DIFFERENTIAL ALGEBRA - A SCHEME THEORY APPROACH.

HENRI GILLET

Abstract. Two results in Differential Algebra, Kolchin's Irreducibility Theorem, and a result on descent of projective varieties (due to Buium) are proved using methods of "modern" or "Grothendieck style" algebraic geometry.

Introduction

The goal of this paper is to approach some results in differential algebra from the perspective, and using the results of, modern algebraic geometry and commutative algebra. In particular we shall see new proofs of two results: Kolchin's Irreducibility Theorem, and Buium's result describing the minimal field of definition of a projective variety over an algebraically closed field of characteristic zero.

The first section of the paper describes the construction of prolongations (which associate to an algebraic variety X over a differential field, the ring of differential polynomial functions on X) from the point of view of adjoint functors. This allows us to give simple proofs of several properties of the "prolongation" operation, especially how the functor behaves with respect to formally smooth and formally étale morphisms. In particular we observe that the prolongation functor is a quasi-coherent sheaf of rings in the étale topology.

In Section 2, we discuss the proof of Kolchin's theorem:

Theorem (Kolchin's irreducibility theorem - [14] Chapter IV, Proposition 10)**.** *Let A be an integral domain, of finite type over a differential field k; then the associated differential variety is irreducible.*

Note that Proposition 10 of Kolchin also contains a computation of the differential dimension polynomial of the associated differential variety. Also here we only consider rings with a single derivation. The key point in the proof given here is that discrete valuation rings containing a field of characteristic zero are formally smooth over that field, and hence have rings of differential polynomial functions which are integral domains. It is interesting to notice that for a more general valuation ring it is still true that the associated ring of differential polynomial functions is an integral domain; however the only proof that I know of this result uses Zariski's uniformization theorem.

In Section 3, we turn to the result of Buium:

Theorem ([4], [3])**.** *Let X be a variety, proper over an algebraically closed field K. Then X is defined over the fixed field of the set of all derivations of K which lift to derivations of the structure sheaf of X.*

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The main tools in the proof given here are Grothendieck's theorem on algebraizability of morphisms between projective varieties over formal schemes, and Artin's approximation theorem.

Finally in Section 4, two questions which arise from the techniques used in the paper are posed.

The initial genesis for this paper was a seminar at UIC organized by David Marker, Lawrence Ein and myself, in which we and some graduate students read the book [2] of Buium. In addition to my talk at the Rutgers Newark workshop, I gave talks on this material at the conference on model theory, algebraic and arithmetic at MSRI in 1998, and Columbia University and CCNY in 1999.

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1. Differential Rings

1.1. **Some Commutative Algebra.**
In this section we shall review some basic facts about commutative rings and derivations.

All rings are commutative with unit. If k is a ring, then a k-algebra A is simply a ring A together with a homomorphism $k \to A$.

1.1.1. *Étale homomorphisms.*
See [19] for details on this section.

Definition 1.1. *Recall that a ring homomorphism $R \to S$ is* formally étale *if, given a ring C and a square zero ideal $I \lhd C$, together with a commutative diagram of ring homomorphisms:*

$$\begin{array}{ccc} S & \longrightarrow & C/I \\ \uparrow & & \uparrow \\ R & \longrightarrow & C \end{array}$$

there is a unique *homomorphism $S \to C$ making the diagram commute. If instead there exists* at least one *such homomorphism, we say that $R \to S$ is* formally smooth*, while if there exists* at most one *such homomorphism, we say that $R \to S$ is* formally unramified*. If in addition to satisfying one of the above conditions, S is an R-algebra of finite type, we remove the adjective "formally" and say that the morphism is* étale, smooth*, or* unramified *as appropriate.*

The following exercises are straightforward.

Exercise 1.2. *The composition of two formally étale (resp. unramified, resp. smooth) maps is again formally étale, (resp. unramified, resp. smooth).*

Exercise 1.3. *If $S \subset R$ is a multiplicative subset of R, and $S^{-1}R$ is the localization of R with respect to S, then the localization map $R \to S^{-1}R$ is formally étale, and is étale if S is finitely generated. This follows immediately from the fact that an element $x \in C$ is a unit if and only if it is a unit modulo I, since $I \lhd C$ is a nilpotent ideal.*

Exercise 1.4. *If $S = R[t]/(f(t))$ with f monic, and (the image of) $f'(t)$ is a unit in S, then S/R is étale.*

Exercise 1.5. *If $R \to S$ is formally étale, then so is $A \otimes R \to A \otimes S$ for all R-algebras A.*

Exercise 1.6. *If* $(R_n, \theta_n : R_n \to R_{n+1})$ *for* $n \geq 1$ *is a direct system of rings, with* R_{n+1} *formally smooth over* R_n *for all* n*, then the direct limit* $\varinjlim_n R_n$ *is formally smooth over* R*.*

1.1.2. *Derivations.* If R is a commutative ring, and M an R-module, then recall that a derivation $\delta : R \to M$ is an additive map $\delta : R \to M$, such that $\delta(ab) = a\delta(b) + b\delta(a)$.

To give a derivation $\delta : R \to M$ is equivalent to giving a homomorphism of rings,

$$\delta_* : R \to R \oplus M\varepsilon$$
$$\delta_* : r \longmapsto r + \delta(r)\varepsilon$$

where $R \oplus M\varepsilon$, with $\varepsilon^2 = 0$, is the ring of dual numbers over R with coefficients in M, *i.e.* as an abelian group $R \oplus M\varepsilon$ is just the direct sum $R \oplus M$, and the multiplication law is defined by $(r + m\varepsilon).(r' + m'\varepsilon) = (rr' + (rm' + r'm)\varepsilon)$. Note that we require that the composition of the augmentation

$$\begin{array}{ccc} R \oplus M\varepsilon & \to & R \\ r + m\varepsilon & \longmapsto & r \end{array}$$

with δ_* is the identity. The set $\mathcal{D}er(R, M)$, of all derivations $\delta : R \to M$ is an R-module, indeed a sub-module of the module of all functions from R (viewed as a set) to M (viewed as an R-module), with addition and scalar multiplication defined pointwise. If R is a k-algebra, *i.e.,* there is a homomorphism $\phi : k \to R$, we write $\mathcal{D}er_k(R, M)$ for the submodule of $\mathcal{D}er(R, M)$ consisting of all δ for which $\delta(\phi(x)) = 0$ for all $x \in k$; note that $\mathcal{D}er(R, M) = \mathcal{D}er_{\mathbb{Z}}(R, M)$. Given a derivation, the set $\{r | \delta(r) = 0\}$ is clearly a subring of R (resp. a subfield if R is a field) which is called the *ring (resp. field) of constants* of δ. Observe that if $\delta \in \mathcal{D}er_k(R, M)$, then k is contained in the constants of δ, so that $\mathbb{Z}$ is always in the constants, and also that if R is a $\mathbb{Q}$-algebra, then the ring of constants is a $\mathbb{Q}$-algebra.

The covariant functor $M \mapsto \mathcal{D}er_k(R, M)$ from R-modules to R-modules is represented by the R-module $\Omega_{R/k}$ of Kähler differentials of R over k, [17] §25, *i.e.* $\mathcal{D}er_k(R, M) \simeq \mathrm{Hom_R}(\Omega_{R/k}, M)$. This isomorphism is induced by the universal derivation:

$$\begin{array}{ccc} R & \to & \Omega_{R/k} \\ r & \mapsto & dr \end{array}$$

If R is a k-algebra, $f : R \to S$ is a homomorphism of k-algebras, and M is an S-module, there is an exact sequence of S-modules:

$$0 \to \mathcal{D}er_R(S, M) \to \mathcal{D}er_k(S, M) \to \mathcal{D}er_k(R, M) \tag{1.1}$$

which is obtained by applying the functor $\mathrm{Hom}(\_, M)$ to the natural exact sequence:

$$S \otimes_R \Omega_{R/k} \overset{v_{S/R/k}}{\to} \Omega_{S/k} \to \Omega_{S/R} \to 0$$

Proposition 1.7. *If* S *is formally smooth over* R*, this sequence becomes split exact (*i.e. *the map* $v_{S/R/k}$ *has a left inverse), as does the sequence 1.1 for any* S *module* M*.*

Proof. This is a standard result, see EGA 0, Théorème 20.5.7, ([10]) for example. However, for completeness, we shall sketch the proof here. Suppose that S is formally smooth over R. If M is an S-module, a derivation $d : R \to M$ (where we view

M as an R-module by restriction of scalars) is equivalent to a ring homomorphism $R \to R \oplus M\varepsilon$ (compatible with the augmentation from $R \oplus M\varepsilon \to R$). This induces a ring homomorphism from $R \to S \oplus M\varepsilon$. Thus we have a commutative square:

$$\begin{array}{ccc} S & \longrightarrow & S \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \oplus M\varepsilon \end{array}$$

and hence, since $R \to S$ is formally smooth, there is a lifting of of the identity on S to a map $S \to S \oplus M\varepsilon$ compatible with the map $R \to S \oplus M\varepsilon$. *I.e.,* the derivation $d : R \to M$ has an extension to a derivation $S \to M$. Note that if S is formally étale over R, this extension is unique. Thus $\mathcal{D}er_k(S, M) \to \mathcal{D}er_k(R, M)$ is surjective.

Now consider the universal derivation $R \to \Omega_{R/k}$. This induces a derivation from R to the S-module $S \otimes \Omega_{R/k}$, which by the previous discussion extends to a derivation $S \to S \otimes \Omega_{R/k}$, and which, by the universal property of the Kähler differentials, is therefore induced by a homomorphism of S-modules $\Omega_{S/k} \to S \otimes \Omega_{R/k}$. It is easily checked that since this extends the universal derivation of R, that it is a left inverse of the map $v_{S/R/k}$, and that, by taking $\mathrm{Hom}(\_, M)$ it induces the splitting of 1.1. □

Proposition 1.8. *If $R \to S$ is a formally unramified homomorphism of k-algebras, and M is an S-module, then $\mathcal{D}er_R(S, M) \simeq 0$.*

Proof. Again consider the square

$$\begin{array}{ccc} S & \longrightarrow & S \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \oplus M\varepsilon \end{array}$$

where now the map $R \to S \oplus M$ is $r \mapsto r + 0\varepsilon$. The zero derivation $S \to M$ provides one lifting in the square, and since R is unramified over S, this is the only lifting; *i.e.*, there are no derivations $S \to M$ which induce the zero derivation on R. □

Combining these two results we immediately get:

Proposition 1.9. *If $R \to S$ is a formally étale homomorphism of k-algebras, and M is an S-module, then*

$$\mathcal{D}er_k(S, M) \simeq \mathcal{D}er_k(R, M) \ .$$

The reader may observe that there is a resemblance between the definitions, via lifting properties, of a formally smooth morphism and of a projective module. This is not a total coincidence:

Proposition 1.10. *Let A be a ring, V a projective A-module. Then the symmetric algebra $B = \mathbb{S}_A(V)$ is formally smooth over A,*

Proof. This follows immediately from the fact that

$$\mathrm{Hom}_{A-\mathrm{algebras}}(\mathbb{S}_A(V), C) \simeq \mathrm{Hom}_{A-\mathrm{modules}}(V, C) \ ,$$

so that a lifting exists, even when the ideal $I \subset C$ is not nilpotent. □

Proposition 1.11. *Let A be a ring, B an A-algebra. Then if B is formally smooth over A, $\Omega_{B/A}$ is a projective (but not necessarily finitely generated) B-module.*

Proof. EGA IV, Proposition 16.10.2, ([11]) - where this is deduced from EGA 0, Corollary 19.5.4, ([10]). □

1.1.3. *Definition and Elementary Properties.*

Definition 1.12. *A* differential ring *consists of a pair* (R, δ) *in which* R *is a commutative ring with unit, and* $\delta : R \to R$ *is a derivation.*

If (R, δ) and (R', δ') are differential rings, then a ring homomorphism $f : R \to R'$ is said to be a *differential* homomorphism if it is compatible with the two derivations, *i.e.* if $\delta' \cdot f = f \cdot \delta$. The kernel of a differential homomorphism $(R, \delta) \to (R', \delta')$ is a *differential ideal, i.e.* an ideal $\mathfrak{p} \lhd R$ which is closed under δ. Note that the intersection of two differential ideals in a differential ring (R, δ) is again a differential ideal, and hence any subset $X \subset R$ is contained in a smallest differential ideal, which we denote $[X]$.

The following lemma, which is easily proved by induction on k, will be useful:

Lemma 1.13. *If* $\delta : R \to R$ *is a derivation, then for all* $a, b \in R$*, and* $k \geq 1$*,*

$$\delta^k(ab) = \sum_{j=0}^{k} \binom{k}{j} \delta^j(a)\delta^{k-j}(b) .$$

¿From this we deduce:

Proposition 1.14. *If* R *is a* $\mathbb{Q}$*-algebra, then a derivation* $\delta : R \to R$ *induces a ring homomorphism:*

$$\begin{aligned} \exp(\delta) &: R \to R[[t]] \\ \exp(\delta) &: R \mapsto \sum_{k=0}^{\infty} \frac{\delta^k(r)}{k!} t^k . \end{aligned}$$

Proof. Since δ, and therefore δ^k for any k, is additive, we need only check that $\exp(\delta)$ preserves products, which follows immediately from the previous lemma. □

For any R, not just a $\mathbb{Q}$-algebra, recall that giving a derivation $\delta : R \to R$ is equivalent to giving a homomorphism of rings (augmented towards R):

$$\begin{aligned} \delta_* = \exp_{\leq 1}(\delta) &: R \to R[\varepsilon] \\ \delta_* = \exp_{\leq 1}(\delta) &: r \longmapsto r + \delta(r)\varepsilon . \end{aligned}$$

where $R[\varepsilon]$, with $\varepsilon^2 = 0$, is the ring of dual numbers over R. More generally, if $n!$ is a unit in R, then we get a truncated exponential $\exp_{\leq n}(\delta) : R \to R[t]/t^{n+1}$ by dropping all the terms of degree greater than n from the exponential map.

This interpretation leads to the following lemma:

Lemma 1.15. *Let* $f : R \to S$ *be a formally étale ring homomorphism. Then if* $\delta_R : R \to R$ *is a derivation, there is a* unique *derivation* $\delta_S : S \to S$ *extending* δ*. Furthermore if* $\sigma : S \to \Lambda$ *is a ring homomorphism, with* $\Lambda = (\Lambda, \delta_\Lambda)$ *a differential ring, such that* $\rho = \sigma \cdot f : R \to \Lambda$ *is a differential homomorphism, then* σ *is a differential homomorphism with respect to* δ_S *and* δ_Λ*.*

Proof. The first assertion follows from Proposition 1.9. The second assertion follows by considering the diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & \Lambda \\ f\Big\uparrow & & \Big\uparrow \\ R & \xrightarrow{\tilde{\rho}\cdot\exp_{\leq 1}(\delta_R)} & \Lambda[\epsilon] \end{array}$$

Here we use $\tilde{\rho}$ to denote the map $R[\varepsilon] \to \Lambda[\varepsilon]$ induced by ρ. There are two possible ways of filling this diagram in with a homomorphism from S to $\Lambda[\varepsilon]$:

(1) $\exp_{\leq 1}(\delta_\Lambda) \cdot \sigma$
(2) $\tilde{\sigma} \cdot \exp_{\leq 1}(\delta_S)$ (here we use $\tilde{\sigma}$ to denote the map $S[\varepsilon] \to \Lambda[\varepsilon]$ induced by σ).

Since f is formally étale, and in particular formally unramified, these maps must be equal, and we are done. □

Since localization is étale, we automatically get:

Corollary 1.16. *If $R = (R, \delta_R)$ is a differential ring, and $S \subset R$ is a multiplicative set, the derivation δ_R has a unique extension to the localization $S^{-1}R$.*

(Of course this result can also be proved explicitly, by using the quotient rule to define the derivation.)

Lemma 1.17. *If R is a differential $\mathbb{Q}$-algebra, then its nilradical is a differential ideal.*

Proof. This is a standard result, which may be found in [2] for example. Here is a short proof using the exponential map. We must show that if $r \in R$ is nilpotent, then so is $\delta(r)$. Since $\exp(\delta)$ is a ring homomorphism, $\exp(\delta)(r)$ is nilpotent in $R[[t]]$. Therefore $\exp(\delta)(r) - r$ is also nilpotent in $R[[t]]$. But $\exp(\delta)(r) - r = \delta(r)t + O(t^2)$ and therefore if $(\exp(\delta)(r) - r)^N = 0$, we have:

$$(\exp(\delta)(r) - r)^N = (\delta(r)t + O(t^2))^N = \delta(r)^N t^N + O(t^{N+1}) = 0$$

and hence $\delta(r)^N = 0$.

□

Lemma 1.18. *If R is a differential $\mathbb{Q}$-algebra, and $\mathfrak{p} \lhd R$ is a minimal prime ideal in R, then $\mathfrak{p}$ is a differential ideal.*

Proof. $\mathfrak{p}$ is the inverse image of $\mathfrak{p}R_\mathfrak{p}$ under the natural map $R \to R_\mathfrak{p}$. The derivation in R extends to $R_\mathfrak{p}$, and $\mathfrak{p}R_\mathfrak{p}$ is the unique minimal prime ideal in $R \to R_\mathfrak{p}$, hence is equal to the nilradical of $R_\mathfrak{p}$, and is therefore a differential ideal in $R_\mathfrak{p}$. Since the inverse image of a differential ideal is a differential ideal, it follows that $\mathfrak{p}$ is a differential ideal. □

1.2. **Prolongation.**

1.2.1. *Existence.* If (k, δ) is a differential ring we may consider the category $\mathbf{Diff}_{(k,\delta)}$ of differential (k, δ)-algebras. There is clearly a forgetful functor $U : \mathbf{Diff}_{(k,\delta)} \to \mathbf{Alg}_k$, which associates to the differential (k, δ)-algebra $(R, \tilde{\delta})$ the k-algebra R.

Proposition 1.19. *The forgetful functor $U : \mathbf{Diff}_{(k,\delta)} \to \mathbf{Alg}_k$ has a left adjoint.*

Proof. It is a general fact in Universal algebra that the forgetful functor between two categories of algebras induced by forgetting one or more operations or equations has a left adjoint. More generally, it is a result of Lawvere that "algebraic functors" have left adjoints. It is difficult to give a nice reference for the proof, in part because of variations in the way that a *variety of algebras* can be defined. However one can find, in various references, proofs of the existence of free algebras, i.e. of left adjoints for the forgetful functors from a category of algebras, such as differential k-algebras or k-algebras, to the category of sets. For example, see [15] Section V.6, and in particular the discussion following Theorem 3. The more general case of the lemma then follows immediately from the existence of free algebras via Theorem 28.12 in [13].

Let us give a more detailed version of this proof, which will also be useful later. First, we need two lemmas:

Lemma 1.20. *Let k be a ring, X a set, $k[X]$ the associated polynomial ring, and M a $k[X]$-module. Suppose given a derivation $\delta : k \to M$. Let $\mathcal{D}er_{(k,\delta)}(k[X], M)$ be the set of derivations $k[X] \to M$ which extend δ. Then the restriction map*

$$\mathcal{D}er_{(k,\delta)}k(k[X], M) \to Hom_{Sets}(X, M)$$

which associates to a derivation $\tilde{\delta}$ extending δ, the restriction of $\tilde{\delta}$ to X, is a bijection.

Proof. Giving a derivation $\tilde{\delta} : k[X] \to M$ extending δ is equivalent to giving a homomorphism of k-algebras $k[X] \to k[X] \oplus M\varepsilon$ (of which the first component is the identity), where $k[X] \oplus M\varepsilon$ is a k-algebra via the homomorphism $k \to k[X] \oplus M\varepsilon$ determined by δ. Since $k[X]$ is the free k-algebra on the set X, the restriction map from the set of such homomorphisms to the set of functions $X \to k[X] \oplus M\varepsilon$ which have as their first component the inclusion $X \to k[X]$, *i.e.,* the set of functions $X \to M$, is bijective. □

Lemma 1.21. *If X is a set, the functor from (k, δ)-algebras to sets:*

$$(R, \delta_R) \mapsto R^X$$

is representable - i.e. $\mathbf{Diff}_{(k,\delta)}$ *has free objects, and hence the forgetful functor from (k, δ)-algebras to sets has a left adjoint.*

Proof. Define a differential ring $(k\{X\}, \delta_X)$ as follows. Let $\mathbb{Z}_{\geq 0}$ be the set of non-negative integers, and set $k\{X\}$ equal to the polynomial ring on $X \times \mathbb{Z}_{\geq 0}$. By the lemma, the set of derivations $k\{X\} \to k\{X\}$ extending $\delta : k \to k \subset k\{X\}$ is bijective with the set of functions $X \times \mathbb{Z}_{\geq 0} \to k\{X\}$. We fix $\tilde{\delta}$ to be the derivation induced by the function

$$X \times \mathbb{Z}_{\geq 0} \to X \times \mathbb{Z}_{\geq 0}$$
$$(x, n) \mapsto (x, n + 1)\ .$$

Now we identify X with a subset of $k\{X\}$ via the function

$$\eta : X \to k\{X\}$$
$$\eta : x \mapsto (x, 0)\ .$$

We must show that $\eta : X \to k\{X\}$ is universal for maps from X to (k, δ)-algebras. Given a (k, δ)-algebra (R, δ_R), and a function $f : X \to R$, let $\tilde{f}$ be the function $\tilde{f} : X \times \mathbb{Z}_{\geq 0} \to R$ by $f : (x, n) \mapsto \delta_R^n(f(x))$; note that via the identification of X

with the subset $X \times \{0\} \subset X \times \mathbb{Z}_{\geq 0}$ $\tilde{f}$ extends f. The function $\tilde{f}$ determines a unique ring homomorphism $\rho_f : k\{X\} \to R$. To check that ρ_f is a homomorphism of differential rings, we must check that the two derivations

$$\rho_f \cdot \tilde{\delta} : k\{X\} \to R$$

and

$$\delta_R \cdot \rho_f : k\{X\} \to R$$

coincide. By Lemma 1.20 it is enough to observe that they both induce the same function $X \times \mathbb{Z}_{\geq 0} \to R$:

$$\rho_f \cdot \tilde{\delta}((x, n)) = \rho_f((x, n+1)) = \delta_R^{n+1}(f(x))$$

and

$$\delta_R \cdot \rho_f((x, n)) = \delta_R(\delta_R^n(f(x))) = \delta_R^{n+1}(f(x)) \ .$$

It remains to show that ρ_f is the unique extension of f to a differential homomorphism. If σ is another extension, since polynomial rings are free, $\rho_f = \sigma$ if they have the same restriction to $X \times \mathbb{Z}_{\geq 0}$, but then the requirement that ρ_f and σ be differential homomorphisms forces $\rho_f((x, n)) = \rho_f(\tilde{\delta}^n(x) = \delta_R^n(x)$ which equals $\sigma((x, n))$ by symmetry. □

Note that one usually writes $x^{(n)}$ in place of (x, n), and we shall do so in the rest of the paper.

Let us now return to the proof of the theorem:

Given any k-algebra R, we form the ring of differential polynomials $k\{R\}$, *i.e.* the free (k, δ)-algebra on the *set* R. Let $\mathfrak{I}_R$ be the ideal which is the kernel of the canonical surjection $k[R] \to R$. Notice that the triple

$$\mathfrak{I}_R \subset k[R] \subset k\{R\}$$

is functorial in R and hence we get a functor $R \mapsto k\{R\}/[\mathfrak{I}_R]$ from k-algebras to (k, δ)-algebras. We claim that this is the left adjoint that we seek. First note that the inclusion $k[R] \subset k\{R\}$ induces a homomorphism $\theta_R : R \simeq k[R]/\mathfrak{I}_R \to k\{R\}/[\mathfrak{I}_R]$ which is functorial in R, *i.e.* it is a natural transformation of functors from k-algebras to k-algebras. We must show that given a (k, δ)-algebra (A, δ_A) and a homomorphism of k-algebras $f : R \to A$, that it has a unique factorization $f = \sigma_f \cdot \theta_R$ through a homomorphism of (k, δ) algebras σ_f.

Viewing f simply as a function between sets, it has a unique factorization via a differential homomorphism $\rho_f : k\{R\} \to A$ and through the function $R \to k\{R\}$, by the earlier discussion of free algebras. Since $f : R \to A$ is a homomorphism of *rings*, ρ_f must map the image of $\mathfrak{I}_R$ in $k\{R\}$ to zero in A, and hence ρ_f factors through a homomorphism $\sigma_f : k\{R\}/[\mathfrak{I}_R] \to A$. Since ρ_f is uniquely determined by f, so is σ_f, and we are done. □

Let us denote the left adjoint of the proposition:

$$(\ )^\infty : R \mapsto R^\infty$$

Thus if $(k, \delta = \delta_k)$ is a differential ring, $(\ )^\infty$ associates to each k-algebra R a differential (k, δ)-algebra R^∞, and a ring homomorphism $\eta : R \to R^\infty$with the following universal property: given a homomorphism $\phi : (k, \delta) \to (S, \delta)$ of differential rings, and a ring homomorphism $f : R \to S$, there is a unique homomorphism $f^\infty : R^\infty \to S$ of differential rings which makes the following diagram commute:

$$\begin{array}{ccc} R^\infty & \xrightarrow{f^\infty} & S \\ {\scriptstyle\eta}\uparrow & & \| \\ R & \xrightarrow{f} & S \\ \uparrow & & \uparrow \\ k & = & k \end{array}$$

Lemma 1.22. *The functor* $(\ )^\infty : \mathbf{Alg}_k \to \mathbf{Diff}_{(k,\delta)}$ *commutes with direct (or inductive) limits.*

Proof. This is a standard property of left adjoints. See [15] V.6 for example. ☐

Lemma 1.23. *Let* k *be a differential field and let* X *be a set. Then* $(k[X])^\infty \simeq k\{X\}$*.*

Proof. The composition of the forgetful functor from (k, δ)-algebras to k-algebras with the forgetful functor from k-algebras to sets is the forgetful functor from (k, δ)-algebras to sets, Hence the left adjoint of the composition is the composition of the two left adjoints. ☐

Definition 1.24. $k\{X\}$ *is called the ring of differential polynomials on the set* X *over* (k, δ)

1.2.2. *An Alternative Construction.* We can also give another construction of R^∞ using the universal properties of the module of differentials. An advantage of this approach will be that we will see that when R is a formally smooth k-algebra and an integral domain, then R^∞ is also an integral domain.

We start with:

Lemma 1.25. *Suppose that* A *is a commutative ring, that* B *is an* A*-algebra, and that we are given a derivation* $\delta : A \to B$*. The functor from the category of* B*-algebras to the category of sets, which assigns to the* B*-algebra* R *the set of all derivations* $\tilde{\delta} : B \to R$ *which extend the derivation* δ*, is representable. If* B *is formally smooth over* A*, the object this functor is isomorphic to the symmetric algebra on a projective* B*-module.*

Proof. If we omit the requirement that $\tilde{\delta} : B \to R$ extend δ, then we are simply representing the set of all derivations from B to R. The set of such derivations is represented by the set of B-module homomorphisms $\Omega_B \to R$. Since R is a B-algebra, this is equivalent to the set of all B-algebra homomorphisms $\mathbb{S}_B(\Omega_B) \to R$. The condition that $\tilde{\delta} : B \to R$ extend the derivation δ is that the homomorphism $\mathbb{S}_B(\Omega_B) \to R$ map all elements of the form da for $a \in A$ to $\delta(a)$. Thus we divide $\mathbb{S}_B(\Omega_B)$ by the ideal I generated by all elements of the form $da - \delta(a)$ for $a \in A$. If we now assume that B is formally smooth over A, then we know that $\Omega_{B/A}$ is a projective B-module, and that Ω_B is non-canonically isomorphic to the direct sum of $\Omega_{B/A}$ and Ω_A. Therefore the quotient $\mathbb{S}_B(\Omega_B)/I$ is isomorphic to $\mathbb{S}_B(\Omega_{B/A})$, and is formally smooth over B. ☐

Suppose that (k, δ) is a differential ring, and that R is a k-algebra. Consider the category $\mathbf{T}_{(k,\delta,R)}$ with objects sequences $(R^{\{n\}}, \rho_n, \delta^{\{n\}})$ for $n \geq -1$, in which:

(1) $R^{\{-1\}} = k$.
(2) $R^{\{0\}} = R$.
(3) For all $n \geq -1$, ρ_n is a ring homomorphism $R^{\{n\}} \to R^{\{n+1\}}$, and $\delta^{\{n\}}$ is a derivation : $R^{\{n\}} \to R^{\{n+1\}}$ (with respect to the $R^{\{n\}}$ module structure on $R^{\{n+1\}}$ via ρ_n). ρ_{-1} is the structure map for the k-algebra R, and $\delta^{\{-1\}}$) is the composition $\rho_{-1} \cdot \delta$.
(4) For each $n \geq 0$, we have a commutative diagram:

$$\begin{array}{ccc} R^{\{n\}} & \xrightarrow{\delta^{\{n\}}} & R^{\{n+1\}} \\ \uparrow & & \uparrow \\ R^{\{n-1\}} & \xrightarrow{\delta^{\{n-1\}}} & R^{\{n\}} \end{array}$$

A morphism between two sequences $(R^{\{n\}}, \rho_n, \delta^{\{n\}})$ and $(R'^{\{n\}}, \rho'_n, \delta'^{\{n\}})$ is a sequence of ring homomorphisms $f_n : R^{\{n\}} \to R'^{\{n\}}$ for $n \geq 1$, compatible with the homomorphisms and derivations in each sequence.

Note that if $g : R \to A$ is a homomorphism of k-algebras with (A, δ_A) a (k, δ)-algebra, then we get an object in the category by setting $R^{\{n\}} := A$ for $n \geq 1$, $\delta^{\{n\}} := \delta_A$ for $n \geq 1$, and $\delta^{\{0\}} := \delta_A \cdot g$ when $n = 0$. Finally observe that the $\delta^{\{n\}}$) induce a derivation on $\varinjlim_n R^{\{n\}}$ which makes it a (k, δ)-algebra.

Proposition 1.26. *The category $\mathbf{T}_{(k,\delta,R)}$ has an initial object $(R^{\{n\}}, \rho_n, \delta^{\{n\}})$, and there is an isomorphism $\varinjlim_n R^{\{n\}} \simeq R^\infty$. If R is formally smooth over k, then $R^{\{n\}}$ is formally smooth over $R^{\{n-1\}}$ (indeed it is isomorphic to the symmetric algebra on a projective $R^{\{n-1\}}$ module) for all $n \geq 1$, and hence R^∞ is formally smooth over R by Exercise 5 in 1.1.*

Proof. We proceed by induction on $n \geq 0$. The case $n = 0$ is already given to us. For the inductive step, given a sequence of algebras and derivations, $(R^{\{k\}}, \rho_k, \delta^{\{k\}})$, for $k < n$ (which we could refer to as a truncated sequence of length n), there is, by Lemma 1.25, a $R^{\{n-1\}}$-algebra $R^{\{n\}}$ together with a derivation $R^{\{n-1\}} \to R^{\{n\}}$ which is universal and therefore defines a universal truncated sequence of length $n+1$. If R is formally smooth over k, then again by Lemma 1.25, and induction on n, $R^{\{n\}}$ is formally smooth over $R^{\{n-1\}}$.

Since a homomorphism $g : R \to A$ of k-algebras with (A, δ_A) a (k, δ)-algebra defines an object in the category $\mathbf{T}_{(k,\delta,R)}$, there is unique map of sequences from the universal sequence $(R^{\{n\}}, \rho_n, \delta^{\{n\}})$ to the constant sequence defined by (A, δ_A), and hence a map of R-algebras limit $\varinjlim_n R^{\{n\}} \to A$, which is also a differential homomorphism of (k, δ) algebras. □

Corollary 1.27. *If (k, δ) is a differential ring, and R is an integral domain which is formally smooth over k, then $R^{\{n\}}$ is an integral domain for all $n \geq 0$, and hence R^∞ is an integral domain. Furthermore $R \to R^\infty$ is injective.*

Proof. Since the direct limit of a sequence $(R^{\{n\}}, \rho_n : R^{\{n\}} \to R^{\{n+1\}})_{n\geq 1}$ of ring homomorphisms between integral domains is an integral domain, it suffices to show that $R^{\{n\}}$ is an integral domain for all n. Since we assume that R is an integral domain, and all the ρ_n are formally smooth, it is enough to know, following Lemma

1.25, that if A is a commutative ring, and P is a projective A-module, then the symmetric algebra on P over R is an integral domain. But we know that a projective module is a direct summand of a free module. Hence the symmetric algebra on P is isomorphic to a subring of a polynomial ring over A, and is therefore an integral domain. Furthermore, $\rho_n : R^{\{n\}} \to R^{\{n+1\}}$ is then clearly injective for all n. □

1.2.3. *Etale Base Change.*

Lemma 1.28. *Let (k, δ) be a differential ring. Suppose that $f : R \to S$ is a formally étale homomorphism of k-algebras. Then*

$$S^\infty \simeq S \otimes_R R^\infty \ .$$

Proof. Suppose that we are given a (k, δ) algebra (Λ, δ) and ring homomorphism $\phi : S \to \Lambda$. We wish to show that this has a unique factorization through a differential homomorphism $S \otimes_R R^\infty \to \Lambda$. Writing $i : R \to S$ for the inclusion, the induced homomorphism $\phi \cdot i : R \to \Lambda$ has a canonical factorization $\phi \cdot i = (\phi \cdot i)^\infty \cdot \eta_R$, where $(\phi \cdot i)^\infty : R^\infty \to \Lambda$ is a differential homomorphism. Hence we have a commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & \Lambda \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle (\phi \cdot i)^\infty} \\ R & \xrightarrow{\ \eta_R\ } & R^\infty \end{array}$$

and an induced homomorphism of commutative rings

$$S \otimes_R R^\infty \to \Lambda \ .$$

Since $R^\infty \to S \otimes_R R^\infty$ is formally étale and $R^\infty \to \Lambda$ is a differential homomorphism, it follows from Lemma 1.15 that the induced map $S \otimes_R R^\infty \to \Lambda$ is a differential homomorphism. The rest of the details are left to the reader. □

Definition 1.29. *The conclusion of the lemma could be phrased as "the functor $R \mapsto R^\infty$ is a quasi-coherent sheaf in the étale topology". Thus we can define,* for any scheme *X, the sheaf (in the étale topology) $\mathcal{O}_X^\infty$ of differential polynomial functions on X, by setting (if $U = \mathrm{Spec}(R)$ is affine and étale over X:*

$$\Gamma(U, \mathcal{O}_X^\infty) := R^\infty$$

Definition 1.30. *Let $k = (k, \delta)$ be a differential ring. Suppose that X is scheme over k. Then we define $X^\infty := \boldsymbol{Spec}(\mathcal{O}_X^\infty)$. (See* [12] *Ch. II, Exercise 5.17 for the construction of* ***Spec****.) X^∞ is a "differential scheme" in the sense that $\mathcal{O}_{X^\infty}$ is equipped with a derivation. The passage from X to X^∞ is known as* prolongation*; when the derivation δ on k is zero, and X is smooth over k, X^∞ is the (infinite) jet bundle over X. However, X^∞ is not a differential scheme in the sense of other authors* - i.e.*, it does* not *use the "differential spectrum" as a local model.*

1.2.4. *Prolongation of fields and valuation rings.*

Proposition 1.31. *If $k = (k, \delta)$ is a differential field, and $k \subset K$ is a finitely generated separable extension of k, then K^∞ is a polynomial ring over K. If we assume in addition that $k = (k, \delta)$ has characteristic zero, then we can remove the assumption of finite generation.*

Proof. By [6] Chapter V, §16.7 Corollary to Theorem 5, we know that K has a separating transcendence basis $\{x_1, \ldots, x_n\}$ over k. Thus K is a finite separable algebraic (and hence étale) extension of $k(x_1, \ldots, x_n)$. Since localizations (including passage to the field of fractions) are formally étale, it follows that K is formally étale over the polynomial ring $k[x_1, \ldots, x_n]$. Hence, by Lemmas 1.21 and 1.28,

$$K^\infty \simeq K \otimes_{k[x_1,\ldots,x_n]} k\{x_1, \ldots, x_n\} \simeq K[\{x_1, \ldots, x_n\} \times \mathbb{N}] .$$

If k has characteristic zero, then by Steinitz Theorem ([6] Chapter V, Theorem 1 of §14.2) K has a transcendence basis $X \subset K$ over k, and the extension $k(X) \subset K$ is algebraic, hence (since we assume characteristic zero) a composite of finite separable extensions, and hence we again have that K is formally étale over the polynomial ring $k[X]$ (see also *op. cit.* V.16.3.). □

If k has characteristic p, and the extension is not finitely generated, then the extension need not have a separating transcendence basis; see [6] Chapter V §16, Exercises 1 and 2. I will ignore the question of whether the lemma might be still true in the absence of a separating basis.

This proposition, combined with Lemma 1.28, gives another way of proving Corollary 1.27, though under the stronger hypothesis that R is a smooth (not just formally smooth) k-algebra:

Corollary 1.32. *If R is a smooth k-algebra, with (k, δ) a differential ring, and if R is an integral domain, then R^∞ is an integral domain.*

Proof. Since R is smooth over k, $X = \mathrm{Spec}(R)$ is locally isomorphic in the étale topology to affine space $\mathbb{A}^n_k$ (where n is the dimension of R over k). Hence by Lemma 1.21, R^∞ is locally isomorphic, again in the étale topology, to the ring of differential polynomials $k\{x_1 \ldots x_n\}$. Since flatness is a local condition, and any polynomial ring is flat over its ring of coefficients, R^∞ is flat over R. Since R is a domain, it injects into its fraction field F. Since R^∞ is flat over R, we have that R^∞ injects into $R^\infty \otimes_R F$. But we know that $R \mapsto R^\infty$ commutes with localization, hence $R^\infty \otimes_R F \simeq F^\infty$, which by Proposition 1.31 is a polynomial ring over F, and is therefore a domain. Thus R^∞ injects into a domain, and is itself a domain. □

We can globalize this as follows - the proof is left to the reader:

Corollary 1.33. *If X is a variety smooth and integral over a differential field of characteristic zero, then the differential algebraic variety X^∞ associated by prolongation to X is integral, and in particular irreducible.*

The following will be useful in the next section:

Corollary 1.34. *Let k be a differential field of characteristic zero, and let $E \subset F$ be extensions of k. Then the natural map $E^\infty \to F^\infty$ is injective.*

Proof. Since any transcendence basis of E can be extended to a transcendence basis of F, the corollary follows immediately from Proposition 1.31. □

We can also give an alternate proof of the first assertion of Proposition 1.31 using Proposition 1.26.

Proposition 1.35. *Suppose that (k, δ) is a differential field of characteristic zero, and that $k \subset K$ is an extension of fields. Then K^∞ (formed relative to (k, δ)) is an integral domain.*

Proof. By a theorem of Cohen, we know that K is formally smooth over k (see [10] Ch. 0, §Théorème 19.6.1) and so we can apply Proposition 1.26. □

We turn now from fields to valuation rings, and in particular to discrete valuation rings. First a technical lemma:

Lemma 1.36. *Let k be a field of characteristic zero, and R a k-algebra which is a discrete valuation ring. Then R is a formally smooth k algebra.*

Proof. Let K be the residue field of R. The induced map from k to K is an inclusion, which makes K a separable extension of k, and hence using Cohen's theorem again, (see EGA Ch. 0, Corollary §19.6.1, in [10]) K is formally smooth over k. If M is the maximal ideal of R, then $M/M^2 \simeq K$ and is therefore a projective K-module. Therefore by Corollary §19.5.4, in *op. cit.*, R is formally smooth over k. □

¿From the lemma, together with Proposition 1.26 we immediately get:

Corollary 1.37. *Let k be a differential field of characteristic zero, and R a k-algebra which is a discrete valuation ring. Then R^∞ is an integral domain.*

More generally, by using uniformization, we have:

Lemma 1.38. *Let k be a differential field of characteristic zero, and R a k algebra which is a valuation ring. Then R^∞ is an integral domain.*

Proof. By the *Uniformization Theorem* of Zariski, [20], see also Popescu [18] for a more general result, we know that any valuation ring R containing a field k of characteristic zero is the direct limit of smooth k-algebras (which we may assume to be integral domains):

$$R = \varinjlim_\alpha A_\alpha \ .$$

Since $R \mapsto R^\infty$ commutes with direct limits, we have that

$$R^\infty = \varinjlim_\alpha (A^\alpha)^\infty$$

is a direct limit of integral domains, and is therefore an integral domain. □

2. Kolchin's Irreducibility Theorem

In the last section we saw that if X is a variety smooth and integral over a differential field, then the differential algebraic variety X^∞ associated by prolongation to X is integral, and in particular irreducible. Kolchin's Irreducibility Theorem tells us that in characteristic zero, the irreducibility conclusion remains true even if X is not smooth. Note however that X^∞ will not in general be reduced, *i.e.*, its coordinate ring may contain nilpotent elements.

Theorem 1. *Let $K = (K, \delta)$ be a differential field of characteristic zero, and suppose that R is a K-algebra which is an integral domain. Then R^∞ has a unique minimal prime ideal.*

Remark. In Kolchin [14], Proposition 10 of Chapter IV, this is phrased as: "Let $\mathfrak{p}_0$ be a prime ideal of $\mathcal{F}[y_1, \ldots, y_n]$ ... . Then $\{\mathfrak{p}_0\}$ is a prime differential ideal ... ." Here $\{\mathfrak{p}_0\}$ denotes the smallest radical differential ideal containing $\mathfrak{p}_0$. If we set $R := \mathcal{F}[y_1, \ldots, y_n]/\mathfrak{p}_0$, then the image of $\{\mathfrak{p}_0\}$ in $R^\infty \simeq \mathcal{F}\{y_1, \ldots, y_n\}/[\mathfrak{p}_0]$ is the nilradical of R^∞. Thus $\mathfrak{p}_0$ is prime if and only if R^∞ has a unique minimal prime. In general, the nilradical of a ring is prime if and only if the spectrum of the ring is irreducible; thus the theorem tells us that if X is a reduced irreducible affine scheme, then X^∞ is also irreducible.

The following definition will be useful in the proof of the theorem:

Definition 2.1. *Let R be a K-algebra, with K a differential field of characteristic zero. Suppose that $\mathfrak{p}$ is a prime ideal in R. Denote by $[[\mathfrak{p}]]$ the prime differential ideal in R^∞ which is the kernel of the homomorphism $R^\infty \to k(\mathfrak{p})^\infty$ where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$. Note that $[[\mathfrak{p}]]$ is a prime ideal because $k(\mathfrak{p})^\infty$ is an integral domain by Proposition 1.31.*

With this definition, we can rephrase Kolchin's theorem as asserting that any prime ideal $\mathfrak{p} \lhd R^\infty$ contains $[[0]]$.

2.1. **Proof of the theorem.** We start by assuming that R is noetherian. The proof will then consist the following five steps:

(1) Show that it is enough to prove that any prime *differential* ideal $\mathfrak{p} \lhd R^\infty$ contains $[[0]]$.
(2) Show that if $\eta : R \to R^\infty$ is the natural map, then for any prime differential ideal $\mathfrak{p} \lhd R^\infty$, $[[\eta^{-1}(\mathfrak{p})]] \subset \mathfrak{p}$. Therefore it will be enough to prove that for any prime ideal $\mathfrak{q} \lhd R$, $[[0]] \subset [[\mathfrak{q}]] \lhd R^\infty$.
(3) Show that if R is a one dimensional local domain, with maximal ideal $\mathfrak{p}$, then $[[0]] \subset [[\mathfrak{p}]] \lhd R^\infty$.
(4) Show that if $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals in R, with $\mathfrak{q}$ of height one above $\mathfrak{p}$, then $[[\mathfrak{p}]] \subset [[\mathfrak{q}]] \lhd R^\infty$.
(5) Show that if $\mathfrak{p} \subset \mathfrak{q}$ are arbitrary prime ideals in R, then $[[\mathfrak{p}]] \subset [[\mathfrak{q}]] \lhd R^\infty$. In particular for any prime ideal $\mathfrak{q} \lhd R$, $[[0]] \subset [[\mathfrak{q}]] \lhd R^\infty$.

2.1.1. *Proof of Step 1.* By Zorn's lemma any prime ideal in R^∞ contains a minimal prime ideal, which by Lemma 1.18 is a differential ideal. Thus it suffices to show that any *prime* differential ideal contains $[[0]]$.

2.1.2. *Proof of Step 2.*

Lemma 2.2. *Let K and R be as above. Suppose that $\mathfrak{p}$ is a prime differential ideal in R^∞. If we write $\mathfrak{p}_0$ for the prime ideal $\eta^{-1}\mathfrak{p}$, where η is the natural map from R to R^∞, then $[[\mathfrak{p}_0]] \subset \mathfrak{p}$.*

Proof. Since the map $R \to R^\infty/\mathfrak{p}$ factors through $R/\mathfrak{p}_0$, there is an induced map $\theta : (R/\mathfrak{p}_0)^\infty \to R^\infty/\mathfrak{p}$, and the quotient map $R^\infty \to R^\infty/\mathfrak{p}$ factors through θ. Since the prolongation functor $(\ )^\infty$ commutes with localization, if k is the fraction field of $R/\mathfrak{p}_0$, the natural map $(R/\mathfrak{p}_0)^\infty \otimes_{R/\mathfrak{p}_0} k \to k^\infty$ is an isomorphism. Therefore there is a commutative diagram:

$$\begin{array}{ccccc}
R & \xrightarrow{\eta} & R^\infty & = & R^\infty \\
\downarrow & & \downarrow & & \downarrow \\
R/\mathfrak{p}_0 & \longrightarrow & (R/\mathfrak{p}_0)^\infty & \xrightarrow{\theta} & R^\infty/\mathfrak{p} \\
\downarrow & & \downarrow & & \downarrow \\
k & \longrightarrow & k^\infty \simeq (R/\mathfrak{p}_0)^\infty \otimes_{R/\mathfrak{p}_0} k & \longrightarrow & R^\infty/\mathfrak{p} \otimes_{R/\mathfrak{p}_0} k
\end{array}$$

Since the bottom left hand vertical map in this diagram is a localization, the other two bottom vertical maps are also localizations. Therefore, since $R^\infty/\mathfrak{p}$ is an integral domain, the right bottom vertical map is injective. Hence:

$$\mathfrak{p} = \mathrm{Ker}(R^\infty \to R^\infty/\mathfrak{p}) = \mathrm{Ker}(R^\infty \to R^\infty/\mathfrak{p} \otimes_{R/\mathfrak{p}_0} k)$$

and therefore:

$$[[\mathfrak{p}_0]] = \mathrm{Ker}(R^\infty \to k^\infty) \subset \mathfrak{p} = \mathrm{Ker}(R^\infty \to R^\infty/\mathfrak{p} \otimes_{R/\mathfrak{p}_0} k)$$

□

2.1.3. *Proof of Step 3.*

Lemma 2.3. *Let K and R be as above, and assume in addition that R is a one dimensional noetherian local domain. If $\mathfrak{p}$ is the maximal ideal in R, then with the notation above,*

$$[[0]] \subset [[\mathfrak{p}]]$$

Proof. Let A be the normalization of R, *i.e.* the integral closure of R in its fraction field F. By [5], V§2.1, Proposition 1, for any maximal ideal $\mathfrak{m} \lhd A$, $\mathfrak{m} \cap R = \mathfrak{p}$. Then by the Krull-Akizuki Theorem, (Proposition 5 of Chapter VII, §2.5, ([5])), A is a Dedekind domain. Hence for any maximal ideal $\mathfrak{m} \lhd A$, $A_\mathfrak{m}$ is a discrete valuation ring. Therefore, we have a commutative diagram:

$$\begin{array}{ccc}
R & \longrightarrow & R/\mathfrak{p} = k_0 \\
\downarrow & & \downarrow \\
A & \longrightarrow & A/\mathfrak{m} = k \\
\downarrow & & \\
F & &
\end{array}$$

in which all the vertical maps are injective, with $k_0 \subset k$ a separable field extension, and the inclusion $A \subset F$ is a localization. Since A is a discrete valuation ring, A^∞ is an integral domain, by Lemma 1.36, and $A^\infty \to F^\infty = A^\infty \otimes_A F$ is injective. Therefore $[[0]] = \mathrm{Ker}(R^\infty \to F^\infty) = \mathrm{Ker}(R^\infty \to A^\infty)$. Since $k_0 \to k$ is injective and k has characteristic zero, the field extension $k|k_0$ is formally smooth, has finite degree by the Krull-Akizuki Theorem, and is therefore étale. Hence, by étale base change, $k^\infty \simeq (k_0)^\infty \otimes_{k_0} k$ and so the map $(k_0)^\infty \to k^\infty$ is injective. Therefore $[[\mathfrak{p}]] = \mathrm{Ker}(R^\infty \to k^\infty)$. Since the map $R^\infty \to k^\infty$ factors through A^∞, we get that $[[0]] \subset [[\mathfrak{p}]]$. □

2.1.4. *Proof of Step 4.*

Lemma 2.4. *Suppose that $\mathfrak{p} \subset \mathfrak{b} \lhd R$ when $\mathfrak{b}$ has height 1 above $\mathfrak{p}$. Then $[[\mathfrak{p}]] \subset [[\mathfrak{b}]] \lhd R^\infty$*

Proof. We know by the previous lemma that, that since $R_\mathfrak{b}/(\mathfrak{p}R_\mathfrak{b}) \simeq (R/\mathfrak{p})_\mathfrak{b}$ is a one dimensional local ring, there is an inclusions of kernels:

$$\mathrm{Ker}((R_\mathfrak{b}/(\mathfrak{p}R_\mathfrak{b}))^\infty \to k(\mathfrak{p})^\infty) \subset \mathrm{Ker}((R_\mathfrak{b}/(\mathfrak{p}R_\mathfrak{b}))^\infty \to k(\mathfrak{b})^\infty)\ .$$

Hence there is also an inclusion,

$$\mathrm{Ker}(R^\infty \to k(\mathfrak{p})^\infty) \subset \mathrm{Ker}(R^\infty \to k(\mathfrak{b})^\infty)$$

as desired. □

2.1.5. *Proof of Step 5.* Since we are assuming that R is noetherian, by Krull's principal ideal theorem, (Theorem §13.5 of [17] or Section 12.E of [16]) the prime ideals in R satisfy the descending chain condition. In particular, if $\mathfrak{p} \lhd R$ is a prime ideal, there are only finitely many prime ideals between $0 \lhd R$ and $\mathfrak{p}$, and so step 5 follows by induction from Step 4.

2.1.6. *The non-noetherian case.* We shall give two proofs of the non-noetherian case. The first replaces Steps 3 and 5 with the following argument:

Lemma 2.5. *Let Λ be a differential ring, and R be a Λ-algebra which is a local domain. If $\mathfrak{p}$ is the maximal ideal in R, then with the notation above,*

$$[[0]] \subset [[\mathfrak{p}]] \, .$$

Proof. Now, rather than using integral closure and the Krull-Akizuki Theorem, we shall use Lemma 1.38 (which depends on uniformization), together with the existence of valuations with a given center.

Let F be the fraction field of R. By [5], Chapter VI, the Corollary in §1.2, there is a valuation ring $A \subset F$ which dominates R, *i.e.*, $R \subset A$, and if $\mathfrak{m} \lhd A$ is the maximal ideal of A, $\mathfrak{m} \cap R = \mathfrak{p}$. Then, we have a commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{p} = k_0 \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/\mathfrak{m} = k \\ \downarrow & & \\ F & & \end{array}$$

in which all the vertical maps are injective, and the inclusion $A \subset F$ is a localization. Since A is a valuation ring, A^∞ is an integral domain, by Lemma 1.38, and $A^\infty \to F^\infty = A^\infty \otimes_A F$ is injective. Therefore $[[0]] = \mathrm{Ker}(R^\infty \to F^\infty) = \mathrm{Ker}(R^\infty \to A^\infty)$. Since $k_0 \to k$ is injective and k is characteristic zero, the map $(k_0)^\infty \to k^\infty$ is injective by Lemma 1.28. Therefore $[[\mathfrak{p}]] = \mathrm{Ker}(R^\infty \to k^\infty)$. Since the map $R^\infty \to k^\infty$ factors through A^∞, we get that $[[0]] \subset [[\mathfrak{p}]]$. □

The second proof does not use general valuation rings, but rather deduces the non-noetherian case from the finitely generated case. Suppose therefore that R is an arbitrary integral domain in the category of k-algebras, *i.e.,* R is not necessarily finitely generated over k. Since the functor $(\ )^\infty$ commutes with direct limits, R^∞ is the direct limit of A^∞, as A runs through the partially ordered set of finitely generated subalgebras $A \subset R$. Since each such subalgebra A is an integral domain which is finitely generated over k, we know that each A^∞ has a single minimal prime. Note that the assertion that a ring has a single minimal prime is equivalent to saying that its nilradical is prime. Suppose now that a pair of elements $a, b \in R^\infty$ have nilpotent product: $(ab)^n = 0$ for some $n \in \mathbb{N}$. Then there exists a finitely generated $A \subset R$ such that $a, b \in A^\infty$, and $(ab)^n = 0$ in A^∞. By the finite type case of the theorem, it follows that one or other of a or b is nilpotent in A^∞, and hence in R^∞, and we are done.

This completes the proof of Kolchin's theorem.

3. Descent for Projective Varieties

Theorem 2. *Let X be a proper variety over an algebraically closed field K of characteristic zero. Let Δ be the K-vector space $H^0(X, \mathcal{D}er(\mathcal{O}_X))$ of global sections of the sheaf of derivations of the structure sheaf of X. Let K^Δ be the (algebraically closed) subfield of K consisting of elements fixed under the action of Δ. Then there exists a variety Y, proper over K^Δ, and an isomorphism*

$$X \simeq Y \otimes_{K^\Delta} K$$

i.e., *K^Δ is a field of definition for X. Furthermore, K^Δ is the minimal field of definition for X, in the sense that any other algebraically closed subfield $L \subset K$, which is a field of definition for X, contains K^Δ.*

3.1. Without loss of generality, we assume that X is connected. The strategy of the proof is as follows:

- Reduce to the case where K has finite transcendence degree over $\mathbb{Q}$.
- Reduce to the case where X is the generic fibre of a smooth proper family $\mathcal{X}$ over the spectrum of a Henselian discrete valuation ring $R^h \subset K$. The special fiber X_0 of $\mathcal{X}$ will then be defined over a field with smaller transcendence degree than that of K, and therefore it will be enough to show that if there is a horizontal vector field on $\mathcal{X}$, then $\mathcal{X} \simeq X_0 \times \operatorname{Spec}(R^h)$.
- Use the horizontal vector field to construct an isomorphism of formal schemes over the completion of R^h.
- Algebrize this formal isomorphism, *i.e.*, show that it is induced by an isomorphism of schemes (not just formal schemes) over the completion of R^h.
- Use Artin Approximation to deduce the existence of an isomorphism $\mathcal{X} \simeq X_0 \times \operatorname{Spec}(R^h)$ over R^h.

3.1.1. *Step 1.*

Lemma 3.1. *K^Δ is contained in all other fields of definition for X.*

(Hence it is enough to show that X is defined over K^Δ.)

Proof. Suppose that $F \subset K$ is a field of definition for X. Thus there is a variety Y, proper over F, and an isomorphism

$$X \simeq Y \otimes_F K\ .$$

On the category of F-algebras, $A \to \Omega_{A/F}$ commutes with direct limits (see, for example, [7] theorems 16.5 and 16.8).

Thus we have an isomorphism of sheaves on $X \simeq Y \otimes_F K$:

$$\Omega_{(\mathcal{O}_Y \otimes_F K)/F} \simeq (\mathcal{O}_Y \otimes_F \Omega_{K/F}) \oplus (K \otimes_F \Omega_{\mathcal{O}_Y/F})$$

and therefore there is an isomorphism

$$H^0(X, \mathcal{D}er_F(\mathcal{O}_X)) \simeq \mathcal{D}er_F(K) \oplus (K \otimes_F H^0(Y, \mathcal{D}er_F(\mathcal{O}_Y))\ .$$

It follows that $\mathcal{D}er_F(K) \subset \Delta = H^0(X, \mathcal{D}er(\mathcal{O}_X))$, and it therefore acts naturally on $K \simeq H^0(X, \mathcal{O}_X)$ with fixed field F, and hence $K^\Delta \subset F = K^{\mathcal{D}er_F(K)}$. □

Lemma 3.2. *There exists a field of definition F for X which is of finite transcendence degree over the prime field $\mathbb{Q}$.*

Proof. This is a standard argument: as a variety of finite type over K, X may be defined by a finite set of equations. We may then take F to be the algebraic closure of the subfield of K generated by these equations. □

3.1.2. *Step 2.* From the two lemmas of the previous sections, we see that we need only consider fields of definition K which have finite transcendence degree over $\mathbb{Q}$, and if such a field is not equal to K^{Δ}, then it must contain K^{Δ} as a subfield with $\mathrm{trdeg}(K/K^{\Delta}) > 0$.

Lemma 3.3. *In order to show that X may be defined over K^{Δ} it suffices to show that if there is a nonzero element $\xi \in H^0(X, \mathcal{D}er(\mathcal{O}_X))$, which acts non-trivially on $K \simeq H^0(X, \mathcal{O}_X)$ with X defined over a field K which is finitely generated over the prime field, then X is defined over a field F which has strictly smaller transcendence degree over the prime field than K does.*

Proof. Since, by the previous lemma, there exist fields of definition for X which have finite transcendence degree over $\overline{\mathbb{Q}}$, we know that we can choose a field of definition F for X which has minimal transcendence degree over K^{Δ}. Since F is a field of definition for X, we may write $X \simeq Y \otimes_F K$, with Y proper over F, If $F \neq K^{\Delta}$, then by definition of K^{Δ} there exists a nonzero element $\delta \in H^0(Y, \mathcal{D}er(\mathcal{O}_Y))$ which has a non-trivial action on F. Hence if we can show that this implies that Y is defined over a smaller field than F, this will contradict the minimality of F. □

3.1.3. *Step 3.* Next we "spread X out" over a discrete valuation ring $R \subset K$.

Proposition 3.4. *Suppose that X is proper and connected over an algebraically closed field K of finite transcendence degree over $\overline{\mathbb{Q}}$, and that there is a $\delta \in H^0(X, \mathcal{D}er(\mathcal{O}_X))$, such that $\mathrm{trdeg}(K/K^{\delta}) > 0$. (Note that δ acts on $K = H^0(X, \mathcal{O}_X)$). Then there exists a subfield $F \subset K$, with $\mathrm{trdeg}(K/F) > 0$ such that X is defined over F.*

In order to prove the proposition, we first need a lemma:

Lemma 3.5. *Let X and K be as above. Then there exists a discrete valuation ring $R \subset K$, such that:*

1. *R is a localization of an algebra which is smooth and of finite type over $\overline{\mathbb{Q}}$.*
2. *there is a scheme $\mathcal{X}$ which is proper, with geometrically connected fibres, over $S = \mathrm{Spec}(R)$ such that $\mathcal{X} \otimes_R K = X$.*
3. *there is a $\xi \in H^0(\mathcal{X}, \mathcal{D}er(\mathcal{O}_{\mathcal{X}}))$ which restricts to $\delta \in H^0(X, \mathcal{D}er(\mathcal{O}_X))$.*
4. *If we write $\overline{\xi}$ for the derivation of R induced by ξ, and f for the generator of the maximal ideal of R, then $\overline{\xi}(f)$ is a unit in R.*
5. *R contains a subfield E, such that the residue field k of R is a finite algebraic extension of E under the natural inclusion of $E \subset k$.*

Proof. Arguing as in the proof of Lemma 3.2, we first assume that there is a subring $\Lambda \subset K$ which is of finite type over $\overline{\mathbb{Q}}$, and a scheme $\mathcal{X}$ which is proper over $S = \mathrm{Spec}(\Lambda)$ such that $\mathcal{X} \otimes_\Lambda K = X$ and that δ extends to a derivation ξ of $\mathcal{O}$.

By generic smoothness of varieties over algebraically closed fields, we may assume after localization that $S = \mathrm{Spec}(\Lambda)$ is smooth over $\overline{\mathbb{Q}}$. Since S is smooth, and the geometric generic fibre X of $\mathcal{X}$ over S is connected, we know that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Lambda$, and hence ξ induces a derivation $\overline{\xi}$ of Λ (which agrees with δ via the inclusion $\Lambda \subset K$). If $\overline{\xi}$ is trivial, then $\Lambda \subset K^{\delta}$ and we can take $F = K^{\delta}$.

If $\overline{\xi}$ is non-zero, then there is a closed point $x \in S$ at which $\overline{\xi}$ (which may now be viewed as a section of the tangent bundle of S over $\overline{\mathbb{Q}}$) does not vanish. Since S is smooth at x, there is a regular system of parameters $\{z_1, \ldots, z_d\}$ of $\mathcal{O}_{S,x}$, and since $\overline{\xi}$ does not vanish at x, there is at least one i for which $\overline{\xi}(z_i)$ does not vanish at x. Without loss of generality, we may take $i = 0$. After further localization, we may assume that all the z_i are regular on S, that the induced map $\pi = (z_1, \ldots, z_d) : S \to \mathbb{A}^d_{\overline{\mathbb{Q}}}$ is étale, and that $\overline{\xi}(z_1)$ vanishes nowhere on S.

Let R be the discrete valuation ring which is the localization of Λ at the prime ideal generated by z_1. We may then take $f = z_1$. Then the map $\pi : S \to \mathbb{A}^d_{\overline{\mathbb{Q}}}$ realizes R as an étale extension of the discrete valuation ring $\overline{\mathbb{Q}}[z_1, z_2, \ldots, z_d]_{(z_1)}$, and the residue field k of R is a finite separable extension of $\overline{\mathbb{Q}}(z_2, \ldots, z_d)$ since π is étale. Since R contains $\overline{\mathbb{Q}}[z_2, \ldots, z_d]$ as a subring which injects into the residue field of R, it follows that there is an inclusion $\overline{\mathbb{Q}}(z_2, \ldots, z_d) \subset R$. Finally we set E equal to the algebraic closure of $\overline{\mathbb{Q}}(z_2, \ldots, z_d)$ in R. □

3.1.4. *step 4.* Let R^h be a strict Henselization of R. Then the algebraic closure of E in R^h maps isomorphically onto the residue field of R^h. Since the fraction field of R^h is an algebraic extension of the fraction field of R, and K is algebraically closed, there is an embedding $R^h \subset K$ extending $R \subset K$.

The isomorphism $\mathcal{X} \otimes_R K \simeq X$ extends uniquely to an isomorphism $\mathcal{X} \otimes_R R^h \otimes_{R^h} K \simeq X$. Since $R \subset R^h$ is formally étale, ξ lifts uniquely to a derivation of the structure sheaf of $\mathcal{X} \otimes_R R^h$, and this lift is compatible with base change from R^h to K. Note that $f \in R \subset R^h$ is also the generator of the maximal ideal of R^h.

Lemma 3.6. *Let (A, δ) be a differential ring, and $I = fA$ a principal ideal in A. Let $B := A/I$, and for $n \geq 0$, write A_n for A/I^n. Suppose that $\delta(f)$ is a unit mod I; then the composition of the ring homomorphisms:*

$$\phi : A \stackrel{\exp(\delta)}{\to} A[[t]] \to B[[t]] \ .$$

induces a compatible system of ring isomorphisms:

$$A_n \to B[t]/(t^n)$$

Proof. First observe that the composition of ϕ with the map $B[[t]] \to B$ sending t to 0 maps I to 0, and hence $\phi(I) \subset tB[[t]]$. Let us write $J = tB[[t]]$. To prove the lemma it suffices to show that for all $k \geq 0$ the induced map $I^k/I^{k+1} \to J^k/J^{k+1}$ is an isomorphism for all $k \geq 0$. We start with the case $k = 0$, where it is clear that $A/I \simeq B \to B[[t]]/tB[[t]] \simeq B$ is an isomorphism. For $k \geq 1$, I^k/I^{k+1} is a free rank one B-module generated by $f^k + I^{k+1}$ while J^k/J^{k+1} is a free rank one B-module generated by $t^k + J^{k+1}$. Since $\phi(f) = \delta(f)t + O(t^2)$,with $\delta(f)$ a unit, $\phi(f^k) = (\delta(f))^k t^k + O(t^{k+1})$ is a generator of J^k and it follows that ϕ induces an isomorphism $I^k/I^{k+1} \to J^k/J^{k+1}$ as desired. □

3.1.5. *Algebraization.* The existence of a formal isomorphism between $\mathcal{X}$ and a variety defined over a smaller field.

We may apply the lemma to $\mathcal{X}$ and the sheaf of ideals on $\mathcal{X}$ generated by f. The vector field ξ on $\mathcal{X}$ induces maps

$$\phi_n : \mathcal{X}_n \to Y \otimes_{F=R/(f)} A[t]/(t^n) \qquad ,$$

where Y is the fibre of $\mathcal{X}$ over the closed point of $\mathrm{Spec}(R)$, which by the lemma are isomorphisms, *i.e.*, ξ induces an isomorphism of formal schemes

$$\widehat{\phi} = \exp(\xi) : \widehat{\mathcal{X}} \to Y \widehat{\otimes}_{F=R/(f)} \widehat{\mathbb{A}^1}$$

where $\widehat{\mathcal{X}}$ is the formal scheme which is the formal completion of $\mathcal{X}$ with respect to the sheaf of ideals generated by f. This isomorphism is an isomorphism of formal schemes over the formal scheme $\mathrm{Specf}(\widehat{R} \simeq F[[t]])$, and hence by [9], Ch. III, Th. 5.4.1, since $\mathcal{X}$ and Y are both proper over $\widehat{R} \simeq F[[t]]$, $\exp(\xi)$ is algebraizable, *i.e.*, it is induced by an isomorphism of schemes (not just formal schemes!)

$$\phi : \mathcal{X} \otimes_R \widehat{R} \to Y \otimes_{F=R/(f)} (\widehat{R} \simeq F[[t]]).$$

3.1.6. *Approximation.* Now Artin Approximation [1] implies, since R is Henselian, that there exists an isomorphism

$$\phi' : \mathcal{X} \to Y \otimes_{F=R/(f)} R \qquad .$$

(Note that ϕ' may be chosen to agree with ϕ (and hence $\widehat{\phi}$) to any given finite order, but does not necessarily induce the map $\widehat{\phi}$.) Finally we observe that by taking $\otimes_R K$,we obtain an isomorphism

$$X = \mathcal{X} \otimes_R K \to Y \otimes_F K$$

and we are done.

3.2. **Remark.** The only place that we used that X is proper was in the algebraization of isomorphisms in the category of formal schemes over $\mathrm{Specf}(F[[t]])$. Thus Buium's result will hold for any subcategory of the category of schemes with this property.

4. Complements and Questions

4.1. **Hasse-Schmidt Differentiation.** It is natural to ask to ask to what extent the methods used above to prove Buium's theorems can be used in characteristic p.

Note that the proof of Lemma 3.6 does not use that δ is a derivation, but only that ϕ is a ring homomorphism. In general a homomorphism from a ring R to the ring $R[[t]]$ of formal power series over R which, when composed with the augmentation $R[[t]] \to R$ which sends t to 0, is the identity, is known as a *Hasse-Schmidt Differentiation.* More generally we, define:

Definition 4.1. *A* Hasse-Schmidt Differentiation *or* flow ϕ *on a scheme* X *is a map of formal schemes* $\phi : X\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[[t]] \to X$*, which when composed with the map* $X \to X\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[[t]]$ *given by setting* $t = 0$ *is the identity.*

In order for the differentiation to be the exponential of a derivation, you need additional information:

Definition 4.2. *Let* X *be a scheme over a base* S*, and let* $\mathcal{G}$ *be a one parameter formal group over* S*. A flow* ϕ *on a scheme* X *over* S *is said to be* $\mathcal{G}$-iterative, *with respect to* S*, if* ϕ *is an action of* $\mathcal{G}$ *on* X*. Note that in characteristic zero, all one parameter formal groups are isomorphic to the formal additive group* $\mathcal{G}_a$*, in which case a flow is iterative if and only if it is the integral of a vector field. See* [17].

In Lemma 3.6, we are given a scheme $p : \mathcal{X} \to S$, proper, and geometrically connected over the spectrum $S = \mathrm{Spec}(\Lambda)$ of a discrete valuation ring, and a flow ϕ on with the property that if ξ is the associated vector field, and π is the generator of the maximal ideal in Λ then $\xi(\pi)$ is a unit modulo π. Of course once one no longer assumes that the flow is the exponential of the associated vector field, the flow is not determined by a finite amount of data; thus one cannot argue as in 3.1.3 that one can reduce to a field of finite transcendence degree over the prime field.

Notice that given a flow $\phi : A \to A[[t]]$, if we write the flow as:

$$\phi : a \mapsto \sum_i D_i(a)t^i$$

then each D_i is a differential operator of order i. (See [11], §16 for the definition of a differential operator.) If ϕ is the exponential of a vector field, then $D_i = (D_1)^i/i!$.

Thus one can ask:

Question 1. *Suppose that X is a variety, projective and geometrically connected over a field k, possibly of characteristic greater than zero. Consider the algebra $\mathcal{D} = H^0(X, \mathcal{D}iff_k(\mathcal{O}_X, \mathcal{O}_X))$ of global sections of the sheaf $\mathcal{D}iff_k(\mathcal{O}_X, \mathcal{O}_X)$ of differential operators on $\mathcal{O}_X$. Then $\mathcal{D}$ acts on $H^0(X, \mathcal{O}_X) \simeq k$, and we ask whether X is defined over the field of constants $k^{\mathcal{D}}$, or at least over its algebraic closure.*

4.2. **Derivations and Valuation Rings.** Recall that a valuation ring is a local domain R such that if $x \in F$, F being the fraction field of R, then $x \notin R$ if and only if $1/x \in \mathfrak{m}$, $\mathfrak{m}$ being the maximal ideal of R. The quotient F^*/R^* is an abelian group which is totally ordered by $v(x) \geq 0$ if and only if $x \in R$, where $v : F^* \to F^*/R^*$ is the quotient map. Given a totally ordered abelian group Γ, a *valuation* v on F with values in Γ is a homomorphism $v : F^* \to \Gamma$, such that $v(x + y) \geq \mathrm{Min}(v(x), v(y))$. Given such a valuation, the set $\{x \in F | v(x) \geq 0\}$ is a subring R_v, which is easily seen to be a valuation ring with maximal ideal $\mathfrak{m}_v = \{x \in F | v(x) > 0\}$, and the valuation induces an (order preserving) injection $F^*/R_v^* \subset \Gamma$. Generally when we speak of a valuation on a field we assume that this inclusion is the identity. See [5] Chapter VI, [21], and [8] for more details on valuation rings.

Valuation rings are in general *not* noetherian, however they do still have some nice properties:

Lemma 4.3. *Let R be a valuation ring. Then*

(1) *Any finitely generated torsion free R-module is free.*
(2) *Any torsion free R-module is flat.*

Proof. Exercise; see [5] Lemma 1 of VI.3.6. □

Question 2. *Let R be a valuation ring containing a differential field of characteristic zero. Is there a "simple" proof (or at least a proof that is not equivalent to proving uniformization) that R^∞ is an integral domain?*

For example it would be enough to know that $\Omega_{R/k}$ is torsion free. For by Lemma 4.3 this would imply that $\Omega_{R/k} = \bigcup_\alpha F_\alpha$ is a union of free submodules, and hence that $R^{\{1\}} = \varinjlim_\alpha \mathbb{S}^*_R(F_\alpha)$ is a direct limit of polynomial rings over R, where $R^{\{1\}}$ is the first stage in the sequence of rings of Section 1.2.2. $R^{\{1\}}$ is therefore an integral domain which is formally smooth over R. It is then easy to show that $R^{\{n\}}$ is formally smooth over R for all n. However I am told by at least one of the experts that this statement is quite probably equivalent to uniformization.

Another remark is that since R is a domain, we know by Kolchin's theorem that R^∞ has a unique minimal ideal. Hence it would also suffice to show that R^∞ is reduced.

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Department of Mathematics, Statistics, and Computer Science, (m/c 249), University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607

E-mail address: henri@math.uic.edu