Algebraic Geometry HW Due Monday 2/23/2009

(1) Prove that if $S = \bigoplus_d S_d$ is a $\mathbb{Z}$-graded commutative ring, such that there is a $d > 0$ and a unit $u \in S_d$, then the map between homogeneous prime ideals $\mathfrak{p} \lhd S$ and prime ideals in S_0, $\mathfrak{p} \mapsto \mathfrak{p} \cap S_0$ is a bijection.

(2) If R is a commutative ring, a derivation $\partial : R \to R$ is a map such that for all $r, s \in R$, $\partial(rs) = r\partial(s) + \partial(r)s$. If R is a k-algebra, we say that ∂ is a k-derivation if $\partial(a) = 0$ for all $a \in k$.
   (a) Show that the scheme $\alpha_2 := \mathrm{Spec}(k[\epsilon])$ (where we assume that $\epsilon^2 = 0$) has a natural group scheme structure, induced by $\epsilon \mapsto \epsilon \otimes 1 + 1 \otimes \epsilon$
   (b) Show that to give a k-derivation of a k-algebra R is the same as giving an action (in the category of k-schemes) of α_2 on $\mathrm{Spec}(R)$

(3) In the category of sets we know what we mean by the image of a morphism – it is the unique subset of the target which the morphism factors through. In a general category a subobject $S \subset X$ is an equivalence class of monomorphisms $i : S \to X$. (You should look this up if you don't know what it means).
We can ask if images exist in the category of schemes. *I.e.*, if $f : X \to Y$ is a morphism of schemes, is there a unique *subscheme* $\mathrm{Im}(f) \subset X$, such that f factors through $\mathrm{Im}(f)$?

So, the first question for you to think about:
What is a subscheme of a scheme?
Generally one only considers subschemes of a scheme for which the underlying set is either open, closed, or the intersection of an open and a closed set (Look in Hartshorne).
However is $\mathrm{Spec}(Q)$ *viewed as the generic point of* $\mathrm{Spec}(Z)$ *a subscheme of* $\mathrm{Spec}(Z)$ *in the categorical sense?*
Finally, we ask, given $f : X \to Y$ a morphism of schemes, does f have an image:
in the purely category theory sense?
as a closed subscheme? I.e. is there a smallest closed subscheme of X *with the right universal property (among closed subschemes)?*
as a locally-closed subscheme (i.e., is there a smallest locally closed subscheme through which the morphism factors)?