Math 215 HW Solutions

p117, ♯12. $A \subset \mathbb{Z}$. The statement $P(A) =$"There is a greatest number in the set A"
In symbols:

$$\exists n \in A, \forall m \in A, n \geq m$$

The negation of P:

$$\forall n \in A, \exists m \in A, n < m.$$

p271, ♯1.

$$7|6^n + 1 \Leftrightarrow n \text{ is odd.}$$

Proof $6 \equiv -1 \pmod{7}$
Therefore $6^n + 1 \equiv (-1)^n + 1 \pmod{7}$
Therefore when n is odd, $6^n + 1 \equiv 0$, *i.e.*, $7|6^n + 1$,
while if n is even $6^n + 1 \equiv 2$ and so $7 \nmid 6^n + 1$

p271, ♯5. Find a test for divisibility by 99.

One possibility is just to combine the answers to questions ♯3 and ♯4
I.e., A number is divisible by 99 if and only if it is divisible by both 9 and 11. Hence if and ony if the sum of its digits is divisible by 9 and the alternating sum is divisible by 11.
However I think what the question intended was a single test just using 99. Then observe that 100 is congruent to 1 mod 99. So group the digits of n in pairs: if $n = a_k a_{k-1} \ldots a_1 a_0$ is the decimal expansion of n, then add up the numbers $a_1 a_0 + a_3 a_2 + \ldots$ (working mod 99) and if you get 0 mod 99, then n is divisible by 99.
Example 8217 is divisble by 99 because $82 + 17 = 99 \equiv 0 \bmod 99$

p271, ♯6. The diophantine equation $3x^2 = 4y^2 = 5z^2$ has no non-trivial solutions. *I.e.* if (x, y, z) is a triple of integers satisfying the equation, then they must all equal zero.

We will study this equation mod 5. First notice that mod 5 the only squares are $0^2 = 0, 1^2 = 1, 2^2 = 4, 3^2 = 9 \equiv 4, 4^2 = 16 \equiv 1$ *i.e.*, $0, 1, 4$ Looking at the original equation we see that mod 5, it becomes:

$$3x^2 + 4y^2 \equiv 0$$

, However substituting $\{o, 1, 4\}$ for x^2 and y^2 we find that the only way to get

$$3x^2 + 4y^2 \equiv 0$$

is if $x^2 \equiv y^2 \equiv 0$.

Thus both x^2 and y^2 must be divisible by 5. Hence x and y are divisible by 5: $x = 5x'$, $y = 5y'$, so we get:

$$25 \cdot 3x'^2 + 25 \cdot 4y'^2 = 5z^2$$

Dividing by 5 we get

$$5 \cdot 3x'^2 + 5 \cdot 4y'^2 = z^2$$

and so z^2 is divisible by 5, hence so is z: $z = 5z'$, and we get

$$5 \cdot 3x'^2 + 5 \cdot 4y'^2 = 25z'^2$$

and so

$$3x'^2 + 4y'^2 = 5z'^2.$$

I.e., we have found a smaller solution of the equation. But this cannot go for ever and so there cannot be a non-trivial solution.