Log-Double-Uniform Jump-Diffision Model for Stock Price Dynamics (Working Paper)

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1 Log-Double-Uniform Jump-Diffusion Model

The following constant rate stochastic differential equation (SDE) is used to model the dynamics of the asset price, $S(t)$:

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t) + J(Q)dP(t)\right), \tag{1}$$

where $S_0 = S(0) > 0$, μ is the drift coefficient, σ is the diffusive volatility, $W(t)$ is the Wiener process, $J(Q)$ is the Poisson jump-amplitude, Q is an underlying Poisson amplitude mark process selected so that

$$Q = \ln(J(Q) + 1)$$

for convenience, $P(t)$ is the standard Poisson jump counting process with joint mean and variance

$$\mathrm{E}[P(t)] = \lambda t = \mathrm{Var}[P(t)].$$

Let the density of the jump amplitude mark Q be double-uniformly distributed:

$$\phi_Q(q) = -\frac{p}{a} I_{\{a \le q < 0\}} + \frac{q}{b} I_{\{0 \le q \le b\}}, \tag{2}$$

where $a < 0 < b$ and $0 < p < 1$ represents the probability of downword jumps and $q = 1 - p$ is the probability of upward jumps. The set indicator function is $I_{\{\mathcal{S}\}}$ for set S. The mean of Q is $\mu_j = \frac{1}{2}(pa + qb)$ and the variance of Q is $\sigma_j^2 = \frac{pq}{4}(b-a)^2 + \frac{pa^2+qb^2}{12}$. The third central moment of Q is $M_j^{(3)} \equiv E[(q-\mu_j)^3] = \frac{pq}{4}(b-a)^2(aq+bp)$ and The fourth central moment of Q is $M_j^{(4)} \equiv E[(q-\mu_j)^4] = \mu_j^4 + p/5(a^4 - 5a^3\mu_j + 10a^2\mu_j^2 - 10a\mu_j^3) + q/5(b^4 - 5b^3\mu_j + 10b^2\mu_j^2 - 10b\mu_j^3)$.

According to the Itô stochastic chain rule [5] for jump-diffusions, the log-return process $\ln(S(t))$ satisfies the constant coefficient SDE

$$d\ln(S)(t) = \mu_{ld}dt + \sigma dW(t) + \sum_{i=1}^{dP(t)} Q_i, \tag{3}$$

where $\mu_{ld} \equiv \mu - 0.5\sigma^2$ and the Q_i here are independent identically double-uniformly distributed jump-amplitude marks Q. In the case that the time step Δt is an increment rather than an infinitesimal like dt, the log-return increment $\Delta \ln(S(t))$ satisfies the following SDE

$$\Delta \ln(S)(t) = \mu_{ld} dt + \sigma \Delta W(t) + \sum_{i=1}^{\Delta P(t)} Q_i, \tag{4}$$

2 The Basic Moments of Log-Return Increments $\Delta \ln(S(t))$

For the Moments of Log-Return Increments $\Delta \ln(S(t))$, we have the following theorem:

Theorem 2.1 *If $\Delta \ln(S(t))$ satisfies SDE (4), the first four moments of $\Delta \ln(S(t))$ are the following:*

$$M_1^{(jd)} \equiv \mathrm{E}[\Delta \ln(S(t))] = (\mu_{ld} + \lambda \mu_j)\Delta t;$$

$$\begin{aligned} M_2^{(jd)} &\equiv \mathrm{Var}[\Delta \ln(S(t))] \\ &= (\sigma^2 + \lambda(\sigma_j^2 + \mu_j^2))\Delta t; \end{aligned}$$

$$\begin{aligned} M_3^{(jd)} &\equiv \mathrm{E}\left[(\Delta \ln(S(t)) - M_1^{(jd)})^3\right] \\ &= \frac{pa^3 + qb^3}{4}\lambda \Delta t; \end{aligned}$$

$$\begin{aligned} M_4^{(jd)} &\equiv \mathrm{E}\left[(\Delta \ln(S(t)) - M_1^{(jd)})^4\right] \\ &= \frac{pa^4 + qb^4}{5}\lambda \Delta t \\ &\quad +3(\sigma^2 + \lambda(pa^2 + qb^2)/3)^2(\Delta t)^2. \end{aligned}$$

Proof: From Theorem 5.12 in [5], we know directly that the first two moments $M_1^{(jd)}$ and $M_2^{(jd)}$ are true. Also, from the same Theorem 5.12, $M_3^{(jd)} = (M_j^{(3)} + \mu_j(3\sigma_j^2 + \mu_j^2))\lambda\Delta t$ and $M_4^{(jd)} = \left(M_j^{(4)} + 4\mu_j M_j^{(3)} + 6\mu_j^2\sigma_j^2 + \mu_j^4\right)\lambda\Delta t + 3(\sigma^2 + \lambda(\mu_j^2 + \sigma_j^2))^2(\Delta t)^2$. Then, we put the values of μ_j, σ_j^2, $M_j^{(3)}$ and $M_j^{(4)}$ of the doule-uniform jump amplitude mark Q into the above formulae and do some simplifications, will finally get the formulae as in the theorem for the third and fourth moments. $\square$

3 The density of $\Delta \ln(S(t))$

In order to calculate the density of $\mathbf{\Delta ln(S(t))}$, we need the following Lemmas.

Lemma 3.1 ***Shift Property of the Accumulated Normal Distribution Function:***

$$\Phi^{(n)}(a+x, b+x; \mu+x, \sigma^2) = \Phi^{(n)}(a, b; \mu, \sigma^2).$$

Proof:

$$\begin{aligned}
\Phi^{(n)}(a+x, b+x; \mu+x, \sigma^2) &= \int_{a+x}^{b+x} \frac{e^{-\frac{(z-\mu-x)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dz \\
&\overset{\text{set } y = z - x}{=} \int_a^b \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= \Phi^{(n)}(a, b; \mu, \sigma^2).
\end{aligned}$$

□

Lemma 3.2 ***Distribution Property of the negative sign:***

$$\Phi^{(n)}(-a, -b; -\mu, \sigma^2) = -\Phi^{(n)}(a, b; \mu, \sigma^2).$$

Proof:

$$\begin{aligned}
\Phi^{(n)}(-a, -b; -\mu, \sigma^2) &= \int_{-a}^{-b} \frac{e^{-\frac{(x+\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx \\
&\overset{\text{set } z = -x}{=} \int_a^b \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} - dz \\
&= \int_b^a \frac{e^{-\frac{(z-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dz \\
&= \Phi^{(n)}(b, a; \mu, \sigma^2) \\
&= -\Phi^{(n)}(a, b; \mu, \sigma^2).
\end{aligned}$$

□

Lemma 3.3

$$\begin{aligned}
IB_1(x_1, x_2, A, \sigma) &\equiv \int_{x_1}^{x_2} x \frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx \\
&= \frac{\sigma}{\sqrt{2\pi}} (e^{-\frac{(x_1-A)^2}{2\sigma^2}} - e^{-\frac{(x_2-A)^2}{2\sigma^2}}) + A\Phi^{(n)}(x_1, x_2; A, \sigma^2).
\end{aligned}$$

Proof:

$$
\begin{aligned}
IB_1(x_1, x_2, A, \sigma) &= \int_{x_1}^{x_2} (x - A + A)\frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\
&= \int_{x_1}^{x_2} (x - A)\frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx + A\int_{x_1}^{x_2} \frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\
&= -\sigma^2 \int_{x_1}^{x_2} \frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}d - \frac{(x-A)^2}{2\sigma^2} + A\Phi^{(n)}(x_1, x_2; A, \sigma^2) \\
&= \frac{-\sigma}{\sqrt{2\pi}}e^{-\frac{(x-A)^2}{2\sigma^2}}\sqrt{2\pi\sigma^2}|_{x_1}^{x_2} + A\Phi^{(n)}(x_1, x_2; A, \sigma^2) \\
&= \frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-A)^2}{2\sigma^2}} - e^{-\frac{(x_2-A)^2}{2\sigma^2}}) + A\Phi^{(n)}(x_1, x_2; A, \sigma^2).
\end{aligned}
$$

□

According to (3) and the convolution theorem [5], we get the following theorem:

Theorem 3.1 *Assume $a + b < 0$, the density of $\mathbf{dln}(\mathbf{S}(\mathbf{t}))$ is*

$$
\begin{aligned}
\phi(x) &= \sum_{k=0}^{\infty} p_k(\lambda\Delta t)\phi_{dG}(*\phi_Q)^k(x) && (5)\\
&\approx \sum_{k=0}^{2} p_k(\lambda\Delta t)\phi^{(k)}(x). && (6)
\end{aligned}
$$

where $dG \equiv \mu_{ld}\Delta t + \sigma dW(t)$, $p_k(\lambda\Delta t) = e^{-\lambda\Delta t}(\lambda\Delta t)^k/k!$ and

$$
\begin{aligned}
\phi^{(0)}(x) &= \phi^{(n)}(x; \bar{\mu}, \bar{\sigma}^2), \\
\phi^{(1)}(x) &= \frac{q}{b}\Phi^{(n)}(0, b; x - \bar{\mu}, \bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(a, 0; x - \bar{\mu}, \bar{\sigma}^2), \\
\phi^{(2)}(x) &= \frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg((\frac{p}{a} + \frac{q}{b})^2 e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} + (\frac{p}{a})^2 e^{-\frac{(x-\bar{\mu}-2a)^2}{2\bar{\sigma}^2}} + (\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu}-2b)^2}{2\bar{\sigma}^2}} \\
&\quad -2\left((\frac{p}{a})^2 + \frac{pq}{ab}\right) e^{-\frac{(x-\bar{\mu}-a)^2}{2\bar{\sigma}^2}} - 2\left((\frac{q}{b})^2 + \frac{pq}{ab})\right) e^{-\frac{(x-\bar{\mu}-b)^2}{2\bar{\sigma}^2}} + \frac{2pq}{ab}e^{-\frac{(x-\bar{\mu}-a-b)^2}{2\bar{\sigma}^2}} \Bigg) \\
&\quad +(\frac{p}{a})^2 \left((x - 2a - \bar{\mu})\Phi^{(n)}(2a, a; x - \bar{\mu}, \bar{\sigma}^2) - (x - \bar{\mu})\Phi^{(n)}(a, 0; x - \bar{\mu}, \bar{\sigma}^2)\right) \\
&\quad +\frac{2pq}{ab}\left((x - \bar{\mu})\Phi^{(n)}(0, b; x - \bar{\mu}, \bar{\sigma}^2) - (x - a - \bar{\mu})\Phi^{(n)}(a, a + b; x - \bar{\mu}, \bar{\sigma}^2)\right) \\
&\quad +(\frac{q}{b})^2 \left((x - \bar{\mu})\Phi^{(n)}(0, b; x - \bar{\mu}, \bar{\sigma}^2) - (x - 2b - \bar{\mu})\Phi^{(n)}(b, 2b; x - \bar{\mu}, \bar{\sigma}^2)\right) \\
&\quad -\frac{2pq}{a}\Phi^{(n)}(a + b, b; x - \bar{\mu}, \bar{\sigma}^2),
\end{aligned}
$$

where $\bar{\mu} \equiv \mu_{ld}\Delta t$ and $\bar{\sigma} \equiv \sigma\sqrt{\Delta t}$.

Proof: For the first part (5), please see Chapter 0 in Hanson's book [5]. Now we come to prove the second part (6).
For $k = 0$, $\phi^{(0)}(x) = \phi_{dG}(*\phi_Q)^0(x) = \phi_{dG}(x) = \phi^{(n)}(x; \bar{\mu}, \bar{\sigma}^2)$.
For $k = 1$,

$$\begin{aligned}
\phi^{(1)}(x) &= \phi_{dG}(*\phi_Q)^1(x) \\
&= \phi_{dG} * \phi_Q(x) \\
&= \int_{-\infty}^{\infty} \phi^{(n)}(x-y; \bar{\mu}, \bar{\sigma}^2)\phi_Q(y)dy \\
&= \int_{-\infty}^{\infty} \phi^{(n)}(x-y; \bar{\mu}, \bar{\sigma}^2)(-\frac{p}{a}I_{\{a \le y < 0\}} + \frac{q}{b}I_{\{0 \le y \le b\}})dy \\
&= -\frac{p}{a}\int_a^0 \phi^{(n)}(x-y; \bar{\mu}, \bar{\sigma}^2)dy + \frac{q}{b}\int_0^b \phi^{(n)}(x-y; \bar{\mu}, \bar{\sigma}^2)dy \\
&= \frac{q}{b}\Phi^{(n)}(x-b, x; \bar{\mu}, \bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(x, x-a; \bar{\mu}, \bar{\sigma}^2). \\
&\overset{Lemma}{\underset{3.2}{=}} -\frac{q}{b}\Phi^{(n)}(b-x, -x; -\bar{\mu}, \bar{\sigma}^2) + \frac{p}{a}\Phi^{(n)}(-x, a-x; -\bar{\mu}, \bar{\sigma}^2). \\
&\overset{Lemma}{\underset{3.1}{=}} -\frac{q}{b}\Phi^{(n)}(b, 0; x-\bar{\mu}, \bar{\sigma}^2) + \frac{p}{a}\Phi^{(n)}(0, a; x-\bar{\mu}, \bar{\sigma}^2). \\
&= \frac{q}{b}\Phi^{(n)}(0, b; x-\bar{\mu}, \bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(a, 0; x-\bar{\mu}, \bar{\sigma}^2).
\end{aligned}$$

For $k = 2$, first of all, let us calculate $(\phi_Q * \phi_Q)(y)$.

$$\begin{aligned}
(\phi_Q * \phi_Q)(y) &= \int_{-\infty}^{\infty} \phi_Q(y-z)\phi_Q(z)dz \\
&= \int_{-\infty}^{\infty} (-\frac{p}{a}I_{\{a \le y-z < 0\}} + \frac{q}{b}I_{\{0 \le y-z \le b\}})(-\frac{p}{a}I_{\{a \le z < 0\}} + \frac{q}{b}I_{\{0 \le z \le b\}})dz \\
&= (\frac{p}{a})^2\int_{-\infty}^{\infty} I_{\{y < z \le y-a\ ,\ a \le z < 0\}}dz + (\frac{q}{b})^2\int_{-\infty}^{\infty} I_{\{y-b \le z \le y\ ,\ 0 \le z \le b\}}dz \\
&\quad -\frac{pq}{ab}\int_{-\infty}^{\infty} I_{\{y < z \le y-a\ ,\ 0 \le z \le b\}}dz - \frac{pq}{ab}\int_{-\infty}^{\infty} I_{\{y-b \le z \le y\ ,\ a \le z < 0\}}dz \\
&= (\frac{p}{a})^2(min(y-a, 0) - max(y, a))^+ + (\frac{q}{b})^2(min(y, b) - max(y-b, 0))^+ \\
&\quad -\frac{pq}{ab}(min(y-a, b) - max(y, 0))^+ - \frac{pq}{ab}(min(y, 0) - max(y-b, a))^+.
\end{aligned}$$

But, if $a + b \le 0$, then

$$min(y-a, b) - max(y, 0))^+ = (min(y, 0) - max(y-b, a))^+ = \begin{cases} (y-a)^+ & \text{if } y \le a+b; \\ b & \text{if } a+b \le y < 0; \\ (b-y)^+ & \text{if } y \ge 0. \end{cases}$$

Otherwise, If $a+b>0$, then

$$min(y-a,b)-max(y,0))^{+} = (min(y,0)-max(y-b,a))^{+} = \begin{cases} (b-y)^{+} & \text{if } y \geq a+b; \\ -a & \text{if } 0 \leq y < a+b; \\ (y-a)^{+} & \text{if } y \leq 0. \end{cases}$$

Hence, we have,

$$\begin{aligned} (\phi_Q * \phi_Q)(y) &= (\frac{p}{a})^2(min(y-a,0)-max(y,a))^{+} + (\frac{q}{b})^2(min(y,b)-max(y-b,0))^{+} \\ &\quad -\frac{2pq}{ab}(min(y-a,b)-max(y,0))^{+}. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi^{(2)}(x) &= \phi_{dG}(*\phi_Q)^2(x) \\ &= \int_{-\infty}^{\infty} \phi^{(n)}(x-y;\bar{\mu},\bar{\sigma}^2)(\phi_Q * \phi_Q)(y)dy \\ &= (\frac{p}{a})^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,0)-max(y,a))^{+}dy \\ &\quad -\frac{2pq}{ab}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,b)-max(y,0))^{+}dy \\ &\quad +(\frac{q}{b})^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y,b)-max(y-b,0))^{+}dy. \end{aligned}$$

However,

$$\begin{aligned} I_1 &\equiv \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,0)-max(y,a))^{+}dy \\ &= (\int_{-\infty}^{a} + \int_{a}^{\infty})\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y-a,0)-max(y,a))^{+}dy \\ &= \int_{-\infty}^{a} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-2a)^{+}dy + \int_{a}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(-y)^{+}dy \\ &= \int_{2a}^{a} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(y-2a)dy + \int_{a}^{0} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(-y)dy \end{aligned}$$

$$
\begin{aligned}
&= \int_{2a}^{a} y\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy - \int_{a}^{0} y\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy - 2a\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2) \\
&\overset{\text{Lemma}}{\underset{3.3}{=}} \sqrt{\frac{\bar\sigma^2}{2\pi}}(e^{-\frac{(2a-(x-\bar\mu))^2}{2\bar\sigma^2}} - e^{-\frac{(a-(x-\bar\mu))^2}{2\bar\sigma^2}}) + (x-\bar\mu)\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2) \\
&\quad -\sqrt{\frac{\bar\sigma^2}{2\pi}}(e^{-\frac{(a-(x-\bar\mu))^2}{2\bar\sigma^2}} - e^{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}}) - (x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2) \\
&\quad -2a\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2). \\
&= \sqrt{\frac{\bar\sigma^2}{2\pi}}\left(e^{-\frac{(2a-(x-\bar\mu))^2}{2\bar\sigma^2}} - 2e^{-\frac{(a-(x-\bar\mu))^2}{2\bar\sigma^2}} + e^{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}}\right) + \\
&\quad (x-2a-\bar\mu)\Phi^{(n)}(2a,a;x-\bar\mu,\bar\sigma^2) - (x-\bar\mu)\Phi^{(n)}(a,0;x-\bar\mu,\bar\sigma^2).
\end{aligned}
$$

$$
\begin{aligned}
I_2 &\equiv \int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}(min(y-a,b)-max(y,0))^+dy \\
&\overset{a+b<0}{=} (\int_{-\infty}^{a+b} + \int_{a+b}^{0} + \int_{0}^{\infty})\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}(min(y-a,b)-max(y,0))^+dy \\
&= \int_{-\infty}^{a+b}\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}(y-a)^+dy + b\int_{a+b}^{0}\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy + \int_{0}^{\infty}\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}(b-y)^+dy \\
&= \int_{a}^{a+b}\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}(y-a)dy + b\Phi^{(n)}(a+b,0;x-\bar\mu,\bar\sigma^2) + \int_{0}^{b}\frac{e^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}(b-y)dy \\
&= \int_{a}^{a+b}\frac{ye^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy - \int_{0}^{b}\frac{ye^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy - a\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2) \\
&\quad +b\Phi^{(n)}(a+b,0;x-\bar\mu,\bar\sigma^2) + b\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2) \\
&= \int_{a}^{a+b}\frac{ye^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy - \int_{0}^{b}\frac{ye^{-\frac{(x-y-\bar\mu)^2}{2\bar\sigma^2}}}{\sqrt{2\pi\bar\sigma^2}}dy \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar\mu,\bar\sigma^2) - a\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2) \\
&\overset{\text{Lemma}}{\underset{3.3}{=}} \sqrt{\frac{\bar\sigma^2}{2\pi}}(e^{-\frac{(a-(x-\bar\mu))^2}{2\bar\sigma^2}} - e^{-\frac{(a+b-(x-\bar\mu))^2}{2\bar\sigma^2}}) + (x-\bar\mu)\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2) \\
&\quad -\sqrt{\frac{\bar\sigma^2}{2\pi}}(e^{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}} - e^{-\frac{(b-(x-\bar\mu))^2}{2\bar\sigma^2}}) - (x-\bar\mu)\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar\mu,\bar\sigma^2) - a\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2) \\
&= \sqrt{\frac{\bar\sigma^2}{2\pi}}\left(e^{-\frac{(a-(x-\bar\mu))^2}{2\bar\sigma^2}} + e^{-\frac{(b-(x-\bar\mu))^2}{2\bar\sigma^2}} - e^{-\frac{(a+b-(x-\bar\mu))^2}{2\bar\sigma^2}} - e^{-\frac{(x-\bar\mu)^2}{2\bar\sigma^2}}\right) \\
&\quad +(x-a-\bar\mu)\Phi^{(n)}(a,a+b;x-\bar\mu,\bar\sigma^2) - (x-\bar\mu)\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar\mu,\bar\sigma^2).
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\equiv \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y,b)-max(y-b,0))^+dy \\
&= (\int_{-\infty}^{b}+\int_{b}^{\infty})\frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(min(y,b)-max(y-b,0))^+dy \\
&= \int_{-\infty}^{b} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}y^+dy+\int_{b}^{\infty} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(2b-y)^+dy \\
&= \int_{0}^{b} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}ydy+\int_{b}^{2b} \frac{e^{-\frac{(x-y-\bar{\mu})^2}{2\bar{\sigma}^2}}}{\sqrt{2\pi\bar{\sigma}^2}}(2b-y)dy \\
&\overset{\text{Lemma}}{\underset{3.3}{=}} \sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}-e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}})+(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad -\sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-e^{-\frac{(2b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}})-(x-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad +2b\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2) \\
&= \sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}-2e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}+e^{-\frac{(2b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}) \\
&\quad +(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)-(x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\phi^{(2)}(x) &= \phi_{dG}(*\phi_Q)^2(x) \\
&= (\frac{p}{a})^2\Big(\sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(2a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-2e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}+e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right)+ \\
&\quad (x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2)-(x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\Big) \\
&\quad -\frac{2pq}{ab}\Big(\sqrt{\frac{\bar{\sigma}^2}{2\pi}}\left(e^{-\frac{(a-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}+e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-e^{-\frac{(a+b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}-e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}\right) \\
&\quad +(x-a-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)-(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) \\
&\quad +b\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)\Big) \\
&\quad +(\frac{q}{b})^2\Big(\sqrt{\frac{\bar{\sigma}^2}{2\pi}}(e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}}-2e^{-\frac{(b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}+e^{-\frac{(2b-(x-\bar{\mu}))^2}{2\bar{\sigma}^2}}) \\
&\quad +(x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)-(x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg((\frac{p}{a}+\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} + (\frac{p}{a})^2 e^{-\frac{(x-\bar{\mu}-2a)^2}{2\bar{\sigma}^2}} + (\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu}-2b)^2}{2\bar{\sigma}^2}} \\
& -2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)e^{-\frac{(x-\bar{\mu}-a)^2}{2\bar{\sigma}^2}} - 2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)e^{-\frac{(x-\bar{\mu}-b)^2}{2\bar{\sigma}^2}} + \frac{2pq}{ab}e^{-\frac{(x-\bar{\mu}-a-b)^2}{2\bar{\sigma}^2}}\Bigg) \\
& +(\frac{p}{a})^2\left((x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2)-(x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) \\
& +\frac{2pq}{ab}\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)-(x-a-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
& +(\frac{q}{b})^2\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)-(x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
& -\frac{2pq}{a}\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

□

4 The Bin Probability Distribution for $\Delta\ln(S(t))$

First of all, let us derive the following lemmas.

Lemma 4.1

$$
\begin{aligned}
IB_2(x_1,x_2,A,B) \quad \equiv \quad & \int_{x_1}^{x_2}\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx \\
= \quad & x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\
& +\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+B))^2}{2\sigma^2}}+e^{-\frac{(x_2-(\mu+A))^2}{2\sigma^2}}-e^{-\frac{(x_1-(\mu+A))^2}{2\sigma^2}}-e^{-\frac{(x_2-(\mu+B))^2}{2\sigma^2}}) \\
& +(\mu+B)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2)-(\mu+A)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
IB_2(x_1,x_2,A,B) \quad \stackrel{By}{\underset{IBP}{=}} \quad & x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2}-\int_{x_1}^{x_2}x\frac{d\Phi^{(n)}(A,B;x-\mu,\sigma^2)}{dx}dx \\
= \quad & x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2}-\int_{x_1}^{x_2}x(-\frac{e^{-\frac{(x-\mu-y)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}|_A^B)dx \\
= \quad & x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2}+\int_{x_1}^{x_2}x(\frac{e^{-\frac{(x-\mu-y)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}|_A^B)dx \\
= \quad & x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2}+\int_{x_1}^{x_2}x(\frac{e^{-\frac{(x-\mu-B)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}-\frac{e^{-\frac{(x-\mu-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}})dx
\end{aligned}
$$

$$
\begin{aligned}
&\overset{Lemma}{\underset{3.3}{=}} \quad x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\
&\quad +\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+B))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+B))^2}{2\sigma^2}}) + (\mu+B)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+A))^2}{2\sigma^2}}) - (\mu+A)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) \\
&= \quad x\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\
&\quad +\frac{\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\mu+B))^2}{2\sigma^2}} + e^{-\frac{(x_2-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_1-(\mu+A))^2}{2\sigma^2}} - e^{-\frac{(x_2-(\mu+B))^2}{2\sigma^2}}) \\
&\quad +(\mu+B)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) - (\mu+A)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2).
\end{aligned}
$$

□

Lemma 4.2

$$
\begin{aligned}
IB_3(x_1,x_2,A,\sigma) &\equiv \int_{x_1}^{x_2} x^2 \frac{e^{-\frac{(x-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \\
&= \frac{-\sigma}{\sqrt{2\pi}}(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} + (\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;A,\sigma^2).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
IB_3(x_1,x_2,A,\sigma) &\underset{x=z+A}{\overset{Set}{=}} \int_{x_1-A}^{x_2-A} (z+A)^2 \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \int_{x_1-A}^{x_2-A} (z^2+2Az+A^2) \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \int_{x_1-A}^{x_2-A} z^2 \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz + 2A\int_{x_1-A}^{x_2-A} z\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz + A^2\int_{x_1-A}^{x_2-A} \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}\int_{x_1-A}^{x_2-A} z\,de^{-\frac{z^2}{2\sigma^2}} + \frac{-2A\sigma}{\sqrt{2\pi}}\int_{x_1-A}^{x_2-A} de^{-\frac{z^2}{2\sigma^2}} + A^2\int_{x_1-A}^{x_2-A} \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} - \sqrt{2\pi\sigma^2}\int_{x_1-A}^{x_2-A} \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz\right) \\
&\quad +\frac{-2A\sigma}{\sqrt{2\pi}}e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} + A^2\int_{x_1-A}^{x_2-A} \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz \\
&= \frac{-\sigma}{\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} + 2Ae^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A}\right) + (\sigma^2+A^2)\int_{x_1-A}^{x_2-A} \frac{e^{-\frac{z^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dz
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\sigma}{\sqrt{2\pi}}(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} + (\sigma^2+A^2)\Phi^{(n)}(x_1-A,x_2-A;0,\sigma^2) \\
&\overset{Lemma}{\underset{3.1}{=}} \frac{-\sigma}{\sqrt{2\pi}}(z+2A)e^{-\frac{z^2}{2\sigma^2}}|_{x_1-A}^{x_2-A} + (\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;A,\sigma^2).
\end{aligned}
$$

□

Lemma 4.3

$$
\begin{aligned}
IB_4(x_1,x_2,A,B) &\equiv \int_{x_1}^{x_2} x\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx \\
&= \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad +0.5\Big((\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -(\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + z^2\Phi^{(n)}(A,B;z-\mu,\sigma^2)|_{x_1}^{x_2}\Big).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
IB_4(x_1,x_2,A,B) &= \frac{1}{2}\int_{x_1}^{x_2}\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx^2 \\
&\overset{By}{\underset{IBP}{=}} \frac{1}{2}\left(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} - \int_{x_1}^{x_2} x^2\frac{d\Phi^{(n)}(A,B;x-\mu,\sigma^2)}{dx}dx\right) \\
&= \frac{1}{2}\left(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} + \int_{x_1}^{x_2} x^2(\frac{e^{-\frac{(x-\mu-y)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}|_A^B)dx\right) \\
&= \frac{1}{2}\left(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} + \int_{x_1}^{x_2} x^2(\frac{e^{-\frac{(x-\mu-B)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} - \frac{e^{-\frac{(x-\mu-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}})dx\right) \\
&= \frac{1}{2}\Big(x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2} \\
&\quad + \frac{-\sigma}{\sqrt{2\pi}}(z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)} + (\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad - \frac{-\sigma}{\sqrt{2\pi}}(z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)} - (\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2)\Big) \\
&= \frac{-\sigma}{2\sqrt{2\pi}}\left((z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\right) \\
&\quad +0.5\Big((\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2) \\
&\quad -(\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + x^2\Phi^{(n)}(A,B;x-\mu,\sigma^2)|_{x_1}^{x_2}\Big).
\end{aligned}
$$

$$\stackrel{x\to z}{=} \frac{-\sigma}{2\sqrt{2\pi}}\Bigg((z+2(\mu+B))e^{-\frac{z^2}{2\sigma^2}}\Big|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2(\mu+A))e^{-\frac{z^2}{2\sigma^2}}\Big|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\Bigg)$$
$$+0.5\Bigg((\sigma^2+(\mu+B)^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2)$$
$$-(\sigma^2+(\mu+A)^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + z^2\Phi^{(n)}(A,B;z-\mu,\sigma^2)\Big|_{x_1}^{x_2}\Bigg)\,.$$

□

Based on Lemma 4.3, we get the following corollary.

Corollary 4.1

$$IB_5(x_1,x_2,A,B) \equiv \int_{x_1}^{x_2}(x-\mu)\Phi^{(n)}(A,B;x-\mu,\sigma^2)dx$$
$$= \frac{-\sigma}{2\sqrt{2\pi}}\Bigg((z+2B)e^{-\frac{z^2}{2\sigma^2}}\Big|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2A)e^{-\frac{z^2}{2\sigma^2}}\Big|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\Bigg)$$
$$+0.5\Bigg((\sigma^2+B^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2)$$
$$-(\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + z^2\Phi^{(n)}(A,B;z,\sigma^2)\Big|_{x_1-\mu}^{x_2-\mu}\Bigg)\,.$$

Proof:

$$IB_5(x_1,x_2,A,B) \underset{x=z+\mu}{\stackrel{Set}{=}} \int_{x_1-\mu}^{x_2-\mu} z\Phi^{(n)}(A,B;z,\sigma^2)dz$$
$$\underset{4.3}{\stackrel{Lemma}{=}} \frac{-\sigma}{2\sqrt{2\pi}}\Bigg((z+2B)e^{-\frac{z^2}{2\sigma^2}}\Big|_{x_1-(\mu+B)}^{x_2-(\mu+B)} - (z+2A)e^{-\frac{z^2}{2\sigma^2}}\Big|_{x_1-(\mu+A)}^{x_2-(\mu+A)}\Bigg)$$
$$+0.5\Bigg((\sigma^2+B^2)\Phi^{(n)}(x_1,x_2;\mu+B,\sigma^2)$$
$$-(\sigma^2+A^2)\Phi^{(n)}(x_1,x_2;\mu+A,\sigma^2) + z^2\Phi^{(n)}(A,B;z,\sigma^2)\Big|_{x_1-\mu}^{x_2-\mu}\Bigg)\,.$$

□

Now, we come to calculate the bin probability distribution formula for $\Delta\ln(S(t))$.

Theorem 4.1 *The second order approximation to* $[x_1,x_2]$ *bin probability distribution for the linear jump-diffusion log-return increment* $\Delta\ln(S(t))$ *with log-uniform jump-amplitude is given by*

$$\Phi(x_1,x_2) \approx \frac{\sum_{k=0}^{2} p_k(\lambda\Delta t)\Phi^{(k)}(x_1,x_2)}{\sum_{k=0}^{2} p_k(\lambda\Delta t)}, \tag{7}$$

where,

$$\Phi^{(0)}(x_1,x_2) \equiv \Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2),$$

$$\begin{aligned}
\Phi^{(1)}(x_1,x_2) \;=\;& \frac{q}{b}x\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} - \frac{p}{a}x\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&+\frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg(\frac{q}{b}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}) + (\frac{q}{b}+\frac{p}{a})(e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}}) \\
&+\frac{p}{a}(e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}})\Bigg) \\
&+\frac{p}{a}(\bar{\mu}+a)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + \frac{q}{b}(\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&-(\frac{q}{b}+\frac{p}{a})\bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2),
\end{aligned}$$

$$\begin{aligned}
\Phi^{(2)}(x_1,x_2) \;=\;& 0.5\bar{\sigma}^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + (\frac{p}{a})^2\Phi(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2) \\
&+(\frac{q}{b})^2\Phi(x_1,x_2;\bar{\mu}+2b,\bar{\sigma}^2) - 2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) \\
&-2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) + \frac{2pq}{ab}\Phi(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg) \\
&+0.5\frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg((\frac{p}{a})^2\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
&+(\frac{2pq}{ab})\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+a+b)}^{x_2-(\bar{\mu}+a+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-a-\bar{\mu}}^{x_2-a-\bar{\mu}} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-b-\bar{\mu}}^{x_2-b-\bar{\mu}}\right) \\
&+(\frac{q}{b})^2\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+2b)}^{x_2-(\bar{\mu}+2b)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right)\Bigg) \\
&+0.5\Bigg((\frac{p}{a})^2(z^2\Phi^{(n)}(0,-a;z,\bar{\sigma}^2)|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - z^2\Phi^{(n)}(a,0;z,\bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}) \\
&+\frac{2pq}{ab}\left(z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)|_{x_1-a-\bar{\mu}}^{x_2-a-\bar{\mu}}\right) \\
&+(\frac{q}{b})^2(z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - z^2\Phi^{(n)}(-b,0;z,\bar{\sigma}^2)|_{x_1-2b-\bar{\mu}}^{x_2-2b-\bar{\mu}})\Bigg) \\
&+p^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + q^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&+pq\frac{b}{a}\left(\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) - \Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\right) \\
&-\frac{2pq}{a}\Bigg(z\Phi^{(n)}(a+b,b;z-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} + (\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&-(\bar{\mu}+a+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg),
\end{aligned}$$

and an appropriate 2nd order renormalization is used to preserve the distribution property.

Proof:

$$
\begin{aligned}
\Phi^{(1)}(x_1,x_2) &\equiv \int_{x_1}^{x_2} \phi^{(1)}(x)dx \\
&\overset{Theorem}{\underset{3.1}{\equiv}} \int_{x_1}^{x_2} (\frac{q}{b}\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - \frac{p}{a}\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2))dx \\
&= \frac{q}{b}\int_{x_1}^{x_2} \Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)dx - \frac{p}{a}\int_{x_1}^{x_2} \Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)dx \\
&\overset{Lemma}{\underset{4.1}{\equiv}} \frac{q}{b}\Bigg(x\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&\quad + \frac{\bar{\sigma}}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}} + e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}) \\
&\quad + (\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) - \bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2)\Bigg) \\
&\quad - \frac{p}{a}\Bigg(x\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&\quad + \frac{\bar{\sigma}}{\sqrt{2\pi}}(e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}} + e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}}) \\
&\quad + \bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) - (\bar{\mu}+a)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2)\Bigg) \\
&= \frac{q}{b}x\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} - \frac{p}{a}x\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)|_{x_1}^{x_2} \\
&\quad + \frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg(\frac{q}{b}(e^{-\frac{(x_1-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+b))^2}{2\bar{\sigma}^2}}) + (\frac{q}{b}+\frac{p}{a})(e^{-\frac{(x_2-\bar{\mu})^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_1-\bar{\mu})^2}{2\bar{\sigma}^2}}) \\
&\quad + \frac{p}{a}(e^{-\frac{(x_1-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}} - e^{-\frac{(x_2-(\bar{\mu}+a))^2}{2\bar{\sigma}^2}})\Bigg) \\
&\quad + \frac{p}{a}(\bar{\mu}+a)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + \frac{q}{b}(\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&\quad - (\frac{q}{b}+\frac{p}{a})\bar{\mu}\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2).
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{(2)}(x_1,x_2) \quad &\equiv \quad \int_{x_1}^{x_2} \phi^{(2)}(x)dx \\
&\overset{Theorem}{\underset{3.1}{=}} \int_{x_1}^{x_2} \Bigg(\frac{\bar{\sigma}}{\sqrt{2\pi}} \Bigg((\frac{p}{a}+\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} + (\frac{p}{a})^2 e^{-\frac{(x-\bar{\mu}-2a)^2}{2\bar{\sigma}^2}} + (\frac{q}{b})^2 e^{-\frac{(x-\bar{\mu}-2b)^2}{2\bar{\sigma}^2}} \\
&\quad -2\left((\frac{p}{a})^2+\frac{pq}{ab}\right) e^{-\frac{(x-\bar{\mu}-a)^2}{2\bar{\sigma}^2}} - 2\left((\frac{q}{b})^2+\frac{pq}{ab})\right) e^{-\frac{(x-\bar{\mu}-b)^2}{2\bar{\sigma}^2}} + \frac{2pq}{ab} e^{-\frac{(x-\bar{\mu}-a-b)^2}{2\bar{\sigma}^2}} \Bigg) \\
&\quad +(\frac{p}{a})^2 \left((x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +\frac{2pq}{ab} \left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-a-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad +(\frac{q}{b})^2 \left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right) \\
&\quad -\frac{2pq}{a}\Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)\Bigg) dx \\
&= \quad \bar{\sigma}^2 \Bigg((\frac{p}{a}+\frac{q}{b})^2 \Phi(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + (\frac{p}{a})^2 \Phi(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2) \\
&\quad +(\frac{q}{b})^2 \Phi(x_1,x_2;\bar{\mu}+2b,\bar{\sigma}^2) - 2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) \\
&\quad -2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) + \frac{2pq}{ab}\Phi(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg) \\
&\quad +(\frac{p}{a})^2 \int_{x_1}^{x_2} \left((x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) dx \\
&\quad +\frac{2pq}{ab} \int_{x_1}^{x_2} \left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-a-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right) dx \\
&\quad +(\frac{q}{b})^2 \int_{x_1}^{x_2} \left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-2b-\bar{\mu})\Phi^{(n)}(b,2b;x-\bar{\mu},\bar{\sigma}^2)\right) dx \\
&\quad -\frac{2pq}{a} \int_{x_1}^{x_2} \Phi^{(n)}(a+b,b;x-\bar{\mu},\bar{\sigma}^2)dx
\end{aligned}
$$

But,

$$
\begin{aligned}
IA \quad &\equiv \quad \int_{x_1}^{x_2} \left((x-2a-\bar{\mu})\Phi^{(n)}(2a,a;x-\bar{\mu},\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) dx \\
&\overset{Lemma}{\underset{3.1}{=}} \int_{x_1}^{x_2} \left((x-2a-\bar{\mu})\Phi^{(n)}(0,-a;x-\bar{\mu}-2a,\bar{\sigma}^2) - (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)\right) dx \\
&= \int_{x_1}^{x_2} (x-2a-\bar{\mu})\Phi^{(n)}(0,-a;x-\bar{\mu}-2a,\bar{\sigma}^2)dx - \int_{x_1}^{x_2} (x-\bar{\mu})\Phi^{(n)}(a,0;x-\bar{\mu},\bar{\sigma}^2)dx
\end{aligned}
$$

$$
\begin{aligned}
\overset{Corollary}{\underset{4.1}{=}} \quad & \frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left((z-2a)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)}\right) \\
& +0.5\Bigg((\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) \\
& -\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2) + z^2\Phi^{(n)}(0,-a;z,\bar{\sigma}^2)\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)}\Bigg) \\
& -\frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - (z+2a)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)}\right) \\
& -0.5\Bigg(\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) \\
& -(\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + z^2\Phi^{(n)}(a,0;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\Bigg) \\
= \quad & \frac{\bar{\sigma}}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)}\right) \\
& +0.5\left(z^2\Phi^{(n)}(0,-a;z,\bar{\sigma}^2)\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - z^2\Phi^{(n)}(a,0;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
& -0.5\bar{\sigma}^2\left(\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + \Phi^{(n)}(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2)\right) \\
& +(\bar{\sigma}^2+a^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2).
\end{aligned}
$$

$$
\begin{aligned}
IB \quad \equiv \quad & \int_{x_1}^{x_2}\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-a-\bar{\mu})\Phi^{(n)}(a,a+b;x-\bar{\mu},\bar{\sigma}^2)\right)dx \\
\overset{Lemma}{\underset{3.1}{=}} \quad & \int_{x_1}^{x_2}\left((x-\bar{\mu})\Phi^{(n)}(0,b;x-\bar{\mu},\bar{\sigma}^2) - (x-a-\bar{\mu})\Phi^{(n)}(0,b;x-a-\bar{\mu},\bar{\sigma}^2)\right)dx \\
\overset{Corollary}{\underset{4.1}{=}} \quad & \frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left((z+2b)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
& +0.5\Bigg((\bar{\sigma}^2+b^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
& -\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\Bigg) \\
& -\frac{-\bar{\sigma}}{2\sqrt{2\pi}}\left((z+2b)e^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a+b)}^{x_2-(\bar{\mu}+a+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)}\right) \\
& -0.5\Bigg((\bar{\sigma}^2+b^2)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2) \\
& -\bar{\sigma}^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)\Big|_{x_1-a-\bar{\mu}}^{x_2-\bar{\mu}-a}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\bar\sigma}{2\sqrt{2\pi}}\Bigg((z+2b)e^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+a+b)}^{x_2-(\bar\mu+a+b)} - (z+2b)e^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+b)}^{x_2-(\bar\mu+b)} \\
& +ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu} - ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+a)}^{x_2-(\bar\mu+a)}\Bigg) \\
& +0.5(\bar\sigma^2+b^2)\left(\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2) - \Phi^{(n)}(x_1,x_2;\bar\mu+a+b,\bar\sigma^2)\right) \\
& +0.5\bar\sigma^2\left(\Phi^{(n)}(x_1,x_2;\bar\mu+a,\bar\sigma^2) - \Phi^{(n)}(x_1,x_2;\bar\mu,\bar\sigma^2)\right) \\
& +0.5\left(z^2\Phi^{(n)}(0,b;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu} - z^2\Phi^{(n)}(0,b;z,\bar\sigma^2)|_{x_1-a-\bar\mu}^{x_2-a-\bar\mu}\right).
\end{aligned}
$$

$$
\begin{aligned}
IC \quad \equiv \quad & \int_{x_1}^{x_2}\left((x-\bar\mu)\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2) - (x-2b-\bar\mu)\Phi^{(n)}(b,2b;x-\bar\mu,\bar\sigma^2)\right)dx \\
\overset{Lemma}{\underset{3.1}{=}} \quad & \int_{x_1}^{x_2}\left((x-\bar\mu)\Phi^{(n)}(0,b;x-\bar\mu,\bar\sigma^2) - (x-2b-\bar\mu)\Phi^{(n)}(-b,0;x-2b-\bar\mu,\bar\sigma^2)\right)dx \\
\overset{Corollary}{\underset{4.1}{=}} \quad & \frac{-\bar\sigma}{2\sqrt{2\pi}}\left((z+2b)e^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+b)}^{x_2-(\bar\mu+b)} - ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}\right) \\
& +0.5\Bigg((\bar\sigma^2+b^2)\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2) \\
& -\bar\sigma^2\Phi^{(n)}(x_1,x_2;\bar\mu,\bar\sigma^2) + z^2\Phi^{(n)}(0,b;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu}\Bigg) \\
& -\frac{-\bar\sigma}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+2b)}^{x_2-(\bar\mu+2b)} - (z-2b)e^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+b)}^{x_2-(\bar\mu+b)}\right) \\
& -0.5\Bigg(\bar\sigma^2\Phi^{(n)}(x_1,x_2;\bar\mu+2b,\bar\sigma^2) \\
& -(\bar\sigma^2+b^2)\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2) + z^2\Phi^{(n)}(-b,0;z,\bar\sigma^2)|_{x_1-2b-\bar\mu}^{x_2-2b-\bar\mu}\Bigg). \\
= \quad & \frac{\bar\sigma}{2\sqrt{2\pi}}\left(ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+2b)}^{x_2-(\bar\mu+2b)} - 2ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-(\bar\mu+b)}^{x_2-(\bar\mu+b)} + ze^{-\frac{z^2}{2\bar\sigma^2}}|_{x_1-\bar\mu}^{x_2-\bar\mu}\right) \\
& +0.5\left(z^2\Phi^{(n)}(0,b;z,\bar\sigma^2)|_{x_1-\bar\mu}^{x_2-\bar\mu} - z^2\Phi^{(n)}(-b,0;z,\bar\sigma^2)|_{x_1-2b-\bar\mu}^{x_2-2b-\bar\mu}\right) \\
& -0.5\bar\sigma^2\left(\Phi^{(n)}(x_1,x_2;\bar\mu,\bar\sigma^2) + \Phi^{(n)}(x_1,x_2;\bar\mu+2b,\bar\sigma^2)\right) \\
& +(\bar\sigma^2+b^2)\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2).
\end{aligned}
$$

$$
\begin{aligned}
ID \quad \equiv \quad & \int_{x_1}^{x_2}\Phi^{(n)}(a+b,b;x-\bar\mu,\bar\sigma^2)dx \\
\overset{Lemma}{\underset{4.1}{=}} \quad & z\Phi^{(n)}(a+b,b;z-\bar\mu,\bar\sigma^2)|_{x_1}^{x_2} \\
& +\frac{\bar\sigma}{\sqrt{2\pi}}(e^{-\frac{(x_1-(\bar\mu+b))^2}{2\bar\sigma^2}} + e^{-\frac{(x_2-(\bar\mu+a+b))^2}{2\bar\sigma^2}} - e^{-\frac{(x_1-(\bar\mu+a+b))^2}{2\bar\sigma^2}} - e^{-\frac{(x_2-(\bar\mu+b))^2}{2\bar\sigma^2}}) \\
& +(\bar\mu+b)\Phi^{(n)}(x_1,x_2;\bar\mu+b,\bar\sigma^2) - (\bar\mu+a+b)\Phi^{(n)}(x_1,x_2;\bar\mu+a+b,\bar\sigma^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Phi^{(2)}(x_1,x_2) &= \bar{\sigma}^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + (\frac{p}{a})^2\Phi(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2) \\
&\quad +(\frac{q}{b})^2\Phi(x_1,x_2;\bar{\mu}+2b,\bar{\sigma}^2) - 2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) \\
&\quad -2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) + \frac{2pq}{ab}\Phi(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg) \\
&\quad +(\frac{p}{a})^2 IA + \frac{2pq}{ab}IB + (\frac{q}{b})^2 IC - \frac{2pq}{a}ID \\
&= 0.5\bar{\sigma}^2\Bigg((\frac{p}{a}+\frac{q}{b})^2\Phi(x_1,x_2;\bar{\mu},\bar{\sigma}^2) + (\frac{p}{a})^2\Phi(x_1,x_2;\bar{\mu}+2a,\bar{\sigma}^2) \\
&\quad +(\frac{q}{b})^2\Phi(x_1,x_2;\bar{\mu}+2b,\bar{\sigma}^2) - 2\left((\frac{p}{a})^2+\frac{pq}{ab}\right)\Phi(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) \\
&\quad -2\left((\frac{q}{b})^2+\frac{pq}{ab})\right)\Phi(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) + \frac{2pq}{ab}\Phi(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg) \\
&\quad +0.5\frac{\bar{\sigma}}{\sqrt{2\pi}}\Bigg((\frac{p}{a})^2\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a)}^{x_2-(\bar{\mu}+a)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right) \\
&\quad +(\frac{2pq}{ab})\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+a+b)}^{x_2-(\bar{\mu}+a+b)} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-a-\bar{\mu}}^{x_2-a-\bar{\mu}} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-b-\bar{\mu}}^{x_2-b-\bar{\mu}}\right) \\
&\quad +(\frac{q}{b})^2\left(ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+2b)}^{x_2-(\bar{\mu}+2b)} - 2ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-(\bar{\mu}+b)}^{x_2-(\bar{\mu}+b)} + ze^{-\frac{z^2}{2\bar{\sigma}^2}}\Big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}\right)\Bigg) \\
&\quad +0.5\Bigg((\frac{p}{a})^2(z^2\Phi^{(n)}(0,-a;z,\bar{\sigma}^2)\big|_{x_1-(\bar{\mu}+2a)}^{x_2-(\bar{\mu}+2a)} - z^2\Phi^{(n)}(a,0;z,\bar{\sigma}^2)\big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}}) \\
&\quad +\frac{2pq}{ab}\left(z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)\big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)\big|_{x_1-a-\bar{\mu}}^{x_2-a-\bar{\mu}}\right) \\
&\quad +(\frac{q}{b})^2(z^2\Phi^{(n)}(0,b;z,\bar{\sigma}^2)\big|_{x_1-\bar{\mu}}^{x_2-\bar{\mu}} - z^2\Phi^{(n)}(-b,0;z,\bar{\sigma}^2)\big|_{x_1-2b-\bar{\mu}}^{x_2-2b-\bar{\mu}})\Bigg) \\
&\quad +p^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+a,\bar{\sigma}^2) + q^2\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&\quad +pq\frac{b}{a}\left(\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) - \Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\right) \\
&\quad -\frac{2pq}{a}\Bigg(z\Phi^{(n)}(a+b,b;z-\bar{\mu},\bar{\sigma}^2)\big|_{x_1}^{x_2} + (\bar{\mu}+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+b,\bar{\sigma}^2) \\
&\quad -(\bar{\mu}+a+b)\Phi^{(n)}(x_1,x_2;\bar{\mu}+a+b,\bar{\sigma}^2)\Bigg).
\end{aligned}
$$

□

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