Tests for Convergence of a Series

DEFINITION 1 (Convergence) *An infinite series* $\sum_{i=1}^{\infty} = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ **converges to L** *if the sequence of partial sums*

$$s_n = a_1 + a_2 + \cdots a_n$$

converges to a limit L.

This definition says that as we add the terms in the infinite string above, the answer gets closer and closer to L, and does not "pop around".

TEST 1 (Zero Test) *If the series* $\sum_{i=1}^{\infty} a_i$ *converges, then the terms* $a_i \to 0$.

USE 1 The test says that if the terms a_i *do not go to zero*, then there is **no way** for the series of partial sums to converge. Done. Does NOT converge.

TEST 2 (Integral Test) *Let* $a_i = f(i)$, *where f(x) is a continuous function with* $f(x) > 0$, *and is decreasing. Then*

$$\text{the series } \sum_{i=1}^{\infty} a_i \ \textbf{converges} \text{ if the improper integral } \int_1^{\infty} f(x)dx < \infty.$$

$$\text{the series } \sum_{i=1}^{\infty} a_i \ \textbf{diverges} \text{ if the improper integral } \int_1^{\infty} f(x)dx = \infty.$$

USE 2 One application is the convergence of the "p-series":

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \textbf{converges} \text{ if } \ p > 1, \text{and} \quad \textbf{diverges} \text{ if } \ p \leq 1$$

TEST 3 (Comparison Test) *Suppose that* $\sum_1^{\infty} a_i$ *and* $\sum_1^{\infty}$ *are series with all terms positive - so* $a_i \geq 0$ *and* $b_i \geq 0$.

$$\sum_{i=1}^{\infty} b_i \ \text{ is } \textbf{convergent} \text{ and } \ a_i \leq b_i \ \text{ for all i } \Longrightarrow \ \sum_{i=1}^{\infty} a_i \ \text{ is } \textbf{convergent}.$$

$$\sum_{i=1}^{\infty} b_i \ \text{ is } \textbf{divergent} \text{ and } \ a_i \geq b_i \ \text{ for all i } \Longrightarrow \ \sum_{i=1}^{\infty} a_i \ \text{ is } \textbf{divergent}.$$

USE 3 This is the "squeeze test" for infinite series. Use it to justify the "cover-up" method of guessing whether a series converges or diverges.

TEST 4 (Limit Comparison Test) *Suppose that* $\sum_1^\infty a_i$ *and* $\sum_1^\infty b_i$ *are series with all terms positive.*

$$\lim_{i\to\infty} \frac{a_i}{b_i} = c > 0 \quad \Longrightarrow \quad \sum_{i=1}^{\infty} a_i \quad \text{and} \quad \sum_{i=1}^{\infty} b_i \ \text{ either } \textbf{both converge}, \text{ or } \textbf{both diverge}.$$

$$\lim_{i\to\infty} \frac{a_i}{b_i} = 0 \quad \text{and} \quad \sum_{i=1}^{\infty} a_i \ \textbf{converges} \ \Longrightarrow \quad \text{the series} \quad \sum_{i=1}^{\infty} b_i \quad \textbf{converges}.$$

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \infty \quad \text{and} \quad \sum_{i=1}^{\infty} b_i \ \textbf{diverges} \ \Longrightarrow \quad \text{the series} \quad \sum_{i=1}^{\infty} a_i \quad \textbf{diverges}.$$

USE 4 This is one of the most powerful tests, because it squeezes the two series "in the limit". Just be sure to use it right! Part one is clear, but **don't mix up the second and third parts**.

TEST 5 (Alternating Series Test) *For the* alternating series - *where all* $a_i > 0$

$$\sum_{i=1}^{\infty} (-1)^i a_i \quad = \quad a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

$$a_i \geq a_{i+1} \textit{ for all } i \textbf{ and } \lim_{i\to\infty} a_i = 0 \quad \Longrightarrow \quad \sum_{i} (-1)^i a_i \ \textbf{converges}.$$

DEFINITION 2 (Absolute Convergence)

$$\sum_{i}^{\infty} a_i \ \text{ is } \textbf{absolutely convergent} \quad \Longleftrightarrow \quad \text{the sum of } \textbf{absolute values} \quad \sum_{i}^{\infty} |a_i| \ \text{ is convergent.}$$

TEST 6 (Ratio Test)

$$\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = L < 1 \quad \Longrightarrow \quad \sum_{i=1}^{\infty} |a_i| \quad \textbf{converges} \quad \Longrightarrow \quad \sum_{i=1}^{\infty} a_i \quad \textbf{converges}.$$

$$\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = L > 1 \quad \Longrightarrow \quad \sum_{i=1}^{\infty} a_i \quad \textbf{diverges}.$$

TEST 7 (Root Test)

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} |a_n| \quad \textbf{converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \quad \textbf{converges}.$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \quad \textbf{diverges}.$$