Math 215 MT 2 April 2009

Q1.

(1) Write down the multiplication table for $\mathbb{Z}/7\mathbb{Z}$
(2) Prove that $\forall n \in \mathbb{N},\ 7|(2^n - 1) \Leftrightarrow 3|n$

Q2. The *well ordering principle* states that every non-empty subset of the natural numbers contains a smallest element.

(1) Write the statement of the well ordering principle using symbols.
(2) Prove carefully that the well ordering principle *implies* the principal of mathematical induction. That is, suppose the $P(n)$ is a predicate about natural numbers n. Suppose that $P(1)$ is true, and suppose also that for all $n \in \mathbb{N}$, $P(n+1)$ is true if $P(n)$ is true. Using the well ordering principle prove that then $P(n)$ is true *for all* n. (Hint: consider the set of natural numbers n for which $P(n)$ is *false*.)

Q3. Let $G = (G.\cdot, e)$ be a group. Prove carefully that for all $a \in G$, for all $b \in G$, there exists a *unique* $x \in G$ such that

$$ax = b.$$

Does this x also necessarily satisfy $xa = b$? If yes, prove that it does. If not, give a example of a G for which it will be *not* be true that $ax = b \Rightarrow xa = b$.

Q4. Let p be a prime number. Prove that $\sqrt{p}$ is irrational.
Show all the details of your proof.

Things you may assume.

- A natural number p is prime if and only if $\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, p|ab \Rightarrow p|a$ or $p|b$.
- The axioms for a group: A group consists of $(G, \cdot, e)$ in which G is a set, $\cdot : G \times G \to G$ is a function, and $e \in G$ an element such that:
  (1) $\forall a \in G, \forall b \in G, \forall c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$. (Associativity)
  (2) $\forall a \in G, e \cdot a = a \cdot e = a$. (Identity)
  (3) $\forall a \in G, \exists b \in G, ab = ba = e$ (Existence of inverses).