10 points)

(e) Discuss the effects of the constant dividend yield relative to the results of the non-dividend case. Also, discuss the put-call parity discrepancy, in particular, whether it confirms the lack of general validity for American options.

(10 points)

{*Hint: watch your units carefully.*}

7. Merton-Like Optimal Portfolio and Consumption Problem for Multiple Assets and SVJD:

Let there be n assets in addition to a bond, so that the stochastic dynamics are given by

$$dS_0(t) = S_0(t)(\mu_0 dt + \sigma_0 \sqrt{V} dW_0(t)) + dCP_0(t, S_0(t)\nu_0) \text{ for bond,} \tag{15}$$

$$dS_i(t) = S_i(t)(\mu_i dt + \sigma_i \sqrt{V} dW_i(t)) + dCP_i(t, S_i(t)\nu_i) \text{ for stock } i, \tag{16}$$

for $i = 1 : n$, where $\mu_j = \mu_j(S_j(t), t)$, $\sigma_j = \sigma_j(S_j(t), t)$ (these are "vol-vol" coefficients), and $\nu_j = \nu_j(S_j(t), t, Q_j) = \exp(Q_j) - 1$, for $j = 0 : n$. The compound Poisson terms are independent of the Gaussian terms, but the Gaussian terms are correlated with $\text{Cov}[dW_i(t), dW_j(t)] = \rho_{i,j}(t)dt$ for $i, j = 0 : n$ such that $\rho_{j,j} = 1$, while $\text{E}[dP_j(t) = \lambda_j dt$ for the independent Poisson counting processes. The stochastic variance is the usual square-root diffusion model,

$$dV(t) = \kappa_v(t)(\theta(t) - V(t)dt + \sigma_v(t)\sqrt{V(t)}dW(t), \tag{17}$$

where $V(t + dt) = \max((V + dV)(t), \varepsilon_v)$ and $\varepsilon_v > 0$. Here, $\text{Cov}[dW_j(t), dW_v(t)] = \rho_{j,v}(t)dt$. Let the portfolio fractions satisfy $U_0(t) + \sum_{i=1}^{n} U_i(t) = 1$. The objective is to maximize the expected discounted utility of final wealth $W(T)$ plus the running discounted utility of consumption, starting at the current time t with the state conditions $\mathcal{S} = \{W(t) = w, V(t) = v\}$, control conditions $\mathcal{C} = \{\vec{U}(t) = \vec{u} \equiv [u_i]_{n\times 1}, C(t) = c\}$, differing utilities $\{\mathcal{U}_w(W(t)), \mathcal{U}_c(C(t))\}$, instant discount rate $\beta(t)$ and cumulative discount $\overline{\beta}(t)$, so that,

$$e^{-\overline{\beta}(t)} J^*(w, v, t) = \max_{\{\vec{u},c\}} \left[\text{E}\left[e^{-\overline{\beta}(T)} \mathcal{U}_w(W(T)) + \int_t^T e^{-\overline{\beta}(s)} \mathcal{U}_c(C(s))ds \,\middle|\, \mathcal{C}, \mathcal{S} \right] \right]. \tag{18}$$

(a) For the wealth $W(t)$, derive the stochastic dynamic equation for $dW(t)$, beginning by modifying (10.17) of L10-p37 for the $n+1$ assets, assuming self-financing, less the consumption in $[t, t+dt)$, in dt-precision. {*Hint: Recall, that the portfolio return for a single risky asset and bond is the sum the relative returns (not the absolute returns) weighted by the portfolio fractions, as in the lecture on Merton's BS paper, so* $dW(t)/W(t) = (1-U(t)) * dB(t)/B(t) + U(t) * dS(t)/S(t) - C(t)dt/W(t)$, *less the consumption relative to the wealth.*} (15points)

(b) Justify the final condition at $t = T$ and the absorbing boundary condition as $w \to 0^+$ and $c \to 0^+$, in this multi-asset case. (10 points)

(c) Derive the stochastic dynamic programming PIDE by stochastic calculus in dt-precision, assuming a corresponding form of Bellman's principle of optimality and, in particular, show where and how the expectations of all the stochastic terms are evaluated. (30 points)