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Independent Rigid Secondary Classes for Holomorphic Foliations

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A foliation $\mathscr{F}$ on a smooth manifold M is said to *transversely holomorphic* if $\mathscr{F}$ is locally defined by submersions into $\mathbb{C}^n$, and the associated transition functions are biholomorphic. If M is also a complex manifold and the local submersions are holomorphic, then F is said to be *holomorphic*. The purpose of this paper is to study the characteristic classes of transversely holomorphic foliations. The universal chern classes for transversely holomorphic foliations of codimension n are shown to be independent up to degree $2n$, above which they must vanish. Using this result, we then construct for all even complex codimensions the first examples of transversely holomorphic foliations for which a set of rigid secondary classes is non-trivial. Applications of the independence results we obtain are given to the study of $B\Gamma_2^{\mathbb{C}}$ and to the study of the space of foliations on an open manifold.

Heitsch showed that the secondary classes of a foliation divide into two categories [9]: the *variable* classes are those whose value can change under a deformation of the foliation; *rigid* classes are those invariant under deformation. The examples of Baum-Bott [§ 11; 2] established that all of the variable classes of degree $2n+1$ in codimension n are independently variable in $H^{2n+1}(F\Gamma_n^{\mathbb{C}})$. The Baum-Bott examples are extended in [Theorem 5; 11], so that all of the variable classes in $H^*(W_n)\otimes H^*(\overline{W}_n)$ which are not products are independently variable in $H^*(F\Gamma_n^{\mathbb{C}})$. The examples of Rasmussen [18] show that some of the decomposable classes can also vary. The natural question raised by these examples is whether any rigid classes are non-trivial in $H^*(F\Gamma_n^{\mathbb{C}})$. We give a positive answer, using the dual homotopy techniques of [10] and a detailed study of the topology of the map $\nu: B\Gamma_n^{\mathbb{C}} \to BU_n$.

The rest of the paper is organized as follows. Section 1 gives the general non-triviality results for the chern classes and rigid secondary classes. Background material about secondary classes and dual homotopy invariants is presented in § 2. Section 3 gives specific results about codimension 2, and § 4 is devoted to the space of foliations. The proofs of Theorems 1.3 and 1.4 are deferred until § 5.

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§ 1. Independence Theorems

All manifolds are assumed to be smooth and paracompact. Topological spaces are assumed to have basepoints and maps between them to be continuous and preserve basepoints. The singular cohomology of X with coefficients in $\mathbb{C}$ is denoted $H^*(X)$.

Let $B\Gamma_n^{\mathbb{C}}$ be Haefliger's classifying space of codimension n transversely holomorphic foliations [7], and let $\nu\colon B\Gamma_n^{\mathbb{C}} \to BU_n$ be defined by the differential.

Proposition 1.1. *The map* $\nu^*\colon H^m(BU_n;\mathbb{Q}) \to H^m(B\Gamma_n^{\mathbb{C}};\mathbb{Q})$ *is injective for all* $m \leqq 2n$.

The Bott Vanishing Theorem for transversely holomorphic foliations [3] implies ν^* is the zero map for $m > 2n$, so this result is sharp.

Proof. The foliation by points of a compact, complex n-manifold M determines a map $f\colon M \to B\Gamma_n^{\mathbb{C}}$ such that $\nu \circ f$ classifies the complex tangent bundle of M. Given a non-zero $z \in H^{2n}(BU_n;\mathbb{Q})$, there is a compact, complex n-manifold M with tangent bundle classified by $g\colon M \to BU_n$ so that $g^*(z) \neq 0$. This follows from the independence of the chern numbers in degree n [§ 16; 16]. We can factor $g = \nu \circ f$, so $\nu^*(z) \neq 0$. Since z was arbitrary, ν^* is injective in degree $2n$. For a non-zero $z \in H^{2l}(BU_n;\mathbb{Q})$ with $l < n$, multiplying by c_1^{n-l} yields a non-zero class of degree $2n$ so that $\nu^*(z \cdot c_1^{n-l}) \neq 0$, whence $\nu^*(z) \neq 0$. Therefore ν^* is injective in all degrees $\leqq 2n$. □

The connectivity theorems of Landweber [15] and Adachi [1] along with Proposition 1.1 yield:

Proposition 1.2. *The map* $\nu_\#\colon \pi_m(B\Gamma_n^{\mathbb{C}}) \to \pi_m(BU_n)$ *is an isomorphism for* $m \leqq n$, *onto for* $m = n+1$ *and has finite cokernel for* $m = n+2$.

Proof. It is shown in [1] that the fiber $F\Gamma_n^{\mathbb{C}}$ of ν is n-connected, so the point of (1.2) is to prove $\nu_\#\colon \pi_{n+2}(B\Gamma_n^{\mathbb{C}}) \otimes \mathbb{Q} \to \pi_{n+2}(BU_n) \otimes \mathbb{Q}$ is onto.

In the fibration $F\Gamma_n^{\mathbb{C}} \to B\Gamma_n^{\mathbb{C}} \xrightarrow{\nu} BU_n$, the base is 1-connected and fiber is n-connected. From the Serre exact sequence for homology [Theorem 7.10; 20], we get the commutative diagram with exact rows:

$$\begin{array}{ccccc}
H_{n+2}(B\Gamma_n^{\mathbb{C}};\mathbb{Q}) & \xrightarrow{\nu_*} & H_{n+2}(BU_n;\mathbb{Q}) & \xrightarrow{\delta_*} & H_{n+1}(F\Gamma_n^{\mathbb{C}};\mathbb{Q}) \\
\uparrow{\scriptstyle h_1} & & \uparrow{\scriptstyle h_2} & & \uparrow{\scriptstyle h_3} \\
\pi_{n+2}(B\Gamma_n^{\mathbb{C}}) \otimes \mathbb{Q} & \xrightarrow{\nu_\#} & \pi_{n+2}(BU_n) \otimes \mathbb{Q} & \xrightarrow{\delta_\#} & \pi_{n+1}(F\Gamma_n^{\mathbb{C}}) \otimes \mathbb{Q}.
\end{array}$$

The vertical maps are the Hurewicz homomorphisms. The map ν_* is onto by (1.1), so $\delta_* = 0$. Since h_3 is an isomorphism, $\delta_\# = 0$ also and thus $\nu_\#$ must be onto. □

The non-triviality of $\nu_\#$ in degree $n+2$ is essential to our construction of examples with non-trivial rigid classes. Recall that $k_*\colon H^*(W_n \oplus \overline{W}_n) \to H^*(F\Gamma_n^{\mathbb{C}})$ is the universal secondary characteristic homomorphism for transversely holomorphic foliations with framed normal bundles.

Theorem 1.3. *Let $n=2k-2$ with $k>1$. Define a set of rigid secondary classes $\mathcal{R} \subseteq H^*(W_n) \subseteq H^*(W_n \oplus \overline{W}_n)$ to be*

$$\mathcal{R} \equiv \{y_2 y_{i_2} \dots y_{i_s} c_2^{k-1} \mid 2 < i_2 < \dots < i_s \leqq n\} \cup \{y_k y_{i_2} \dots y_{i_s} c_k \mid k < i_2 < \dots < i_s \leqq n\}.$$

Then k_* maps $\mathcal{R}$ to a linearly independent set in $H^*(F\Gamma_n^{\mathbb{C}})$.

Let $\mathcal{R}_s \subseteq \mathcal{R}$ be the subset $\mathcal{R}_s = \{y_I c_J \in \mathcal{R} \mid 2 i_s \leqq n+2\}$. The elements of the subset $k_*(\mathcal{R}_s)$ are then, in addition, spherically supported:

Theorem 1.4. *The set $k_*(\mathcal{R}_s) \subseteq H^*(F\Gamma_n^{\mathbb{C}})$ is linearly independent on spherical cycles. That is, the elements of $k_*(\mathcal{R}_s)$ take independent values on the cycles in the image of the Hurewicz map $\pi_*(F\Gamma_n^{\mathbb{C}}) \to H_*(F\Gamma_n^{\mathbb{C}}; Z)$.*

The proofs of (1.3) and (1.4) are deferred until § 5.

Remark 1.5. Theorem 1.3 implies there exists a *complex* manifold M with a holomorphic foliation for which the secondary classes in $\mathcal{R}$ take independent values. It is a standard application of the Gromov-Phillips Theorem to show there exists an open analytic manifold U with a transversely holomorphic foliation for which $\mathcal{R}$ is independent [6]. We take M to be the complexification of U, and it follows from Theorem 2.6 of Landweber [15] that the complex manifold M has a framed holomorphic foliation $\mathcal{F}$ with the classes in $\mathcal{R}$ independent for $\mathcal{F}$. Further, the set $\mathcal{R}_s$ can be realized independently on such an M which has the homotopy type of a bouquet of spheres by (1.4).

§ 3. Secondary and Dual Homotopy Invariants

We recall the construction of the secondary classes as given by Bott [3], and their relation with the dual homotopy invariants of [10].

Define a differential algebra $W_n = \Lambda(y_1, \dots, y_n) \otimes \mathbb{C}[c_1, \dots, c_n]_n$ where $\mathbb{C}[c_1, \dots, c_n]_n$ is the truncated polynomial algebra with generators of degree $c_j = 2j$, and we quotient out the ideal generated by monomials of degree $>2n$. The differential is defined by setting $dy_i = c_i$. Also, we need the conjugate algebra $\overline{W}_n$ which is isomorphic to W_n and has algebra generators $\bar{y}_i$ and $\bar{c}_i$.

Given a transversely holomorphic foliation $\mathcal{F}$ on M with a framing s of the normal $Gl_n\mathbb{C}$-bundle to $\mathcal{F}$, a choice of a Bott connection for $\mathcal{F}$ along with the flat connection of s determines a differential map $\Delta^s\colon W_n \to \mathcal{E}_{\mathbb{C}}(M)$ into the complex de Rham algebra [3]. Taking the conjugates of the forms in the image of Δ^s gives rise to a second map which we also denote $\Delta^s\colon \overline{W}_n \to \mathcal{E}_{\mathbb{C}}(M)$. The sum of both yields in cohomology the characteristic map $\Delta^s_*\colon H^*(W_n \oplus \overline{W}_n) \to H^*(M)$ which depends only on the concordance class of $\mathcal{F}$ and the homotopy class of the framing s. This construction is functorial, so there is a universal map which we denote $k_*\colon H^*(W_n \oplus \overline{W}_n) \to H^*(F\Gamma_n^{\mathbb{C}})$.

It is conjectured that k_* is injective; k_* is known to be injective when restricted to the space of variable classes in the direct sum

$$H^*(W_n) \oplus H^*(\overline{W}_n) \subseteq H^*(W_n) \otimes H^*(\overline{W}_n) \cong H^*(W_n \oplus \overline{W}_n).$$

For $n=2$, $k_*: H^*(W_2) \to H^*(F\Gamma_2^{\mathbb{C}})$ is injective by Theorem 3.2 below. The injectivity of $k_*: H^*(W_n) \to H^*(F\Gamma_n^{\mathbb{C}})$ follows similarly if the space $F\Gamma_n^{\mathbb{C}}$ is $(2n-2)$-connected. It seems to be a delicate matter to realize product classes in $H^*(W_n \oplus \overline{W}_n)$; the only known examples are in [18].

A basis over $\mathbb{C}$ of $H^*(W_n)$ has been determined by Vey [5]. For indices $I=(i_1, \ldots, i_s)$ and $J=(j_1, \ldots, j_n)$ we denote by $y_I c_J$ the element of W_n

$$y_I c_J = y_{i_1} \ldots y_{i_s} \otimes c_1^{j_1} \ldots c_n^{j_n}.$$

A pair (I, J) is *admissible*, and the corresponding $y_I c_J$ is an *admissible cocycle*, if it satisfies the conditions:

$$\begin{gathered} 1 \leqq i_1 < \ldots < i_s \leqq n \quad \text{and} \quad j_l \geqq 0 \\ \deg c_J \leqq 2n \quad \text{and} \quad \deg y_{i_1} c_J \geqq 2n+1 \\ l < i_1 \quad \text{implies } j_l = 0. \end{gathered}$$

The set $\{y_I c_J \mid (I, J) \text{ admissible}\}$ is the Vey basis for $H^*(W_n)$.

With respect to the Vey basis, the *rigid* classes in $H^*(W_n)$ consist of the span of the set $\{y_I c_J \mid (I, J) \text{ admissible and } \deg y_{i_1} c_J \geqq 2n+3\}$. Heitsch showed that if z is a rigid class and $(\mathscr{F}, s)$ is a framed foliation on M, then the value of $\Delta_*^s(z) \in H^*(M)$ depends only on the *homotopy* class of $\mathscr{F}$ and s, and thus is constant under a deformation of $\mathscr{F}$ [9].

Associated to a transversely holomorphic foliation $\mathscr{F}$ on M of codimension n is another family of invariants, the dual homotopy classes. We describe their essential properties; complete details are in [10].

The Chern-Weil homomorphism with respect to a Bott connection for $\mathscr{F}$ gives a differential algebra homomorphism $h: I_n \to \mathscr{E}_{\mathbb{C}}(M)$, where $I_n \equiv \mathbb{C}[c_1, \ldots, c_n]_n$ is the truncated polynomial algebra. It is shown in [Theorem 2.11; 10] that the de Rham-Sullivan algebra homotopy class of h depends only on the concordance class of $\mathscr{F}$. Any property of h which depends only on its algebra homotopy class will then be an invariant of the concordance class of $\mathscr{F}$.

Given a differential algebra $\mathscr{A}$ with $H^0(\mathscr{A}) = \mathbb{C}$, the minimal model of $\mathscr{A}$ is denoted $\mathscr{M}_{\mathscr{A}} \xrightarrow{\phi} \mathscr{A}$. Define $\pi^*(\mathscr{A})$ to be the graded vector space of indecomposables in $\mathscr{M}_{\mathscr{A}}$,

$$\pi^*(\mathscr{A}) \simeq \mathscr{M}_{\mathscr{A}} / (\mathscr{M}_{\mathscr{A}}^+)^2$$

where $\mathscr{M}_{\mathscr{A}}^+$ denotes the ideal of elements of positive degree. The correspondence $\mathscr{A} \to \pi^*(\mathscr{A})$ is functorial. The dual homotopy invariants of $\mathscr{F}$ are obtained by applying π^* to the Chern-Weil map [Theorem 2.12; 10]:

Thorem 2.1. *The map of graded vector spaces $h^\#: \pi^*(I_n) \to \pi^*(M)$ depends only on the concordance class of $\mathscr{F}$.*

We use in (2.1) the notation $\pi^*(M)\equiv\pi^*(\mathscr{E}_{\mathbb{C}}M)$. For a simply connected manifold, the space $\pi^*(M)$ is naturally identified with $\mathrm{Hom}(\pi_*(M),\mathbb{C})$. The classes in the image of $h^\#$ form a system of "higher order residues" for the foliation, in the sense of [2, 19]. See also [Remark 4.3; 10].

When $n>1$, the minimal model of I_n has an infinite number of generators and the space of invariants $\pi^*(I_n)$ is infinite-dimensional.

The construction of $h^\#$ is functorial, so there is a universal map

$$h^\#:\pi^*(I_n)\to\pi^*(B\Gamma_n^{\mathbb{C}})\overset{\mathrm{def}}{=}\mathrm{Hom}(\pi_*(B\Gamma_n^{\mathbb{C}}),\mathbb{C}).$$

This map is injective for $n=1$ or 2. For $n>2$, if $F\Gamma_n^{\mathbb{C}}$ is $(2n-2)$-connected, then the universal map $h^\#$ will be injective.

One of the useful features of the dual homotopy classes is their explicit relationship with the secondary classes. In degrees $m>2n$, we can define a map ξ to be the composition

$$\pi_m(B\Gamma_n^{\mathbb{C}})\otimes\mathbb{Q}\simeq\pi_m(F\Gamma_n^{\mathbb{C}})\otimes\mathbb{Q}\to H_m(F\Gamma_n^{\mathbb{C}};\mathbb{Q})$$

where the last map is the Hurewicz homomorphism. The dual to ξ gives a map $\xi^*: H^m(F\Gamma_n^{\mathbb{C}})\to\pi^*(B\Gamma_n^{\mathbb{C}})$.

Theorem 2.2. *There exists an inclusion $\zeta: H^*(W_n)\to\pi^*(I_n)$ such that, for $m>2n$, the diagram commutes:*

$$\begin{tikzcd}
\pi^m(I_n) \arrow[r, "h^\#"] & \pi^m(B\Gamma_n^{\mathbb{C}}) \\
H^m(W_n) \arrow[u, "\zeta"] \arrow[r, "k_*"] & H^m(F\Gamma_n^{\mathbb{C}}). \arrow[u, "\xi^*"']
\end{tikzcd}$$

This is Corollary 3.6 of [10]. On the level of differential forms, the relationship of (2.2) was discussed by Haefliger in [8]. The purpose of introducing $h^\#$ in order to study k_* is that the dual homotopy classes can detect Whitehead products in $\pi_*(B\Gamma_n^{\mathbb{C}})$, and these Whitehead products are often explicitly constructable. We prove Theorems 1.3 and 1.4 by exhibiting an appropriate product in $\pi_{2n+3}(B\Gamma_n^{\mathbb{C}})$ which is detected by a class in the image of $h^\#\circ\zeta$, and then use Theorem 2.2 to conclude the corresponding secondary class in $H^*(F\Gamma_n^{\mathbb{C}})$ is non-trivial.

§ 3. The Structure of $B\Gamma_2^{\mathbb{C}}$

Proposition 1.2 and Theorem 1.4 have many consequences for the study of $B\Gamma_2^{\mathbb{C}}$. First, it follows from the 2-connectivity of $F\Gamma_2^{\mathbb{C}}$ and (1.2) that the loop space $\Omega B\Gamma_2^{\mathbb{C}}$ decomposes after inverting 12 (=localizing away from 12); there is a homotopy equivalence:

$$\Omega(B\Gamma_2^{\mathbb{C}})_{(12)}\simeq\Omega(BU_2)_{(12)}\times\Omega(F\Gamma_2^{\mathbb{C}})_{(12)}. \tag{3.1}$$

This is proved by constructing a section μ

$$\Omega\nu:\Omega B\Gamma_2^{\mathbb{C}}\rightleftarrows\Omega BU_2:\mu$$

where $\Omega\nu \circ \mu$ is multiplication by 12. The number 12 is the determinant of the 2×2 matrix of chern numbers in dimension 2 [P. 194; 16], and enters into (3.1) when constructing a map $\Sigma\mu: S^4 \to B\Gamma_2^{\mathbb{C}}$ which detects $\nu^*(c_2) \in H^4(B\Gamma_2^{\mathbb{C}})$.

The integral groups $\pi_*(B\Gamma_2^{\mathbb{C}})$ and $\pi_*(F\Gamma_2^{\mathbb{C}})$ are essentially the same by (3.1). For $m>4$, information about $H^m(F\Gamma_2^{\mathbb{C}})$ and $\pi_m(F\Gamma_2^{\mathbb{C}})$ is obtained using the secondary and dual homotopy invariants. Recall that the Vey basis for $H^*(W_2)$ is:

$$
\begin{aligned}
V_5 &= \{y_1 c_1^2, y_1 c_2\} && \text{of degree } 5\\
R_7 &= \{y_2 c_2\} && \text{of degree } 7\\
V_8 &= \{y_1 y_2 c_1^2, y_1 y_2 c_2\} && \text{of degree } 8
\end{aligned}
$$

where $V_5 \cup V_8$ is a basis for the variable classes, and R_7 for the rigid classes.

In the next theorem, we regard a class $c \in H^m(F\Gamma_2^{\mathbb{C}})$ as a homomorphism $H_m(F\Gamma_n^{\mathbb{C}}; Z) \to \mathbb{C}$, and by composition with the Hurewicz map, also as a homomorphism $c: \pi_m(F\Gamma_n^{\mathbb{C}}) \to \mathbb{C}$.

Theorem 3.2. *The universal map $k_*: H^*(W_2) \to H^*(F\Gamma_2^{\mathbb{C}})$ is injective. Further, there are epimorphisms of abelian groups:*

(a) $k_*(V_5): \pi_5(F\Gamma_2^{\mathbb{C}}) \to \mathbb{C} \oplus \mathbb{C}$,

(b) $k_*(R_7): \pi_7(F\Gamma_2^{\mathbb{C}}) \otimes \mathbb{C} \to \mathbb{C}$,

(c) $k_*(V_8): \pi_8(F\Gamma_2^{\mathbb{C}}) \to \mathbb{C} \oplus \mathbb{C}$.

Proof. It suffices to show a, b and c hold. (a) follows directly from the Baum-Bott examples [§ 12; 2]. (b) is a restatement of Theorem 1.4 for $n=2$. (c) follows from (a), the existence of a map $g: S^4 \to B\Gamma_2^{\mathbb{C}}$ which detects c_2 (by 1.2) and the analog for $B\Gamma_2^{\mathbb{C}}$ of [Proposition 6.12; 10]. □

It is not known whether $k_*: H^*(W_2 \oplus \overline{W}_2) \to H^*(F\Gamma_2^{\mathbb{C}})$ is injective. As remarked in § 2, k_* is injective on the space of variable classes in $H^*(W_2) \oplus H(\overline{W}_2)$. The class $k_*(y_2 c_2)$ is non-trivial, but the rigid class $(y_2 c_2 - \overline{y_2 c_2})$ may be in the kernel of k_*. In the proof of 1.4, we will construct a map $\tilde{f}: S^7 \to F\Gamma_2^{\mathbb{C}}$ such that the evaluation of $k_*(y_2 c_2)$ on $\tilde{f}$ is a real number (in fact, an integer). Thus, $k_*(y_2 c_2 - \overline{y_2 c_2})(\tilde{f}) = 0$; so we get $k_*(y_2 c_2) \neq 0$, but cannot conclude $k_*(y_2 c_2 - \overline{y_2 c_2})$ is non-trivial.

For the dual homotopy invariants there is an injectivity result parallel to 3.2:

Theorem 3.3. *The universal map $h^\#: \pi^*(I_2) \to \pi^*(B\Gamma_2^{\mathbb{C}})$ is injective.*

Proof. Let $X = S^4 \vee \mathbb{C}P^2$ be the 4-skeleton of the standard CW structure on BU_2. It will suffice to construct a map $f: X \to B\Gamma_2^{\mathbb{C}}$ which induces an isomorphism $(\nu \circ f)^*: H^m(BU_2) \to H^m(X)$ for $m \leqq 4$. For this implies the associated Chern-Weil homomorphism $h^*: I_2 \to H^*(X)$ is an isomorphism, and also $h^\#: \pi^*(I_2) \to \pi^*(X)$ is an isomorphism. Since $h^\#$ factors through $f^\#$, the theorem follows.

To define f, choose a map $g: S^4 \to B\Gamma_2^{\mathbb{C}}$ so that $(\nu \circ g)^*(c_2) \neq 0$. Let $g': \mathbb{C}P^2 \to B\Gamma_2^{\mathbb{C}}$ classify the point foliation. Then $f = g \vee g': X \to B\Gamma_2^{\mathbb{C}}$ has the desired properties. □

On implication of (3.3) is that the rational homotopy groups $\pi_m(B\Gamma_2^{\mathbb{C}})\otimes\mathbb{C}$ are non-trivial for an infinite number of m. More amazingly, from Theorem 3.2a, c and using the ideas of Haefliger [8], we have for all $m>10$, the group $\pi_m(B\Gamma_2^{\mathbb{C}})$ contains an *uncountable* number of free Z-zummands. This derives from the existence of epimorphisms of the *integral* groups $\pi_m(B\Gamma_2^{\mathbb{C}})\twoheadrightarrow\mathbb{C}^{r_m}$, where $r_m\to\infty$. (Precisely, $r_m\neq 0$ for $m=5$, 8, 9 and $m>10$.) These epimorphisms are constructed by showing there is a surjection of Whitehead algebras (the suspensions of graded Lie algebras)

$$\pi_*(B\Gamma_2^{\mathbb{C}})\twoheadrightarrow\pi_*(S^5\vee S^5\vee S^7\vee S^8\vee S^8)\otimes\mathbb{C}$$

except in degree 7. The integer r_m is just the dimension of the right-hand-side vector space in degree m.

§ 4. The Space of Transversely Holomorphic Foliations

Let $\mathscr{F}_n^{\mathbb{C}}(M)$ be the set of codimension n, transversely holomorphic foliations on the smooth manifold M. We assume this set is non-empty and give it the C^∞-topology. The local structure of $\mathscr{F}_n^{\mathbb{C}}(M)$ about a "point" $\mathscr{F}$ is related to the infinitesimal deformations of $\mathscr{F}$, and the associated deformation theory of $\mathscr{F}$ has been studied by Kodaira-Spencer [14] and Duchamp-Kalka [4]. However, very little is known about the global topology of $\mathscr{F}_n^{\mathbb{C}}(M)$. For many open manifolds M, we can apply Theorems 1.3 and 1.4 to find that $\mathscr{F}_n^{\mathbb{C}}(M)$ has an infinite number of path components.

Define a smooth path γ: $[0,1]\to\mathscr{F}_n^{\mathbb{C}}(M)$ to be a smooth foliation $\mathscr{F}'$ on $M\times[0,1]$ of real codimension $2n+1$ such that each restriction $\mathscr{F}'|_{M\times t}$ is a transversely holomorphic foliation of codimension n, and the complex structure of the normal bundle to $\mathscr{F}'|_{M\times t}$ varies smoothly with t. We say $\mathscr{F}_0$ is smoothly homotopic to $\mathscr{F}_1$ if there is a smooth path between them. Let $\pi_0(\mathscr{F}_n^{\mathbb{C}}(M))$ denote the set of components under the equivalence relation generated by smooth homotopies.

Theorem 4.1. *Let M be an open manifold. Assume*

(i) *TM has a trivializable subbundle $Q\subseteq TM$ of rank $q=4k$ for some positive integer k.*

(ii) *M has the homotopy type of a CW complex X of dimension m with $H^m(X)\neq 0$.*

(iii) *One of the following holds:*

$$\begin{aligned} &k=1 \quad \textit{and} \quad m=7\\ &k=2 \quad \textit{and} \quad m=1 \bmod 5 \quad \textit{with} \quad m\geqq 11\\ &k\geqq 3 \quad \textit{with} \quad m>(4k+7)(4k+9). \end{aligned}$$

Then M has a infinite number of transversely holomorphic foliations of complex codimension $n=2k$ whose embedded normal bundles are homotopic to Q, but no two of these foliations are smoothly homotopic. Further, there is a natural *surjection of sets*

$$\pi_0(\mathscr{F}_n^{\mathbb{C}}(M)) \twoheadrightarrow Z^{r_{n,m}}$$

where $\lim_{m\to\infty} r_{n,m} = \infty$ *when* $n \geqq 6$.

Just a sketch of the proof of (3.1) is given as the complete proof is very similar to that of [Theorem 1.2; 12] which can be easily adapted along the lines indicated below.

Let Γ be the topological groupoid of local C^∞ diffeomorphisms of $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ which are of the form $(z,t) \to (\phi(z,t), t)$, where for each t, the map $z \to \phi(z,t)$ is holomorphic. The realization $B\Gamma$ classifies deformations of transversely holomorphic foliations, and the constant deformation defines a natural map $c\colon B\Gamma_n^{\mathbb{C}} \to B\Gamma$. If $\mathscr{F}_0$, $\mathscr{F}_1$ are smoothly homotopic and classified by $f_0, f_1\colon M \to B\Gamma_n^{\mathbb{C}}$, then the smooth homotopy defines a homotopy between $c \circ f_0$ and $c \circ f_1$. Assuming (i)–(iii) of (4.1), the aim is to produce an infinite set of maps

$$\{f_\alpha\colon M \to F\Gamma_n^{\mathbb{C}} \to B\Gamma_n^{\mathbb{C}} \mid \alpha \in Z^{r_{n,m}}\}$$

with $c \circ f_\alpha \cong c \circ f_\beta$ implying $\alpha = \beta$. Since M is open and an embedding $Q \subseteq TM$ is given, the existence theorem of [6, 7] yields a set $\{\mathscr{F}_\alpha\} \subseteq \mathscr{F}_n^{\mathbb{C}}(M)$ with each $\mathscr{F}_\alpha$ in a distinct path component, proving the theorem.

The techniques of [§6; 12] show it suffices to assume $M \simeq S^m$. The codimension $n = 2k$ is even, so Theorem 1.4 implies there is a set $\mathscr{R}_S \subseteq H^*(W_n)$ with $k_*(\mathscr{R}_S)$ spherically supported. Let $\mathscr{R}_S = \{z_1, \ldots, z_\mu\}$ with $n_j = \deg z_j$, and construct the one-point union of spheres $Y = \bigvee_{j=1}^{\mu} S^{n_j}$. The proof in [9] that the rigid classes are constant under smooth homotopy actually shows we can define $k_*(\mathscr{R}_S) \subseteq H^*(F\Gamma)$, where $F\Gamma$ is the homotopy fiber of $B\Gamma \to BU_n$. Then by (1.4), there is a map $c \circ f\colon Y \xrightarrow{f} F\Gamma_n^{\mathbb{C}} \xrightarrow{c} F\Gamma$ so that $(c \circ f)^* \circ k_*(\mathscr{R}_s)$ is a basis for $H^*(Y)$. From the theorem of [§4; 8], we conclude the composition

$$\pi_*(Y) \otimes \mathbb{Q} \xrightarrow{f_\#} \pi_*(F\Gamma_n^{\mathbb{C}}) \otimes \mathbb{Q} \xrightarrow{c_\#} \pi_*(F\Gamma) \otimes \mathbb{Q} \to \pi_*(B\Gamma) \otimes \mathbb{Q} \tag{4.2}$$

is injective. For each $\alpha \in \pi_m(Y)$, define $f_\alpha\colon M \to F\Gamma_n^{\mathbb{C}}$ by $f_\alpha = f \circ \alpha$. We let $r_{n,m} = \dim \pi_m(Y) \otimes \mathbb{Q}$, so the injectivity of (4.2) implies at least $Z^{r_{n,m}}$ of the composites $c \circ f_\alpha$ are homotopicly distinct. (For the general case, let $\beta\colon M \to S^m$ classify a non-trivial element of $H^m M; Z)$ and set $g_\alpha = f \circ \alpha \circ \beta\colon M \to F\Gamma_n^{\mathbb{C}}$. At least $Z^{r_{n,m}}$ of the g_α are homotopicly distinct.) The fact that $r_{n,m} \neq 0$ for the values of m and k listed in (4.1) follows from explicit calculation of $\pi_*(Y) \otimes \mathbb{Q}$.

§5. Whitehead Products in $\pi_*(B\Gamma_n^{\mathbb{C}})$

It is well-known that for each indecomposable chern class $c_i \in H^{2i}(BU_n)$ there exists a map $\bar{g}_i\colon S^{2i} \to BU_n$ such that $\bar{g}_i^*(c_i) \neq 0$. The most important application of Proposition 1.2 is that for $2i < n+2$, the map $\bar{g}_i$ lifts to a map $g_i\colon S^{2i} \to B\Gamma_n^{\mathbb{C}}$

with $(\nu \circ g_i)^*(c_i) \neq 0$. For $2i = n+2$, some multiple of g_i will lift with $(\nu \circ g_i)^*(c_i) \neq 0$.

Let $n = 2k-2$ with $k > 1$ be given. The idea of the proofs of Theorems 1.3 and 1.4 is to use the maps g_2 and g_k to construct Whitehead products $[g_k, g_k]$, $[g_2, \ldots, g_2] \in \pi_{2n+3}(B\Gamma_n^{\mathbb{C}})$. Since $\pi_{2n+3}(BU_n) \otimes \mathbb{Q} = 0$, non-zero multiples of these products come from elements of $\pi_{2n+3}(F\Gamma_n^{\mathbb{C}})$, which in turn determine spherical cycles in $H_{2n+3}(F\Gamma_n^{\mathbb{C}}; Z)$. If we produce dual homotopy classes in $\pi^{2n+3}(B\Gamma_n^{\mathbb{C}})$ which detect these Whitehead products, then Theorem 2.2 asserts the corresponding spherical cycles are detected by secondary classes. In particular, the secondary classes are $k_*(y_k c_k)$ and $k_*(y_2 c_2^{k-1})$. With the notation of (2.2), we further show that $h^\# \circ \zeta(y_2 c_2^{k-1})$ vanishes on the product $[g_k, g_k]$ and hence $k_*(y_2 c_2^{k-1})$ and $k_*(y_k c_k)$ are linearly independent when evaluated on spherical cycles in $H_*(F\Gamma_n^{\mathbb{C}}; Z)$.

Given the independence of $k_*(y_2 c_2^{k-1})$ and $k_*(y_k c_k)$, the rest of the claims of (1.3) and (1.4) follow by the "permanence principle." That is, Theorem 3.3 of [11] shows that if the set $\{y_2 c_2^{k-1}, y_k c_k\}$ is mapped by k_* to an independent set, then all of the classes in the set $\mathscr{R}$ of (1.3) are also mapped by k_* to an independent set. For the subset $\mathscr{R}_s$ of (1.4), we note that [Proposition 6.12; 10] as applied to $B\Gamma_n^{\mathbb{C}}$ implies $\mathscr{R}_s$ is mapped to an independent set by $h^\# \circ \zeta$: $H^*(W_n) \to \pi^*(B\Gamma_n^{\mathbb{C}})$. Theorem 1.4 then follows from Theorem 2.2.

It remains to construct the indicated Whitehead products. First, consider $g_k: S^{2k} \to B\Gamma_n^{\mathbb{C}}$ with $(\nu \circ g_k)^*(c_k) \neq 0$. Choose a point $x_0 \in S^{2k} \times S^{2k}$ which is not the base point, and let $D^{4k} \subseteq S^{2k} \times S^{2k}$ be a closed disc neighborhood of x_0. Set $W = S^{2k} \times S^{2k} - \{x_0\}$, and note there is a homotopy equivalence $W \simeq S^{2k} \vee S^{2k}$. Let F be the composition $W \simeq S^{2k} \vee S^{2k} \xrightarrow{g_k \vee g_k} B\Gamma_n^{\mathbb{C}}$, and define f to be the composition

$$S^{2n+3} = \partial D^{4k} \subseteq W \xrightarrow{F} B\Gamma_n^{\mathbb{C}}.$$

The second order Whitehead product $[g_k, g_k] \in \pi_{2n+3}(B\Gamma_n^{\mathbb{C}})$ is by definition the homotopy class of f.

The evaluation property [4.6; 10] states that the dual homotopy class $h^\# \circ \zeta(y_k c_k)$ takes the value

$$h^\# \circ \zeta(y_k c_k)([g_k, g_k]) = -2\{\int_{S^{2k}} g_k^*(c_k)\}^2 \neq 0. \tag{5.1}$$

Also, the remark following [Theorem 4.4; 10] gives

$$h^\# \circ \zeta(y_2 c_2^{k-1})([g_k, g_k]) = 0$$

for $k > 2$ since $\zeta(y_2 c_2^{k-1})$ has order k, and $[g_k, g_k]$ has order 2.

The integral in (5.1) giving the evaluation of $h^\# \circ \zeta(y_k c_k)$ on $[g_k, g_k]$ corresponds exactly to the residue for f: $W \to B\Gamma_n^{\mathbb{C}}$ which was used by Schweitzer and Whitman [19] to construct analogous products in $\pi_*(B\Gamma_q)$.

Since $\pi_{2n+3}(BU_n) \otimes \mathbb{Q} = 0$, a multiple of the map f representing $[g_k, g_k]$ lifts to a map $\tilde{f}$: $S^{2n+3} \to F\Gamma_n^{\mathbb{C}}$. This is the spherical cycle on which $k_*(y_k c_k)$ is non-zero, and $k_*(y_2 c_2^{k-1})$ vanishes.

Next, we produce a k^{th}-order Whitehead product in $\pi_{2n+3}(B\Gamma_n^{\mathbb{C}})$ using a map $g: S^4 \to B\Gamma_2^{\mathbb{C}}$ chosen so that $(v \circ g)^*(c_2) \neq 0$. This product is of higher order for $k>2$, and its construction is accordingly more delicate. For each positive integer $l<k$, let $X_l = S^4 \times \ldots \times S^4$ be the product of l-copies of S^4. Define f_l: $X_l \to B\Gamma_n^{\mathbb{C}}$ to be the product map $g^l = \overset{l}{\times} g$: $X_l \to \overset{l}{\times} B\Gamma_2^{\mathbb{C}}$ followed by the natural map $\overset{l}{\times} B\Gamma_2^{\mathbb{C}} \to B\Gamma_n^{\mathbb{C}}$. The map g^l determines a singular foliation on X_l of codimension $2l$, and taking the product with the point foliation of $\mathbb{C}^{n-2l}$ defines a singular foliation on $X_l \times \mathbb{C}^{n-2l}$ of codimension n which is classified by f_l.

Now let l be fixed. For each $1 \leqq i \leqq l$, define the map σ_i: $X_{l-1} \to X_l$ to be the inclusion which misses the i^{th}-factor. As both $f_l \circ \sigma_i$ and f_{l-1} determine the same singular foliation on $X_{l-1} \times \mathbb{C}^{n-2l+2}$, and the classifying map of a given foliation is unique up to homotopy, we conclude that $f_l \circ \sigma_i$ and f_{l-1} are homotopic. The transitivity of homotopy yields:

$$f_l \circ \sigma_i \simeq f_l \circ \sigma_j \qquad \text{for all } 1 \leqq i, j \leqq l. \tag{5.2}$$

Choose a point $x_0 \in \overset{k}{\times} S^4 = X_k$ which is not the base point. Let $D^{4k} \subseteq X_k$ be the closed disc neighborhood of x_0. We define the *fat wedge* [17] to be $W = X_k - \{x_0\}$. There is an inclusion of the wedge product $\overset{k}{\bigvee} S^4 \subseteq W$, and the wedge product of k-copies of g gives a diagram

$$\begin{tikzcd} W \arrow[dr, dashed, "F"] & \\ \overset{k}{\bigvee} S^4 \arrow[u, "\cup"] \arrow[r, "\vee g"] & B\Gamma_n^{\mathbb{C}}. \end{tikzcd} \tag{5.3}$$

We claim that $\bigvee g$ extends to a map F making (5.3) commute. There is a general criterion for when a higher order Whitehead product exists (i.e., an extension F of $\bigvee g$ exists), given by Theorem 2.7 of Porter [17]. The assumption of this theorem is that for each fixed $l<k$ and for all possible inclusions $\overset{l}{\bigvee} S^4 \subseteq \overset{k}{\bigvee} S^4$, extensions of the composites

$$\overset{l}{\bigvee} S^4 \subseteq \overset{k}{\bigvee} S^4 \xrightarrow{\vee g} B\Gamma_n^{\mathbb{C}}$$

to maps $\overset{l}{\times} S^4 \to B\Gamma_n^{\mathbb{C}}$ exist, and all of these extensions are homotopic. This is exactly the content of (5.2). We can thus conclude there is an extension F: $W \to B\Gamma_n^{\mathbb{C}}$.

Let f be the composition f: $S^{2n+3} = \partial D^{4k} \subseteq W \xrightarrow{F} B\Gamma_n^{\mathbb{C}}$. The k^{th}-order Whitehead product $[g, \ldots, g] \in \pi_{2n+3}(B\Gamma_n^{\mathbb{C}})$ is defined to be the homotopy class of f. For the dual homotopy class

$$h^{\#} \circ \zeta(y_2 c_2^{k-1}): \pi_{2n+3}(B\Gamma_n^{\mathbb{C}}) \to \mathbb{C},$$

it follows from [4.5; 10] that $h^{\#} \circ \zeta(y_2 c_2^{k-1})([f]) \neq 0$. Let $\tilde{f}$: $S^{2n+3} \to F\Gamma_n^{\mathbb{C}}$ be a lift of some non-zero multiple of f. Then by Theorem 2.2,

$k_*(y_2 c_2^{k-1}) \in H^{2n+3}(F\Gamma_n^{\mathbb{C}})$ is non-zero on the cycle represented by $\hat{f}$. This completes the proofs of Theorems 1.3 and 1.4.

It is interesting to note that the rigid secondary classes $k_*(y_2 c_2^{k-1})$ and $k_*(y_k c_k)$ are shown to be non-trivial by evaluating a "residue" (i.e., $h^\# \circ \zeta(y_2 c_2^{k-1})$ or $h^\# \circ \zeta(y_k c_k)$) about the point omitted in W. This is in exact analogy with the Baum-Bott examples, so that all of the known non-triviality results for secondary classes of transversely holomorphic foliations are obtained using residues.

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