Math 313 , Introduction to Analysis
uniform continuity - worked example 1

Example: Show that the function $f(x) = x^2$ on the closed interval $[1, 3]$ is uniformly continuous.

Solution: Given $\epsilon > 0$ we want to find $\delta > 0$ that works for every point in the interval $[1, 3]$.

This means that for all $1 \le c \le 3$ and any $|h| < \delta$, we have $|f(c+h) - f(c)| < \epsilon$.

Step 1 - use the definition of $f(x)$ to reduce what we are trying to obtain to a simpler form

$$|f(c+h) - f(c)| = |(c+h)^2 - c^2| = |(c^2 + 2ch + h^2) - c^2| = |2ch + h^2| \le |2ch| + |h^2|$$

So it is enough to find $\delta > 0$ small enough so that the two terms on the right side above satisfy $|2ch| < \epsilon/2$ and $h^2 < \epsilon/2$ for every $1 \le c \le 3$.

Step 2 - find a condition that makes $|2ch| < \epsilon/2$. Use that $c \le 3$, so $|2ch| \le |2 \cdot 3 \cdot h| = 6|h|$.

So if $6|h| < \epsilon/2$, or $|h| < \epsilon/12$, then $|2ch| < \epsilon/2$ is satisfied.

Step 3 - find a condition on δ which makes $h^2 < \epsilon/2$. Take square roots of both sides to see that we need $|h| < \sqrt{\epsilon/2} = \sqrt{\epsilon}/\sqrt{2}$. There is no c in this equation, so assume $\delta \le \sqrt{\epsilon}/\sqrt{2}$ and this works for any value of c.

Step 4 - we need both upper bounds on δ to be true, so take $\delta = \min\{\sqrt{\epsilon}/\sqrt{2}, \epsilon/12\}$. That'll do it.

Since $\delta \le \sqrt{\epsilon}/\sqrt{2}$ then $|h| < \delta$ implies $h^2 < \epsilon/2$.

Since $\delta \le \epsilon/12$ then $|h| < \delta$ implies $|6h| < \epsilon/2$ so that $|2ch| < \epsilon/2$ for all $1 \le c \le 3$.

Combine these two, so $|h| < \delta$ implies

$$|f(c+h) - f(c)| = |(c+h)^2 - c^2| = |2ch + h^2| \le |2ch| + |h^2| < \epsilon/2 + \epsilon/2 = \epsilon$$