Math 215 Homework Solutions.

(1) Page 54, number 10

What is wrong with the following proof that 1 is the largest integer?

Let n be the largest integer. Then, since 1 is an integer we must have $1 \leq n$. On the other hand, since n^2 is also an integer we must have $n^2 \leq n$ from which it follows that $n \leq 1$. Thus, since $1 \leq n$ and $n \geq 1$ we must have $n = 1$. Thus 1 is the largest integer as claimed. What does this argument prove?

Solution. If you look at this proof sentence by sentence, you see that every sentence makes sense, *except* the very first. "Let n be the largest integer." *assumes* that such an integer exists! Hence what this argument actually proves that *If* a largest integer existed, it would be 1, which is absurd, since $2 = 1 + 1 > 1$. Hence the assumption leads to a contradiction, and so *there is no largest integer*.

(2) Recall that we defined in class operations of multiplication and addition on the set $\{0, 1\}$:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

(a) Verify the distributive law. (Since a, b, c can each take 2 possible values, there are 8 cases in all). (You should also check that the other field axioms hold – but do not hand that in).

(b) Is there an order on this field satisfying axioms 8-11? Justify your answer.

Solution:

(i) There are eight possible values for a, b, c. We then fill in the following table using the addition and multiplication tables give to us.

a	b	c	ab	ac	$b+c$	$a(b+c)$	$ab+ac$
1	1	1	1	1	0	0	0
1	1	0	1	0	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0

The last two columns are the same; *i.e.*, no matter what the values of a, b, c, $a(b + c) = ab + ac$ and thus the distributive law holds.

(ii) Suppose that we could order the set $(0, 1)$ so that the rules for an ordered field hold. Then we would have, by trichotomy, either $0 < 1$ or $1 < 0$. Thus we have two cases:

(a) Suppose $0 < 1$

then: $0 + 1 < 1 + 1$ by the addition law

hence $1 < 0$ by the addition table.
By trichotomy this contradicts the assumption that $0 < 1$, and hence $0 < 1$ is impossible.

(b) Suppose $1 < 0$
then: $1 + 1 < 0 + 1$ by the addition law
hence $0 < 1$ by the addition table.
By trichotomy this contradicts the assumption that $1 < 0$, and hence $1 < 0$ is impossible.

Therefore neither $0 < 1$ nor $1 < 0$ is possible, and so it is not possible to order the set $\{0, 1\}$ in a way that is compatible with the axioms for an ordered field.

(3) We define $|a| = a$ if $a > 0$ and $-a$ if $a < 0$. Prove that $|a + b| \leq |a| + |b|$ for all a and b. [Hint: You might want to consider 4 cases depending on whether each of a and b is negative or nonnegative. Also we write $a \leq b$ for "$a < b$ or $a = b$".]

Note that here the intention was that you use the axioms for ordered fields.
Solution

There are four cases (and we will see later, that there are *sub-cases* as well).

(a) $a \geq 0$, and $b \geq 0$. Then by additivity, $a + b \geq 0$, and so:

$$\begin{aligned} |a+b| &= a+b \text{ since } a+b \geq 0 \\ &= |a| + |b| \text{ since } a \geq 0, b \geq 0 \end{aligned}$$

Hence $|a + b| = |a| + |b|$.

(b) $a < 0$, and $b < 0$. Then by additivity, $a + b < 0$, and so:

$$\begin{aligned} |a+b| &= -(a+b) \text{ since } a+b \geq 0 \\ &= -a-b (\text{see below}) \\ &= |a| + |b| \text{ since } a < 0, b < 0 \end{aligned}$$

(c) $a < 0$, and $b \geq 0$. Then $b = |b|$, while $|a| = -a > 0$, by definition of the absolute value. Hence:

$$\begin{aligned} a &< |a| \text{ by transitivity} \\ a+b &< |a| + b \text{ by additivity} \\ a+b &< |a| + |b| \text{ since } b \geq 0 \end{aligned}$$

If $a + b \geq 0$, then $|a + b| = a + b < |a| + |b|$ by transitivity.
Since $b \geq 0$, adding $-b$ to this we get $0 \geq -b$ by additivity.
Therefore, if $a + b < 0$, then $|a + b| = -(a + b) = -a - b = |a| - b \leq |a| + b = |a| + |b|$.

(d) $a \geq 0$, and $b < 0$. This follows from the previous case by interchanging a and b

(4) Page 54, number 11
Suppose that there *were* a smallest positive real number; write x for this number.
Since $0 < \frac{1}{2} < 1$, we have $0 < \frac{x}{2} < x$.
Hence we have found a number $(x/2)$ which is both positive and smaller than x, contradicting our assumption, and so there is no positive real number which is smaller than all other positive real numbers.

(5) Page 54, number 14
We proceed by induction on $n \geq 1$, to prove that for all integers $n \geq 1$, for all $x \geq -1$, $(1+x)^n \geq 1 + nx$.
"Base Case" $n = 1$: $(1+x)^1 = 1 + x = 1 + 1 \cdot x$, and so the inequality is true for $n = 1$.
"Inductive step"
Suppose that $n \geq 1$, and that $(1+x)^n \geq 1 + nx$ (this is the *induction hypothesis*).
We want to deduce, given the induction hypothesis that $(1+x)^{n+1} \geq 1 + (n+1)x$.
(**Notice**: we assume the inequality or formula is true for n, and then try to deduce the statement obtained by replacing all the n's by $(n+1)$'s.)
Idea Before writing down the proof of the inductive step, we start by noticing that $(1+x)^{n+1} = (1+x)^n(1+x)$, which suggests that we take the inequality given to us by the induction hypothesis, and multiply it by $(1+x)$:

$$\begin{aligned}
(1+x)^n &\geq 1 + nx \text{ by the induction hypothesis} \\
\therefore (1+x)^n(1+x) &\geq (1+nx)(1+x) \text{ since } (1+x) \geq 0 \\
&= 1 + (n+1)x + nx^2 \\
&\geq 1 + (n+1)x \text{ since } nx^2 \geq 0
\end{aligned}$$

Therefore $(1+x)^1 \geq 1 + 1x$ and

$$\forall n \geq 1, ((1+x)^n \geq 1 + nx) \Rightarrow ((1+x)^{n+1} \geq 1 + (n+1)x,$$

and so by induction on $n \geq 1$, we have that $\forall n \geq 1, ((1+x)^n \geq 1 + nx$.

(6) Page 55, number 17
a_n is defined inductively for $n \geq 1$ by:
$a_1 = 1$ and for $n \geq 1$, $a_{n+1} = \frac{6a_n+5}{a_n+2}$. Prove that for all $n|geq1$, $5 > a_n > 0$.
Proof We proceed by induction on $n \geq 1$.
Base Case $5 > 1 > 0$ so the assertion is true for $n = 0$
Inductive Step Suppose that $n \geq 1$ and that $5 > a_n > 0$.
We want to show, using this assumption that:

$$5 > a_{n+1} = \frac{6a_n + 5}{a_n + 2} > 0.$$

First note that since $a_n > 0$, we have $6a_n + 5 > 0$ and $a_n + 2 > 0$, and hence $\frac{6a_n+5}{a_n+2} > 0$.

Now,

$$\begin{aligned} a_n &< 5 \\ \therefore 6a_n &< 5a_n + 5 \text{ adding } 5a_n \\ \therefore 6a_n + 5 &< 5a_n + 10 \text{ adding } 5 \\ \therefore \frac{6a_n+5}{a_n+2} &< 5 \text{ dividing by } a_n + 2 > 0 \end{aligned}$$

(7) Qn 4 on page 115: Prove by contradiction or otherwise that $A \cap B = A \cap C$ and $A \cup B = A \cup C$ if and only if $B = C$.
Solution. You can prove an if and only if statement by proving each direction separately:
($\Rightarrow$) We argue by contradiction.
I.e., we suppose that the assertion $(A \cup B = A \cup C) \Rightarrow (B = C)$ is *false*.
Therefore then $A \cup B = A \cup C$ is *true* and $B = C$ is *false*. (Think about the truth table for $\Rightarrow$)
Since $B = C$ is *false*, there is an element $x \in B$ with $x \notin C$, or $x \in C$ with $x \notin B$.
Suppose first that $x \in B$ with $x \notin C$.
There are two cases:
First, if $x \in A$, we have $x \in A \cap B$, but $x \notin A \cap C$ (since $x \in A \cap C \Leftrightarrow (x \in A \& x \in C)$).
Hence $x \in A \cap B \neq A \cap C$ in this case. Second, if $x \notin A$ then $(x \in A \cup B) \Leftrightarrow (x \in B)$ (again think about the truth table for "or", since $x \in A \cup B$ if and only if $x \in A$ or $x \in B$)
Hence $x \notin A$ and $x \in B$ and $x \notin C$ implies that $A \cup B \neq A \cup C$.
Hence the assumption that $x \in B$ and $x \notin C$ leads to a contradiction. Switching B and C, we also see that $x \notin B$ and $x \in C$ leads to a contradiction. Hence we have proved that $(A \cup B = A \cup C) \Rightarrow (B = C)$.
($\Leftarrow$) Here you can simply say that since $B = C$, $A \cap B = A \cap C$ and $A \cup B = A \cup C$. (If two objects are equal they may be freely substituted for each other in any formula)

Remark. We could also have used the truth table for "&" and "or", since they reflect the properties of union and intersection of sets.

(8) Qn 8 on page 116
  (i) As discussed in class, you can do this by comparing the truth tables for the two statements $x \in (A \triangle B) \triangle C$ and $x \in A \triangle (B \triangle C)$, which has a row for each of the eight possible truth values for $x \in A$, $x \in B$ and $x \in C$.
  (ii) We guess that $N = \emptyset$ works. Since $A \cup \emptyset = A$, and $A \cap \emptyset = \emptyset$, we see that $\emptyset$ works.
  To check uniqueness. First notice that since union and intersection are commutative, the same is true for symmetric difference.
  Now suppose that N satisfies $\forall A, A \triangle N = A$. Then: $N = N \triangle \emptyset$, since we know $\emptyset$ has the given property.
  But $N \triangle \emptyset = \emptyset \triangle N$ by commutativity,
  which is the equal to $\emptyset$ by the hypothesis on N. Thus any set N which has the given property equals the empty set.

- (iii) The set A itself works. To check uniqueness, suppose $A \triangle B = \emptyset$. By commutativity, $B \triangle A = \emptyset$. Then $B = B \triangle \emptyset = B \triangle (A \triangle A) = (B \triangle A) \triangle A) = \emptyset \triangle A = A$. Hence $B = A$.
- (iv) By now you can tell that this works the same way usual algebra does. Just as $b-a$ is the unique solution to $a+x=b$, we see that $C = B \triangle A$ is the unique solution to $A \triangle C = B$. To check this is a solution, and that it is unique, use the first three properties.

(9) Qn 15 on page 117

First, since the characteristic function of a subset takes the values $0, 1$, $\chi_A(x)\chi_B(x) = 1$ if and only if $\chi_A(x) = 1$ and $\chi_B(x) = 1$ if and only if $x \in A$ and $x \in B$ if and only if $x \in A \cap B$.

Secondly, $\chi_A + \chi_B - \chi_A \chi_B = \chi_{A \cup B}$.