Math 215, Mid-term 1, February 18 2009

Question 1
Prove that if A, B, C are arbitrary sets,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Question 2
Prove that if n is an integer greater than 4, $2^n > n^2 + 1$.

Question 3
Recall that the set of rational numbers $\mathbb{Q}$ is the subset of the set of real numbers $\mathbb{R}$ which are fractions $\frac{a}{b}$ with a, b integers. It is a fact that there are many real numbers which are not rational – *i.e.*, the complement $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$; a number which is not rational is called an *irrational* number.
Assume that the sum of two rational numbers is again a rational number.

(1) Prove that if x is rational, and $x + y$ is irrational, then y is irrational.
(2) Can we conclude that y is irrational if x and $x + y$ are *both* irrational? Justify your answer.

Question 4
Prove carefully, using the axioms for an ordered field, that:

$$1 + x^2 < (1 + x)^2 \Leftrightarrow x > 0.$$

Question 5
Is it possible to find a statement that is logically equivalent to (*i.e.*, has the same truth table as) $(A \Rightarrow B)$, but only involves the operations "or" and "&"? (and does **not** use $\sim$, *i.e.*, negation). Why or why not?

Axioms for an ordered field

(1) (Commutativity) For all a and b, $a + b = b + a$ and $ab = ba$;
(2) (Associativity) For all a, b and c, $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$;
(3) (Distributivity) For all a, b and c, $a(b + c) = ab + ac$;
(4) (Zero) There is an element 0 such that for all a, $a + 0 = a = 0 + a$;
(5) (Unity) There is an element 1 such that for all a, $a1 = a$; furthermore, we assume that $1 \neq 0$
(6) (Subtraction) For all a, the equation $a + x = 0$ has a unique solution $x = -a$. Similarly, the equation $a + x = b$ has a unique solution $b - a$.
(7) (Division) If $a \neq 0$, The equation $ax = 1$ has a unique solution $x = a^{-1}$. Similarly the equation $ax = b$ has a unique solution $x = b/a = ba^{-1}$.
(8) (Trichotomy) Either $a = b$, $a < b$ or $b < a$, and only one these holds.
(9) (Multiplication Law) If $c > 0$, then $ac < bc$ if and only if $a < b$, if $c < 0$, then $ac < bc$ if and only if $b < a$;
(10) (Addition Law) $a < b$ if and only if $a + c < b + c$;
(11) (Transitivity) If $a < b$ and $b < c$, then $a < c$.