Homotopy invariants of foliations

by

S. Hurder and F. W. Kamber*)

1. In this note we propose to study the homotopy groups of $B\Gamma_G^q$, the classifying space of G-foliated microbundles [H1] A foliation F on a manifold X is a G-foliation if it is defined by local submersions into a q-dimensional model manifold B, such that the local transition functions preserve a G-structure on B. With respect to an adapted connection ω, the Chern-Weil homomorphism defines a map $h(\omega):I(G)_\ell \longrightarrow A^\bullet(X)$, where $I(G)_\ell$ is the ring of invariant polynomials on G modulo the ideal of elements of degree $> 2\ell$. The index ℓ depends on G and whether the G-structure is integrable; it can always be taken $\leq q$ [B].

For any commutative DG-algebra A, we let $\varphi:M_A \longrightarrow A$ be a minimal model [S]. Let $\overline{M}_A$ denote the augmentation ideal; the quotient $\pi^*(A) \overset{\text{def}}{=} \overline{M}_A/\overline{M}_A{}^2$ is called the dual homotopy of A. For any (semi-simplicial) manifold X, we set $\pi^*(X) = \pi^*(A^\bullet(X))$ [D], [S], where $A^\bullet(X)$ is the deRham algebra in the sense of Sullivan-Dupont. If X is 1-connected and of finite rational type, then there is a natural isomorphism $\pi^*(X) \cong \mathrm{Hom}(\pi_*(X),\mathbb{R})$. For a DG-algebra A, whose cohomology is of finite type, we define

$$\pi_*(A) = \mathrm{Hom}_{\mathbb{R}}(\pi^*(A),\mathbb{R}).$$

Let X be a G-foliated manifold. The following result is proved in [Hu 1]

1.1. THEOREM. *The Chern-Weil homomorphism induces a map* $h^\#:\pi^*(I(G)_\ell) \longrightarrow \pi^*(X)$ *which depends only on the concordance class of the foliation.*
If X is of finite rational type, the map $h^\#$ induces by transposition a mapping

(1.2) $$h_\#:\Pi(X) \longrightarrow \Pi(I(G)_\ell),$$

*) Partially supported by NSF grant MCS79-00256

where $\Pi(X) = s^{-1}(\pi_*(X)\otimes\mathbb{R})$ and $\Pi(A) = s^{-1}\pi_*(A)$ are the desuspended graded homotopy groups. It is known [B-L] that the functor Π has values in the category of graded Lie algebras. As $h(\omega):I(G)_\ell \longrightarrow A^\bullet(X)$ is a homomorphism of DG-algebras, $h_\#$ is a homomorphism of graded Lie algebras. This construction extends to G-microbundles in an obvious way and therefore defines a homomorphism of Lie algebras

$$\tilde{h}_\#:\Pi(B\Gamma^q_G) \longrightarrow \Pi(I(G)_\ell).$$

If $f:X \longrightarrow B\Gamma^q_G$ denotes the classifying map of the G-foliation F on X, the diagram

(1.3)
$$\begin{array}{ccc} \Pi(X) & \xrightarrow{h_\#} & \Pi(I(G)_\ell) \\ & {\scriptstyle f_\#}\searrow \quad \swarrow{\scriptstyle \tilde{h}_\#} & \\ & \Pi(B\Gamma^q_G) & \end{array}$$

is commutative. It is the purpose of this note to determine the Lie algebra structure of $\Pi(I(G)_\ell)$ (Section 2) and to detect elements in the image of $\tilde{h}_\#$ via appropriate choices of (X,f) (Section 4). In 3 we study the relationship of $h_\#$ with the characteristic homomorphism Δ_* for G-foliations [K-T 1], [K-T 2].

2. *The structure of* $\Pi(I(G)_\ell)$

In this section we determine the structure of the graded Lie algebra $\Pi(I(G)_\ell)$. Let G be a reductive Lie group. In order to simplify the following discussion, we will assume that G is connected in which case $I(G) \cong \mathbb{R}[c_1,\ldots,c_r]$ is a polynomial algebra generated by the characteristic classes c_j of even degree. As before, we denote by

$$I_\ell = I(G)_\ell = \mathbb{R}[c_1,\ldots,c_t]/(\phi(c_1,\ldots,c_t)\,|\deg\phi > 2\ell)$$

the truncated polynomial algebra, where $c_1,\ldots,c_t$ denote the generators of degree $\leq 2\ell$.

Let $A_\ell = \Lambda P_{(2\ell)} \otimes I_\ell$ be the DG-algebra introduced in section 3. The inclusion $0 \longrightarrow I_\ell \longrightarrow A_\ell$ dualizes to give an epimorphism of DG-coalgebras, $A_\ell^* \xrightarrow{j^*} I_\ell^* \longrightarrow 0$. Applying Quillen's L construction [Q], [B-L], we get an exact sequence of free DG-Lie algebras

$$0 \longrightarrow \ker j^* \longrightarrow L(A_\ell^*) \longrightarrow L(I_\ell^*) \longrightarrow 0, \tag{2.1}$$

where $L(C) \equiv \mathbb{L}(s^{-1}\bar{C})$ is the free DG-Lie algebra generated by a suspended reduced DG-coalgebra C [B-L], [N-M]. Passing to cohomology we get an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_\cdot(L(A_\ell^*)) & \longrightarrow & H_\cdot(L(I_\ell^*)) & \xrightarrow{\delta} & H_{\cdot-1}(\ker j^*) & \longrightarrow & 0 \\ & & \| \wr & & \| \wr & & & & \\ & & \Pi(A_\ell) & \xrightarrow{j_\#} & \Pi(I_\ell) & & & & \end{array} \tag{2.2}$$

2.3. THEOREM. *There is an extension of graded Lie algebras*

$$0 \longrightarrow \Pi(A_\ell) \longrightarrow \Pi(I_\ell) \longrightarrow P_{(2\ell)}^* \longrightarrow 0,$$

where $P_{(2\ell)}^*$ *is an abelian Lie algebra and* $\Pi(A_\ell) \cong L(H^\cdot(A_\ell)^*)$ *is a free Lie algebra.*

We remark that the Lie algebra structure of the extension $\Pi(I_\ell)$ is uniquely determined by the induced representation of $P_{(2\ell)}^*$ in the Lie algebra of outer derivations of $\Pi(A_\ell)$. This follows from the fact that the free Lie algebra $\Pi(A_\ell)$ has trivial center and from the general theory of extensions of Lie algebras (compare [Ho] for the ungraded case). The proof of this theorem, culminating in the determination of this induced action, will occupy the rest of this section.

First note that by taking a Λ-minimal model of the KS extension $0 \longrightarrow I_\ell \longrightarrow A_\ell \longrightarrow \Lambda P_{(2\ell)} \longrightarrow 0$ [Ha], there is a long exact dual homotopy sequence with injective coboundary ∂^* and therefore a short exact sequence

$$0 \longrightarrow \partial^* P_{(2\ell)} \longrightarrow \pi^*(I_\ell) \longrightarrow \pi^*(A_\ell) \longrightarrow 0. \tag{2.4}$$

Dualizing this sequence gives the exact sequence of the theorem. The elements of $P^*_{(2\ell)}$ all have odd degree, so as a Lie algebra this must be abelian. We want to analyze how the elements in $P^*_{(2\ell)}$ act on the image of $\Pi(A_\ell)$, and this will show that $\Pi(A_\ell)$ is an ideal in $\Pi(I_\ell)$.

The algebra A_ℓ admits a subalgebra $Z_\ell \subset A_\ell$ with trivial differential and products, which induces an isomorphism in cohomology $Z_\ell \xrightarrow{\cong} H^\bullet(A_\ell)$ [K-T3]. Therefore A_ℓ is biformal and we have isomorphisms

$$(2.5) \qquad L(Z^*_\ell) \cong H_\bullet(L(Z^*_\ell)) \xrightarrow{\cong} H_\bullet(L(A^*_\ell)) \cong \Pi(A_\ell).$$

It follows that $\Pi(A_\ell)$ is a free graded Lie algebra generated by $s^{-1}Z^*_\ell$.

The algebra Z_ℓ and the isomorphism $Z_\ell \cong H(A_\ell)$ have been described in [K-T1], [K-T3]. We use here a slightly different notation, which is more convenient in the present context. For ordered sequences $I = (i_1 < \cdots < i_s)$, $J = (j_1 \le \cdots \le j_m)$, the symbol (I/J) is called *admissible* if it satisfies

$$(2.6) \qquad \deg c_J \le 2\ell, \quad c_J = \prod_{\alpha=1}^{m} c_{j\alpha};$$

$$(2.7) \qquad \deg c_{i_1} c_J > 2\ell;$$

$$(2.8) \qquad i_1 \le j_1.$$

For an admissible symbol (I/J), the cochain

$$z_{(I/J)} = y_I \otimes c_J = y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes c_{j_1} \cdots c_{j_m} \in A_\ell$$

is clearly a cocycle and the product of any two such cocycles $= 0$. The algebra Z_ℓ is then given by the linear space spanned by 1 and the cocycles $z_{(I/J)}$ for (I/J) admissible.

Let $I = \mathbb{R}[c_1, \ldots, c_t]$; the canonical quotient map $I \longrightarrow I_\ell$ dualizes to an inclusion $L(I^*_\ell) \subset L(I^*)$. Since I^* has a canonical Hopf algebra structure and trivial differential, we find easily that

$$H_\bullet(L(I^*)) \cong P^*_{(2\ell)} = \operatorname{span}\{Y_1, \ldots, Y_t\},$$

where $Y_j = s^{-1}c_j^*$, $j = 1,\ldots,t$. Hence all the cycles in $L(I^*)$ of degree $\geq 2\ell$ are boundaries. Thus all the cycles in $L(I_\ell^*)$ of degree $\geq 2\ell$ are boundaries in $L(I^*)$. With this observation in mind, we produce explicitly a set of cycles in $L(I_\ell^*)$, which will generate $H(L(I_\ell^*))$.

For any monomial c_K, $K = (k_1 \leq \cdots \leq k_m)$ in I, we set $Y_K = s^{-1}c_K^* \in I^*$. The diagonal Δ in I^* is given by

$$\Delta(c_K^*) = c_K^* \otimes 1 + 1 \otimes c_K + \frac{1}{2}\sum_{(\alpha,\beta)} (c_\alpha^* \otimes c_\beta^* + c_\beta^* \otimes c_\alpha^*), \tag{2.9}$$

where (α,β) runs over all ordered proper partitions of the set $\{k_1,\ldots,k_m\}$. By definition, the differential d_L of $L(I^*)$ is determined by the formula

$$d_L Y_K = \frac{-1}{2}\sum_{(\alpha,\beta)} [Y_\alpha, Y_\beta] \in L(I^*). \tag{2.10}$$

For an admissible symbol (I/J) with $I = (i)$, it follows that $d_L Y_{(i,J)} \in L(I_\ell^*)$ is a cycle of degree $\geq 2\ell$, and we define

$$s^{-1}u^*_{(i/J)} = -d_L Y_{(i,J)} = \frac{1}{2}\sum_{(\alpha,\beta)} [Y_\alpha, Y_\beta] \tag{2.11}$$

If (I/J) is an arbitrary admissible symbol, we set

$$s^{-1}u^*_{(I/J)} = \mathrm{ad}(Y_{i_s})\circ\cdots\circ\mathrm{ad}(Y_{i_2})s^{-1}u^*_{(i_1/J)}, \quad s > 1, \tag{2.12}$$

where $\mathrm{ad}(Y) = [Y,-]$ denotes the adjoint representation. Clearly the Y_j and the $s^{-1}u^*_{(I/J)}$ are cycles in $L(I_\ell^*)$. Their corresponding homology classes in $H_*(L(I_\ell^*))$ are denoted by the same symbol. Observe that the elements $s^{-1}u^*_{(i/J)}$ correspond exactly to a minimal set of relations $c_i c_J \sim 0$ for the quotient algebra I_ℓ. For $z_{(I/J)} \in Z_\ell$, denote by $z^*_{(I/J)}$ the corresponding dual basis element of Z_ℓ^*.

2.13. LEMMA. *The injective homomorphism* $j_\#\colon \Pi(A_\ell) \longrightarrow H(I_\ell)$ *of Theorem 2.3 is induced by the homomorphism of free DG-Lie algebras* $L(Z_\ell^*) \longrightarrow L(I_\ell^*)$, *which is determined on the generators by* $s^{-1}z^*_{(I/J)} \longrightarrow s^{-1}u^*_{(I/J)}$ *for* (I/J) *admissible.*

It follows that the homology classes $s^{-1}u^*_{(I/J)}$ generate a free subalgebra in $L(I^*_\ell)$. The formulas in the following Proposition have to be understood with the convention: Whenever a symbol (I'/J') is not admissible, the term in which the symbol occurs must be replaced by 0.

2.14. PROPOSITION. Let (I/J) be admissible and $k \leq j$ in $\{1,\dots,t\}$. Then the following formulas hold in $H_\bullet(L(I^*_\ell))$:

(2.15) $\mathrm{ad}(Y_k)(Y_j) = s^{-1}u^*_{(k/j)}$;

(2.16) $\mathrm{ad}(Y_k)s^{-1}u^*_{(I/J)} = s^{-1}u^*_{(i_1\cdots i_s\,k|J)}$, for $k > i_s$;

(2.17) $$\mathrm{ad}(Y_k)s^{-1}u^*_{(I/J)} = \sum_{\beta=\alpha}^{s}(-1)^{s-\beta}\mathrm{ad}(Y_{i_s})\circ\cdots\circ\mathrm{ad}(Y_{i_{\beta+1}})[s^{-1}u^*_{(k/i_\beta)}, s^{-1}u^*_{(i_1\ i_{\beta-1}|J)}]$$
$$+ (-1)^{s-\alpha+1}s^{-1}u^*_{(i_1\cdots i_{\alpha-1}k i_\alpha i_s|J)},\quad \text{for } i_{\alpha-1} < k < i_\alpha,\ 1 < \alpha \leq s;$$

(2.18) $$\mathrm{ad}(Y_k)s^{-1}u^*_{(I/J)} = \sum_{\beta=\alpha}^{s}(-1)^{s-\beta}\mathrm{ad}(Y_{i_s})\circ\cdots\circ\mathrm{ad}(Y_{i_{\beta+1}})[s^{-1}u^*_{(k/i_\beta)}, s^{-1}u^*_{(i_1\ i_{\beta-1}|J)}]$$
$$+ (-1)^{s+1}\mathrm{ad}(Y_{i_s})\circ\cdots\circ\mathrm{ad}(Y_{i_2})\circ\mathrm{ad}(Y_k)s^{-1}u^*_{(i_1|J)},\quad \text{for } k < i_1;$$

(2.19) $$\mathrm{ad}(Y_k)s^{-1}u^*_{(i/J)} = -\sum_{\beta=0}^{m} s^{-1}u^*_{(k,j_\beta/j_0\cdots\hat{j}_\beta\cdots j_m)},\quad \text{for } k < j_0 = i.$$

Together with corresponding formulas for $k = i_\alpha$, $\alpha = 1,\dots,s$, (2.15) to (2.19) completely determine the Lie algebra extension in Theorem 2.3 and hence the structure of $H(I_\ell) = H(L(I^*_\ell))$. They also show that the subalgebra $L(Z^*_\ell) \subset H(L(I^*_\ell))$ is an ideal. We denote by $D(Y_k)$ the derivation on $L(Z^*_\ell)$ induced by $\mathrm{ad}(Y_k)$, $k = 1,\dots,t$. (2.15) implies

(2.20) $$[D(Y_k), D(Y_j)] = D(Y_k)\circ D(Y_j) + D(Y_j)\circ D(Y_k) = \mathrm{ad}(s^{-1}z^*_{(k|j)}).$$

D therefore induces a representation $\tilde{D}$ of the abelian Lie algebra $\Gamma^*_{(2\ell)}$ in $\mathrm{Der}(L(Z^*_\ell))/\mathrm{IntDer}$. This is the representation canonically associated to the extension (2.3).

As an example, we describe $\Pi(I_2)$ for $I = I(\underline{\underline{gl}}(2)) = \mathbb{R}[c_1,c_2]$. We have $I_2 = [c_1,c_2]/(c_1^3,c_1c_2,c_2^2)$. The Lie algebra $\Pi(I_2) \cong H(L(I^*))$ is generated by $Y_j = s^{-1}c_j^*$, $j = 1, 2$ and $s^{-1}u^*_{(1/11)} = [Y_1,Y_{(1,1)}]$, $dY_{(1,1)} = [Y_1,Y_1]$. The free subalgebra $L(Z_2^*) \subset H(L(I_2^*))$ is generated by $s^{-1}u^*_{(1,11)}$ and the elements $s^{-1}u^*_{(1/2)} = [Y_1,Y_2]$, $s^{-1}u^*_{(2/2)} = [Y_2,Y_2]$, $s^{-1}u^*_{(12/11)} = [Y_2,[Y_1,Y_{(1,1)}]]$ and $s^{-1}u^*_{(12/2)} = [Y_2,[Y_1,Y_2]]$. The non-zero brackets in Prop. 2.14 are given by

$$[Y_1,s^{-1}u^*_{(2/2)}] = -\,2s^{-1}u^*_{(12/2)},$$

$$[Y_1,s^{-1}u^*_{(12/11)}] = = [s^{-1}u^*_{(1/2)},s^{-1}u^*_{(1/11)}],$$

$$[Y_1,s^{-1}u^*_{(12/2)}] = [s^{-1}u^*_{(1/2)},s^{-1}u^*_{(1/2)}],$$

$$[Y_2,s^{-1}u^*_{(12/11)}] = \frac{1}{2}\,[s^{-1}u^*_{(2/2)},s^{-1}u^*_{(1/11)}]$$

and

$$[Y_2,s^{-1}u^*_{(12/2)}] = \frac{1}{2}\,[s^{-1}u^*_{(2/2)},\ s^{-1}u^*_{(1/2)}].$$

Theorem 2.3. has the following consequence

2.21. THEOREM. *The minimal algebra* $M(I_\ell)$ *appears as an extension of* DG-*algebras*

(2.22)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & Id(c_1,\dots,c_t) & \longrightarrow & M(I_\ell) & \xrightarrow{Mj} & M(A_\ell) & \longrightarrow & 0 \\
 & & \| & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & Id(c_1,\dots,c_t) & \longrightarrow & C^\bullet(\Pi(I_\ell)) & \longrightarrow & C^\bullet(L(Z_\ell^*)) & \longrightarrow & 0,
\end{array}$$

where $C^\bullet(L) \equiv S^\bullet(sL)$ *denotes the cochain complex of a graded Lie algebra. As* A_ℓ *is biformal, the isomorphism* $M(A_\ell) \cong C^\bullet(L(Z_\ell^*))$ *preserves differentials. For* I_ℓ, *which is only formal, the cochain differential* d_C *describes only the quadratic terms of* d_M.

The 1-cochains $u_{(I/J)} \in M(I_\ell)$, dual to the basis elements $u^*_{(I/J)}$ described earlier, are mapped to the cocycles $z_{(I/J)} \in Z_\ell \subset A_\ell$. The obvious relation in $M(I_\ell)$

(2.23) $$d_M u_{(i/J)} = c_i c_J$$

shows that d_M has non-quadratic terms (for c_J decomposable); hence I_ℓ is not coformal for $\ell > 1$. The minimal algebras of the form $C\,(L(Z_\ell^*))$ have the homotopy type of a finite wedge of spheres, and their study goes back to P. J. Hilton (compare e.g. [H1], [H2]). By contrast the minimal algebra $M(I_\ell)$ appears to be quite complicated as far as the differential is concerned. Details of these constructions will appear elsewhere [K-T4] .

3. *The relationship of* $h^\#$ *with* Δ_*

Let X have a G-foliation; we assume that the G-frame bundle F(Q) of the normal bundle admits an H-reduction, where $H \subseteq G$ is closed and (G,H) is a reductive, CS-pair [K-T2]. Let $P \subseteq \Lambda \underline{\underline{g}}^*$ be the space of primitives: $P \cong \text{span}\{Y_1,\ldots,Y_r\}$ where Y_j is the cohomology suspension of c_j. Let $P = \hat{P} \oplus \tilde{P}$ be a Samelson decomposition. Denote the image of the transgression mapping by $V = \tau_g P = \hat{V} \oplus \tilde{V} \subseteq I(\underline{\underline{g}})$, so that $\text{Ideal}(\hat{V}) = \ker(i^*: I(\underline{\underline{g}}) \longrightarrow I(\underline{\underline{h}}))$. We denote by $V^{(2\ell)}$, resp. $V_{(2\ell)}$ the subspace generated by the elements of degree $> 2\ell$, resp. $\leq 2\ell$. Similarly we decompose $P = P^{(2\ell)} \oplus P_{(2\ell)}$.

The complex A_ℓ used in section 2 is defined by $A_\ell = \Lambda P_{(2\ell)} \otimes I(\underline{\underline{g}})_\ell$, with differential defined by the transgression. A similar relative complex is defined by $\hat{A}_\ell = \Lambda \hat{P}_{(2\ell)} \otimes I(\underline{\underline{g}})_\ell$. The relative Weil algebra of the pair then has cohomology [K-T 3]

$$H^*(W(\underline{\underline{g}},\underline{\underline{h}})_\ell) \cong H^*(\Lambda P \otimes I(\underline{\underline{g}})_\ell \otimes I(\underline{\underline{h}}))$$

$$\cong \Lambda \hat{P}^{(2\ell)} \otimes H^*(\hat{A}_{(2\ell)}) \underset{I(\underline{\underline{g}})}{\otimes} I(\underline{\underline{h}}).$$

The Chern-Weil theory gives a characteristic homomorphism for G-foliations [K-T,2]

$$\Delta : W(\underline{\underline{g}},\underline{\underline{h}})_\ell \longrightarrow A^\bullet(F(Q)/H) \xrightarrow{s^*} A^\bullet(X),$$

giving a commutative diagram of minimal models

$$(3.1)\qquad \begin{array}{ccc} M(W(\underline{\underline{g}},\underline{\underline{h}})_\ell) & \xrightarrow{M\Delta} & M(F(Q)/H) \\ {\scriptstyle Mj}\uparrow & & \downarrow{\scriptstyle Ms} \\ M(I(G)_\ell) & \xrightarrow{Mh} & M(X) \end{array}$$

We deduce two results from (3.1): First, there is a relationship between $h^{\#}$ and Δ_*, given by [Hu1]

3.2. THEOREM. *The diagram*

$$(3.3)\qquad \begin{array}{ccccc} \pi^*(I(G)_\ell) & & \xrightarrow{h^{\#}} & & \pi^*(X) \\ {\scriptstyle \zeta}\uparrow & & & & \downarrow{\scriptstyle \mathcal{H}^*} \\ H^*(\hat{A}_\ell) & & \xrightarrow{\Delta_*} & & H^*(X) \\ & \searrow & & \nearrow {\scriptstyle \Delta_*} & \\ & & H^*(W(\underline{\underline{g}},\underline{\underline{h}})_\ell) & & \end{array}$$

naturally commutes, where $\mathcal{H}^*$ *is the dual Hurewicz map and* ζ *is the inclusion mapping* $z_{(I/J)} \longrightarrow u_{(I/J)}$.

This result gives a new method for showing the non-triviality of Δ_*: a class which is non-zero in the image of $h^{\#} \circ \zeta$ is mapped to a non-zero class by Δ_*. Conversely, the non-triviality of Δ_* for a given X can be used to show $h^{\#}$ is non-trivial, if the map $\mathcal{H}^*$ is known. Section 4 will indicate what can be shown using these techniques.

Let $F\Gamma^q_G$ be the classifying space of trivialized, G-foliated micro-bundles; let $K \subseteq G$ be a maximal compact group. By the functoriality of $h_{\#}$ and $\Delta_{\#}$, dualizing (3.1) gives

3.4. THEOREM. *Let* $f : X \longrightarrow B\Gamma_G^q$ *classify a* G-*foliation on* X . *In a natural way, there are defined maps so that the diagram commutes:*

(3.5)
$$\begin{array}{ccc}
\Pi(F\Gamma_G^q) & \xrightarrow{\tilde{\Delta}_\#} & \Pi(W(\underline{\underline{g}},\underline{\underline{e}})_\ell) \\
\downarrow & & \downarrow \\
\Pi(B\Gamma_G^q) & \xrightarrow{\tilde{\Delta}_\#} & \Pi(W(\underline{\underline{g}},\underline{\underline{k}})_\ell) \\
f_\# \uparrow & \searrow \tilde{h}_\# & \downarrow j_\# \\
\Pi(X) & \xrightarrow[h_\#]{} & \Pi(I(G)_\ell)
\end{array}$$

The cokernel of $j_\#$ is $\tilde{P}^*$; (3.5) forces $\tilde{h}_\#$ to have cokernel $\supseteq \tilde{P}^*$. The obvious question is whether equality holds: Does image $\tilde{h}_\#$ = image $j_\#$?

4. *The homotopy of* $B\Gamma_G^q$

For the three standard types of G-foliation, we indicate the extent to which $\tilde{h}_\#$ is known.

Let $G = G\ell(q,\mathbb{R})$. Mather and Thurston [T] have shown that $\nu : B\Gamma^q \longrightarrow BO(q)$ is $(q+2)$ connected. Therefore

(4.1) $\tilde{h}_\#$ maps onto Y_{2j} for $4j \leq q + 2$.

(4.2) $\pi_m(B\Gamma^q) \otimes Q \longrightarrow H_m(B\Gamma^q;Q)$ is an isomorphism (resp. onto) for $m \leq 2q + 2$ (resp. $m = 2q + 3$).

By (4.1) we see that $\tilde{h}_\#$ is onto $s^{-1}u^*_{(2m,2m)}$, for $q = 4m - 2 > 3$ or $q = 4m - 1$. Theorem 3.2 implies $\Delta_*(Y_{2m}c_{2m}) \neq 0$ in $H^{8m-1}(F\Gamma^q)$. This is a rigid class for q even. Other Whitehead products are similarly in the image of $\tilde{h}_\#$ [Hu1].

Many more results follow from the Theorems of Heitsch [He] or Fuks [F] on the variability of the classes in the image of Δ_*.

Using (4.2) we conclude there is a surjection of $\pi_{2q+1}(B\Gamma^q) \longrightarrow \mathbb{R}^d$, for some $d > 0$. For example, by Fuks we have $\Pi_{2q}(B\Gamma^q) \xrightarrow{\tilde{h}_\#} \Pi_{2q}(I_q)$ is onto. The homotopy of $B\Gamma^q$ therefore maps onto a rather large Lie subalgebra of $\Pi(I_q)$. The $2q$ connectivity of ν would imply $\tilde{h}_\#$ is almost onto, within the restrictions of (3.5).

When $G = G\ell(n,\mathbb{C})$, the classes $Y_1,\ldots,Y_s$ are in the image of $\tilde{h}_\#:\Pi(B\Gamma_n^{\mathbb{C}}) \longrightarrow \Pi(I_n)$, where $s = [\sqrt{n}]$. Coupled with the Theorem of Baum and Bott [B-B], this shows $\Pi(B\Gamma_n^{\mathbb{C}}) \longrightarrow \Pi(I_n)$ is onto a much larger subalgebra than originally considered in [H3] Further details are in [Hu1].

When $G = SO(q)$ the map $\tilde{h}_\#:\Pi(BR\Gamma^q) \longrightarrow \Pi(I_{q'})$, $q' = [q/2]$, is onto, and complete variation occurs [Hu2]. In this case the Lie algebra $\Pi(I_{q'})$ is injected into $\Pi(BR\Gamma^q)$. The variability of the classes implies there are uncountably many distinct ways of choosing a section $\Pi(I_q) \longrightarrow \Pi(BR\Gamma^q)$.

References

[B-L]: H. J. Baues and J. M. Lemaire, *Minimal models in homotopy theory*, Math. Ann. 225, 219-242 (1977).

[B-B]: P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Diff. Geom. 7, 279-342 (1972).

[B]: R. Bott, *On a topological obstruction to integrability*, Proc, Symp. Pure Math., AMS vol. 16, 127-131 (1970).

[D]: J. L. Dupont, *Simplicial de Rham cohomology and characteristic classes of flat bundles*, Topology 15, 233-245 (1976).

[F]: D. B. Fuks, *Non-trivialité des classes caractéristiques de* $\underline{\underline{g}}$*-structures*, C. R. Acad. Sci. Paris 284, 1017-1019 and 1105-1107 (1977).

[H1]: A. Haefliger, *Feuilletages sur les variétés ouvertes*, Topology 9, 183-194 (1970).

[H2]: ____________, *Sur la cohomologie de l'algèbra de Lie des champs de vecteurs*, Ann. Sci. École Norm. Sup. 9, 503-532 (1976).

[H3]: ____________, *Whitehead products and differential forms*, Proceedings-Rio de Janeiro, LNM vol. 652, 13-24 (1978).

[Ha]: S. Halperin, *Lectures on Minimal Models*, Université de Lille I (1978).

[He]: J. Heitsch, *Independent variation of secondary classes*, Annals of Math. 108, 421-460 (1978).

[Ho]: G. Hochschild, *Lie algebra kernels and cohomology*, Amer. J. Math 76, 698-716 (1954).

[Hu1]: S. Hurder, *Dual homotopy invariants of* G*-foliations*, University of Illinois preprint (1979).

[Hu2]: ____________, *Some remarks on the homotopy and cohomology of* $B R\Gamma^q$, University of Illinois preprint (1979).

[K-T1]: F. Kamber and Ph. Tondeur, *Foliated bundles and characteristic classes*, LNM vol. 493 (1975).

[K-T2]: ____________, G*-foliations and their characteristic classes*, Bulletin AMS 84, 1086-1124 (1978).

[K-T3]: ____________, *Characteristic classes and Koszul complexes*, Proc. Symp. Pure Math. vol 32, AMS, 159-166 (1978).

[K-T4]: ____________, *The rational homotopy type of truncated relative Weil algebras*, to appear.

[McL]: S. Mac Lane, *Homology*, Springer-Verlag (1963).

[N-M]: J. Neisendorfer and T. Miller, *Formal and coformal spaces*, Illinois Journal of Math. 22, 565-579 (1978).

[Q]: D. Quillen, *Rational homotopy theory*, Ann. Math. 90, 205-295 (1969).

[S]: D. Sullivan, *Infinitessimal computations in topology*, IHES 47, 269-331 (1977).

[T]: W. Thurston, *Foliations and groups of diffeomorphisms*, Bulletin AMS 80, 304-307 (1974).

Department of Mathematics
University of Illinois
Urbana, IL 61801

USA