APPLICATIONS ORIENTED MATHEMATICS PRELIMINARY EXAMINATION

Monday, September 10, 2001 1:00-4:00pm

Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

1. Find the leading term in the asymptotic expansion of the following integral $I(\lambda)$ as (a) $\lambda \to \infty$ and (b) $\lambda \to -\infty$

$$I(\lambda) = \int_0^1 e^{\lambda t^2} t(1-t)^2 dt.$$

2. Find the leading term in the asymptotic expansion of the following integral as $\lambda \to \infty$

$$I(\lambda) = \int_C e^{i\lambda(z+z^3/3)} dz.$$

Here C is the real axis in the complex z plane (hence $z = x$ with $-\infty < x < \infty$).

   (a) Find all saddle points in the complex z plane.

   (b) At each saddle, find the local steepest descent (SD) and steepest ascent (SA) directions.

   (c) Find the (global) constant phase contours through each saddle, and identify the SD and SA contours.

   (d) By an appropriate contour deformation, obtain the leading term for $I(\lambda)$ as $\lambda \to +\infty$.

3. Consider the ODE $y''(x)+2xy'(x)+x^2y(x) = 0$. Find the leading term in the expansions of all solutions as $x \to \infty$ (show your work!).

4. Consider the eigenvalue problem

$$y''(x) + \lambda e^x y(x) = 0, \quad 0 < x < 2, \quad y(0) = 0, \quad y(2) = 0.$$

Find, using the WKB method, the leading term in the expansion of the large eigenvalues λ_n as $n \to \infty$. Also obtain the corresponding eigenfunctions.

5. Use singular perturbation methods to construct the inner, outer and composite expansion to the solution $y(x)$ of the problem

$$\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 1/4, \quad y(1) = 1.$$

6. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x - \frac{1}{2}xy, \quad x(0) = 1/4 \\ \varepsilon\frac{dy}{dt} &= x - y + 1, \quad y(0) = 0 \end{aligned}$$

   (a) Find the leading term in the composite or uniform expansion as $\varepsilon \to 0^+$.

   (b) Make a rough sketch of the solution in the xy phase plane.

7. Consider the ODE

$$y'' + y' + y(9y^2 + 4\lambda^2) = y, \quad t > 0, \;\; \lambda \text{ is real.}$$

Determine the bifurcation points, find the steady-state solutions (include a bifurcation diagram), and determine the stability of each branch.

8. Use the method of multiple scales to find the leading term of the solution to the following problem that is valid for $t = \mathcal{O}(1/\varepsilon)$

$$y'' - \varepsilon(y')^3 + y = 0, \quad y(0) = 1, \;\; y'(0) = 0.$$