MATHEMATICAL SCIENCE PRELIMINARY EXAMINATION

Monday, April 19, 2004 1:00-4:00pm

The exam is based on questions from the areas: **Appled Optimal Control** and **Computational Finance**. There are 4 questions in each area. Each question is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

Computational Finance

1. Let $S(t)$ be the price of a stock index that pays a continuous yield q and is modelled by the SDE:

$$dS = \mu S dt + \sigma S dW.$$

Here $W(t)$ is Brownian motion, μ is the real world return, and σ is the volatility. Also, assume r is the constant risk-free interest rate.

   (a) *Derive* a Black-Scholes type equation for $V(S,t)$, the price of a European derivative on this index with expiration T and payoff $\Lambda(S)$. Be sure to state the final pricing problem completely.

   (b) If the derivative is a European call option with strike E. State explicitly the formula for its price $c(S,t)$.

   (c) State explicitly the replicating portfolio for the call option described in (b). You must give the quantities of each instrument.

2. Consider a three step multiplicative binomial tree for pricing a European put option on an asset with price S today. The time step is δt and r is the constant risk-free interest rate. The multiplicative jump factors are $u > 1$ (up) and $d < 1$ (down), where $u \times d = 1$.

   (a) Find the risk-neutral probabilities q (up) in terms of u, d, δt, and r.

   (b) Sketch the asset tree for *only* 3 steps and label the asset prices.

   (c) Give an explicit formula (in terms of q and the other parameters) for the price of the put option today, if the strike price is E and the asset price is *at-the-money* today.

   (d) If the option is an American put, find the price of the option at the node with the smallest price in the second time step. Explain the condition that determines whether early exercise is feasible. Assume the strike price is E and the asset price is *at-the-money* today.

3. The price of an asset satisfies

$$dS = \mu dt + \sigma dW, \quad S(0) = S_0$$

where W is Brownian motion.

   (a) Construct the strong solution to the SDE and find the density function of S.

   (b) Compute $\mathrm{E}[S]$ and Var(S).

4. The risk-neutral dynamics of an asset with price $S(t)$ is described by

$$dS = rSdt + \sigma S dW$$

where $W(t)$ is Brownian motion, σ is the volatility, and r is the constant risk-free rate.

   (a) Derive a formula for the density function $p(\hat{S}, T; S, t)$, i.e the probability that $S(T) = \hat{S}$ given that $S(t) = S$ where $t < T$.

   (b) Show that $\mathrm{E}^Q[S(T)|S(t) = S]$ is the forward price of the asset at time t

5. Show that
$$\int_0^t e^{aP(s)}dP(s) = \left\{ \begin{array}{cc} \dfrac{e^{aP(t)}-1}{e^a-1}, & a \neq 0 \\ P(t), & a = 0 \end{array} \right\} ,$$
for real constant a.

6. Solve the following (Itô) diffusion SDE
$$dX(t) = \left(\alpha_0\sqrt{X(t)} + \beta_0^2/4\right)dt + \beta_0\sqrt{X(t)}dW(t) ,$$
for $X(t)$, $\mathrm{E}[X(t)]$ and $\mathrm{Var}[X(t)]$, where α_0 and β_0 are constants, and $X(0) = x_0 > 0$, with probability one.

7. Solve the linear jump-diffusion (Itô) SDE for $X(t)$,
$$dX(t) = \mu(t)X(t)dt + \sigma_0 X(t)dW + \nu_0 X(t)dP(t),$$
for $t > 0$, $X(0) = x_0 > 0$, $\mathrm{E}[W(t)] = 0$, $\mathrm{Var}[W(t)] = t$, and $\mathrm{E}[P(t)] = \lambda_0 t = \mathrm{Var}[P(t)]$. The $\{\sigma_0, \nu_0, \lambda_0\}$ are constants. Using this solution show that
   (a) the expectation of the state is $\mathrm{E}[X(t)] = x_0\exp(m(t) + \lambda_0\nu_0 t)$, where $m(t) \equiv \int_0^t \mu(t)dt$;
   (b) the corresponding *squared coefficient of variation* is $\mathrm{Var}[X(t)]/\mathrm{E}^2[X(t)] = \exp((\sigma_0^2 + \lambda_0\nu_0^2)t) - 1$.

8. For a linear quadratic Gaussian (LQG) problem, the (Itô) linear dynamics equation is
$$dX(t) = (\mu_0 X(t) + \beta_0 U(t))dt + \sigma_0 X(t)dW(t),$$
for $t > 0$, $X(0) = x_0$, $\mu_0 \neq 0$, $\beta_0 \neq 0$, $\sigma_0 > 0$, where the control process $U(t)$ in unconstrained, and the quadratic criterion is
$$V[X(t), U(t)] = \frac{1}{2}\int_t^{t_f} \left(q_0 X^2(t) + r_0 U^2(t)\right)dt, + \frac{1}{2}S_f X^2(t_f)$$
for $q_0 > 0$, $r_0 > 0$, and $S_f > 0$,
   (a) find the PDE of Stochastic Dynamic Programming for the optimal expected value:
$$v^*(x,t) = \min_u \left[\mathrm{E}\left[V[X(t), U(t)]\,|X(t) = x, U(t) = u\right]\right],$$
   and find the optimal (unconstrained) control $u^*(x,t)$ in terms of the *shadow price* $v_x^*(x,t)$;
   (b) show that this PDE of SDP formally admits a pure quadratic form solution $v^*(x,t) = \frac{1}{2}S(t)x^2$ by deriving the resulting final value problem for a Riccati equation that determines the coefficient $S(t)$ (do not solve) and find the *linear feedback control law* for $u^*(x,t)$ in terms of x, $S(t)$ and other parameters.