Ergodic Theory of Foliations and a Theorem of Sacksteder

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Abstract

We introduce the leafwise geodesic flow of a foliation, a flow on the unit tangent bundle to the leaves which preserves the natural foliation on this manifold, and use it to study the ergodic theory of foliations. The topological entropy of a foliation is defined to be the topological entropy of this flow relative to the invariant foliation, and the corresponding relative metric entropies are the metric entropies of the foliation. The topological entropy dominates the metric entropies, and the supremum of the metric entropies yields the foliation topological entropy. Upper estimates of foliation metric entropies via transverse Lyapunov exponents are given, from which we deduce a generalization of a theorem of Sacksteder concerning the existence of linearly contracting holonomy along resilient leaves in codimension-one.

*This work was supported in part by a Grant from the Sloan Foundation and the NSF. The hospitality of the Mathematical Institute at Oxford and the I.H.E.S. is gratefully acknowedged.

Introduction. The dynamical theory of foliations has three milestones from the period 1956 to 1965, and all for codimension one (cf. [L1]): Haefliger's Theorem on the non-existence of analytic foliations on 3-manifolds with finite fundamental group; Novikov's Theorem on the existence of toral leaves in foliations of S^3; and Sacksteder's Theorem that a C^2-foliation on a compact manifold with an exceptional minimal set must have a resilient leaf with linearly contracting holonomy. The theorems of Haefliger and Novikov are proved with techniques from the study of singular flows on surfaces. Sacksteder's Theorem is a result about the behavior of pseudo-groups of C^2-diffeomorphisms of the line, and its proof resembles more closely the approach developed in smooth dynamical systems, especially the study of stable manifolds for partially hyperbolic systems in Pesin Theory. The work described in this paper originated, in part, from our attempt to better understand this similarity and to prove extensions of the Sacksteder Theorem.

Let us recall the theorem of Sacksteder: Let $\mathcal{G}$ be a pseudo-group of orientation-preserving, C^2-diffeomorphisms of the real line $\mathbf{R}$, with a finite generating set $\{g_1, g_2, \ldots, g_N\}$. We denote the domain of g_i by $U(g_i)$, an open connected subset of $\mathbf{R}$. A nowhere dense, perfect compact set $K \subset \mathbf{R}$ is called an *exceptional minimal set* if $K \subset \bigcup_{i=1}^{N} U(g_i)$, each $g_i : K \cap U(g_i) \to K$, and K is a minimal, non-empty set with respect to these properties. A *gap* in K is a closed interval J of $\mathbf{R}$ such that $K \cap J = \{x, y\}$ consists of the endpoints of J, and x or y will be called the endpoints of a gap.

Theorem. (Sacksteder [S]). Suppose $\mathcal{G}$ is a finitely generated pseudo-group of orientation preserving C^2-diffeomorphisms of the line with an exceptional minimal set K. Then for each endpoint $x \in K$ of a gap, there is a sequence of points $\{x_p\} \subset K$ tending to x and elements $\{\gamma_p\} \subset \mathcal{G}$ so that $\gamma_p(x_p) = x_p$ and $0 < \gamma_p'(x_p) < 1$. □

An important application of this result is to the qualitative dynamics of C^2-foliations of codimension-one on compact manifolds. Such a foliation $\mathcal{F}$ on M defines for each open transversal $T \subset M$ a finitely generated pseudo-group $\mathcal{G}(T, \mathcal{F})$. A compact subset $K \subset T$ is an exceptional minimal set for $\mathcal{G}(T, \mathcal{F})$ if there is a compact set $K_0 \subset M$ so that $K = K_0 \cap T$, with K_0 a union of leaves, K_0 nowhere dense and perfect and K_0 is minimal with respect to these properties. We say that K_0 is an exceptional minimal set for $\mathcal{F}$. A leaf $L \subset M$ is *resilient* if there is a transversal $T \subset M$ and $x \in T \cap L$ so that x is a limit point of $T \cap L$ *and* there is a closed loop in L, based at x, which generates a local holonomy map γ that is a contradiction in a neighborhood of x in T. If $0 < \gamma'(x) < 1$, then we say L is a linearly contracting resilient leaf.

Theorem′. (Sacksteder). Let $\mathcal{F}$ be a C^2-foliation of codimension-one on a compact manifold M. If $\mathcal{F}$ has an exceptional minimal set K_0 then for every endpoint of a gap of $K = K_0 \cap T$, there is a sequence of linearly contracting resilient leaves in K_0 containing the endpoint in their closure. □

The proof of the Sacksteder Theorem uses the linear geometry of the one-dimensional Cantor set K and the hypothesis that the local diffeomorphisms are C^2 to establish uniform estimates on the rates of contraction for certain elements of $\mathcal{G}$, from which the existence of hyperbolic fixed-points follows. In comparison with Pesin Theory, it is this uniformity of the contractions which yields a proof that avoids the complexity of the methods associated with stable manifold theory of Pesin. Recent expositions of the proof of the Sacksteder Theorem by Cantwell-Conlon [C4] and Godbillon [G] make this comparison more evident. For the case when the pseudo-group is generated by diffeomorphisms which are only $C^{1+\alpha}$, $0 < \alpha < 1$, the uniform estimates no longer hold, and in fact, there are counter examples to the conclusion of the theorem. M. Herman has studied the modulous of continuity for Denjoy examples in (Chapter 10, [Her 2]) and shows, in particular, that for $0 < \alpha < 1$, there is a $C^{1+\alpha}$-diffeomorphism of S^1 with an exceptional minimal set. An exceptional minimal set of a diffeomorphism cannot contain a resilient leaf. Therefore, for $C^{1+\alpha}$ we must add an additional hypopthesis on the dynamics of an exceptional minimal set in order to obtain hyperbolic contractions; i.e., resilient orbits. Investigating this question leads naturally to the ideas of topological and metric entropies for foliations, and to introduce the extremely powerful tools of Lyapunov Exponents and Pesin Theory into foliation dynamics. We obtain the following extension of Sacksteder's Theorem to foliations of class $C^{1+\alpha}$, for

$0 < \alpha \leq 1$:

Theorem 7.4. Let $\mathcal{F}$ be a codimension-one, $C^{1+\alpha}$-foliation of a compact manifold M, for $0 < \alpha \leq 1$. Let K_0 be an exceptional minimal set for $\mathcal{F}$, and suppose $gr(Dh, K) > 0$. Then K_0 contains a linearly contracting resilient leaf which is dense in K_0. □

The extra hypothesis $gr(Dh, K) > 0$ asserts that the linear holonomy cocycle for $\mathcal{F}$ on K has some hyperbolicity, but with no uniformity implied. The idea behind Theorem 7.4 is that with the condition $gr(Dh, K)$ and the compactness of M, there will be sequences of points in K and elements of the holonomy pseudogroup of $\mathcal{F}$ on K so that the points converge, and the derivatives of these elements at these points grow exponentially. Compactness implies again the existence of a transversally hyperbolic geodesic in K, generic with respect to the hyperbolicity, so that a Pesin stable manifold exists along it. From this, one deduces the existence of hyperbolic fixed-points of holonomy in K. Note that in this approach, we have replaced the uniform estimates of Sacksteder's theorem with genericity used in Pesin Theory, but the rest of the proof is quite similar.

A second application of foliation Pesin Theory is to obtain a result first proved by Ghys-Langevin-Walczak using the Poincaré-Bendixson Theory of codimension-one C^2-foliations [GLW]:

Theorem 6.5. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation of codimension-one of a compact manifold M, for some $0 < \alpha \leq 1$. If the foliation topological entropy satisfies $h(M/\mathcal{F}; f_t) > 0$, then $\mathcal{F}$ has a linearly contracting resilient leaf. □

The converse of this result is easily shown: for $\mathcal{F}$ a C^1-foliation, if $\mathcal{F}$ has a resilient leaf, then the topological entropy is positive. This is because a resilient leaf represents the foliation version of homoclinic orbits for flows.

The above discussion concerns one of the applications of the theory we will outline in this paper. Other applications are given in [H7] and [H8], along with proofs. For the rest of the Introduction, we discuss the nature of our program. First, note that definitions of topological and metric entropies for Z^m-actions on compact Hausdorff spaces have been previously given (cf. Introduction to [E]).

The fact that the topological entropy is the supremum of the metric entropies was observed to hold, also. However, these entropies have the unfortunate property that they yield always zero for groups of diffeomorphisms of a compact manifold, unless $m = 1$ when they agree with the usual definition of entropy for a diffeomorphism. The reason for this vanishing is that the usual term "$\ell n H\left(\bigvee_{|\alpha|<r} T^{\alpha}\mathcal{U}\right)$" is divided by the *number* of elements in a ball of radius r. However, elementary geometry and *compactness* imply that this term grows at most linearly in r. Therefore, one should divide not by the volume of a ball, but by its radius to get the topological entropy of a group of diffeomorphisms. This definition then makes sense for any finitely generated group acting by diffeomorphisms on a compact manifold. The entropy depends a priori on a choice of generating set, but for an amenable group one can show the entropy is independent of this choice. In the approach below, we always work with a foliated manifold with a given leafwise metric, and the normalization used is the radius of a geodesic ball in the leaves. For foliations obtained via the suspension of a group action, one can easily see that the above definition agrees with the foliation entropy when the Riemannian metric is chosen so that the generators correspond to geodesics of length one.

A second advantage of using the normalization given by the radius of the ball, is that the entropy of the group action then corresponds to the relative entropy of the geodesic flow on leaves, taken with respect to the natural invariant foliation. The geodesic flow always admits invariant probability measures, so that even for non-amenable group actions, one can obtain the Theorem that the topological entropy is the supremum of the appropriate metric entropies. One next asks whether these metric entropies can be estimated via appropriate Lyapunov exponents, as is the case for diffeomorphisms by the Margulis Theorem. The answer to this is positive, but one must first understand what form the Lyapunov exponents for a foliation or group action should take.

The Lyapunov exponent theory for foliations developed using the leafwise geodesic flow is a natural continuation of the methods of [HK1] where a key technical point was to use Lyapunov adapted metrics to study the regularity (or tempering) of G-cocycles over a foliation. Given the possibility of forming adapted metrics, the logical next step, when $G \subset GL(N, \mathbf{R})$ for some N, is to produce Lyapunov exponents for the cocycle and show these yield asymptotic approximations to the cocycle. However, an immediate difficulty is that producing the exponents for a group action requires the existence of "exponent homomorphisms" of the group into the real numbers, and for foliations we require the more general idea of a homomorphism of the foliation groupoid into the real numbers. There is no reason why such homomorphisms should exist, and for interesting, "large" groups, one knows that they cannot. The solution we propose here for *foliations* is based on an observation about the methods of proof in foliation theory: a common technique is to "choose two points on a leaf and a path (or holonomy element) between them", then look at value of the linear holonomy along the path. The leaves of a foliation on a compact manifold are complete, so every such path can be replaced by a leafwise geodesic segment. It becomes clear that the fundamental object for cocycle theory is the collection of all such leafwise geodesic segments, which form a groupoid, the geodesic flow along the leaves. Let V denote the $(2m + n - 1)$-manifold of unit tangent vectors to the leaves of $\mathcal{F}$, where $\mathcal{F}$ has dimension m and codimension n. A typical point $(x, v) \in V$ consists of $x \in M$, and a unit vector v tangent to the leaf L_x through x. We denote the flow by $f_t : V \to V$. The natural fibration $\pi(x, v) = x$, $\pi : V \to M$ lifts $\mathcal{F}$ to a foliation $S\mathcal{F}$ on V, again of codimension n, and invariant under $\{f_t\}$. The study of the ergodic theory and dynamics of this flow yields a direct approach to the study of foliation dynamics and cocycles over $\mathcal{F}$. In support of this view, we observe that: the cocycles over $\mathcal{F}$ lift to cocycles over the flow; the flow $\{f_t\}$ always has invariant probability measures when M is compact; with respect to an ergodic probability measure ν^* for $\{f_t\}$, one can use the usual Lyapunov Theory for the lifted cocycle over the flow to obtain exponents, which are functions of ν^*. When applied to the linear holonomy cocylce of a $C^{1+\alpha}$-foliation, one obtains exponents for the infinitesimal transverse behavior of $\mathcal{F}$ along generic leafwise geodesics, and when these exponents are not all zero there will also exist *transverse* stable and unstable manifolds along the geodesic. The existence of such hyperbolic behavior is a strong statement about the dynamics of a foliation, and can be deduced simply from the asymptotic behavior of the linear holonomy of $\mathcal{F}$. Since the linear holonomy cocycle is *the* data used to construct secondary classes for a foliation, this begins to yield a direct method of converting cohomological invariants of a foliation into dynamical properties. In the present paper, we restrict our attention to showing how the Lyapunov Exponent Theory can be applied to obtain the two theorems cited above, and other related results.

We remark that the introduction of the leafwise geodesic flow solves the "homomorphism problem" for Lyapunov Exponents over a foliation by "unramifying" the problem over the unit tangent bundle. For each tangent vector $(x, v) \in V$, we get exponents of $\mathcal{F}$, which can be thought of as maps $\mathbf{R} \to \mathbf{R}$ sending 1 to the exponent, and depending upon (x, v). More precisely, the homomorphism (i.e., exponents) depend upon the choice of an ergodic invariant measure ν^* supported in the ω-limit set of the geodesic segment $\{f_t(x, v) | t \geq 0\}$. The solution leaves unanswered the question of how these exponents depend upon the invariant measures ν^* for a given $\mathcal{F}$. Even more basic is how the set of such invariant measures depend upon the geometry of $\mathcal{F}$. It is clear that every transverse invariant measure μ for $\mathcal{F}$ gives rise to an invariant ν^* for $\{f_t\}$, but the converse is false. The f_t-invariant measures are more properly thought of as the atoms of the $\mathcal{F}$-harmonic measures on M (cf. [H 10]). A good understanding of the set of measures ν^* for $\mathcal{F}$, and how the exponents depend upon them are very important aspects of the study of foliation dynamics. A second related problem is that the above procedure gives a method of approximating the lift of a cocycle to $\{f_t\}$ by the exponents along the orbits, but the speed of the approximation will be a measurable function of $(x, v) \in V$. One can ask for some regularity in the speed of approximation with respect to $(x, v) \in V$, or the choice of measure ν^*. Such results would have applications to developing a *metric* classification theory for amenable foliations, for example.

The last section of this paper, §8, lists ten problems which develop in the course of the exposition, and are collected together with some additional remarks.

There is an Appendix in which a brief history of the development of the "hyperbolic" theory of foliations of dimension $m > 1$ is outlined, especially those topics relevant to the task of developing relations between cohomological invariants of a foliation (secondary classes) and its transverse dynamics. The viewpoint is strictly that of the author, but the Appendix offers the reader some of the essential motivation for our approach to foliation ergodic theory, so seemed to have sufficient usefulness to include it. It is impossible to be comprehensive in such a short affair, and the author apologizes in advance for omissions or undue biases, and will welcome comments of suggested inclusions.

We note here that of the many topics not included, some very important ones where hyperbolicity is an (indirect) key factor, are: the rigidity theory of Lie group actions developed by Zimmer (cf. [Z]); the theory of transverse G-structures for foliations and their ergodic theory; the extensive literature on measurable foliations and measurable groupoid theory, especially the works inspired by the Mackey program [M]; the von Neuman algebras approach to classification developed by Feldman-Moore, on which Moore has written an extensive survey [Mo 2]; and the relations between zeta functions for flows and the topology of the ambient manifold, especially the work developed by Franks and Ruelle, and recent works of Fried and Pollicott.

These notes are an expanded treatment of a talk given at the Conference on Smooth Ergodic Theory at College Park, MD during March 9–15, 1987, and of a sequence of lectures given during April and May, 1987 at the Mathematical Institute, Oxford. Complete proofs of the theorems stated here will appear in the papers [H7], [H8]. Also, there are applications of the leafwise geodesic flow and the methods developed here to the study of the spectral theory and cyclic cocycle theory of leafwise elliptic differential operators on foliations. This is discussed in [H9].

Thanks are due to J. Heitsch, A. Katok and F. Ledrappier for helpful comments and discussions during the developement of this work. The Mathematical Institute at Oxford and the I. H. E. S. provided excellent environment for the development of this work, and their hospitality is gratefully acknowledged.

§1. The leafwise geodesic flow*

Let M be a smooth (C^∞) manifold without boundary and let $\mathcal{F}$ be a codimension n foliation of M, whose leaves are at least C^2-submanifolds of M. Let m denote the dimension of the leaves. Choose a Riemannian metric g on TM such that g has bounded geometry, and for each leaf L of $\mathcal{F}$, the restriction $g|L$ on TL defines a complete metric $D_L : L \times L \to R^+$. For M compact, both hypotheses are automatically satisfied for every g.

Let $V \subseteq T\mathcal{F}$ denote the bundle of unit vectors tangent to leaves of $\mathcal{F}$, with $\pi : V \to M$ the natural fibration having fiber S^{m-1}. The foliation $\mathcal{F}$ on M lifts to a foliation denoted $S\mathcal{F}$ on V, where a typical leaf of $S\mathcal{F}$ is T^1L, the unit tangent bundle to a leaf L of $\mathcal{F}$. Note that for each leaf L of $\mathcal{F}$, g restricts to a Riemannian metric $g|L$ on TL. Define the *leafwise geodesic flow* for $(\mathcal{F}, g)$, $f : R \times V \to V$, by

$$f_{t_0}(x,v) = (\exp_x t_0 v,\ \frac{d}{dt}\exp_x tv|_{t=t_0})$$

where $\exp_x : T_xL \to L$ is the geodesic flow of L for $g|L$. Note that a typical orbit $\{f_t(x,v)|t \in \mathbf{R}\}$ of this flow will be a geodesic in the leaf through x, but need *not* be a geodesic for the metric g on TM.

§2. The contact structure of f_t

Given a Riemannian manifold $(L, g|L)$, $L \subset M$, the cotangent bundle T^*L has a natural symplectic form Ω_L with associated contact from ω_L such that $d_L\omega_L = \Omega_L$. The energy functional $H_L : T^*L \to R$ induces a Hamiltonian flow $\{f^*_{L,t}\}$ on T^*L which leaves invariant the energy level sets $H_L^{-1}(c)$, $c > 0$. The contact form ω_L restricts to a contact form on each $H_L^{-1}(c)$, so that $\omega_L \wedge (d_L\omega_L)^{m-1} = \mathrm{Vol}_L$ is a volume form invariant under the flow $\{f^*_{L,t}\}$. Moreover, the duality induced by the metric $g|L$ defines a natural diffeomorphism $T^1L \cong H_L^{-1}(1)$ and conjugates $f_t|L$ to $f^*_{L,t}$, conjugates ω_L to a contact form denoted θ_L on T^1L, and the volume form Vol_L

*Added in Proof: After this was written, the author received a preprint from P. Walczak, "Dynamics of the geodesic flow for foliations", which also introduces the leafwise geodesic flow, then uses it to study metric properties of the leaves of $\mathcal{F}$, discussing problems complementary to the work in this paper.

to the Liouville measure, also denoted by Vol_L, on T^1L. All of these statements are completely standard for a Riemannian manifold (cf. [We]).

The leafwise geodesic flow $\{f_t\}$ defines a foliated contact flow, a notion which we now make precise. A 1-form θ on V is a *Leafwise contact form* for $\mathcal{F}$ if for each leaf L, the restriction $\theta \wedge (d\theta)^{m-1}|T^1$ is the volume form for the natural metric on T^1L. A flow $\{\tilde{f}_t\}$ on V is a foliated contact flow if $\{\tilde{f}_t\}$ preserves the leaves of $\mathcal{F}$ and leaves invariant some leafwise contact form θ. In our case above, for each $T^1L \subset V$, we obtain a 1-form θ_L from $H_L^{-1}(1)$. Then using the metric on TV obtained from g on TM, we extend the leafwise forms θ_L to a 1-form on V by declaring θ to vanish on vectors perpendicular to $S\mathcal{F}$. Similarly, the volume forms Vol_L extend to a global $(2m-1)$-form $d\nu$ on V which satisfied

$$f_t^*(d\nu|_L) = d\nu|_L.$$

$$d\nu|L = \theta \wedge (d\theta)^{m-1}|L = \mathrm{Vol}_L.$$

Associated to the triple (V, f_t, θ), one can define "foliated" symplectic geometry. This allows the construction of the prequantization machinery of Kostant-Sorieau in a foliated context, which for Riemannian foliations can be quantized [H9].

§3. Spectral theory of foliations

The ergodic theory and analysis of a foliation can be divided into two parts – the unitary and the hyperbolic. We discuss here briefly the unitary aspects and some notions of the spectrum of a foliation. The rest of this paper will then be concerned with the hyperbolic aspects, leaving the development of the unitary theory to [H9].

The flow $\{f_t\}$ defines an action of $\mathbf{R}$ on several Banach algebras naturally associated to $\mathcal{F}$, and each of these actions can be used to define a spectrum. First, consider the continuous functions on V, $C(V)$, equipped with the sup norm. For V non-compact, one can also consider the algebra $C_0(V)$ of functions which vanish at infinity. Then $C(V)$ and $C_0(V)$ are complete Banach algebras, and the flow $\{f_t\}$ induces a norm-preserving action

$$Cf_t : C(V) \to C(V).$$

The nature of the action $\{Cf_t\}$ is exactly tied-in with both the global dynamics of $\{f_t\}$ transverse to the leaves of $\mathcal{F}$, *and* the dynamics of the flows $\{f_t|L\}$ in the leaves.

There is a second class of actions induced by $\{f_t\}$ which more closely reflects the transverse dynamics of the flow and of $\mathcal{F}$. Basic to defining these actions is to introduce the convolution algebra $C_c^\infty(V/\mathcal{F})$ of smooth, compactly supported kernels on the leaves of $S\mathcal{F}$. This algebra was introduced by A. Connes in his study [Co1], [Co2] of the index theory of elliptic operators along the leaves of a foliation. Let us recall how to define this algebra in an elementary way.

For $U \subset V$ an open set and $\phi : U \to (-1,1)^{2m+n-1}$ a coordinate system, we say (U, ϕ) is a *foliation chart* if for each $y \in (-1,1)^n$, the set (the *plaque* for y)

$$P_\phi(y) = \phi^{-1}((-1,1)^{2m-1} \times \{y\})$$

is the connected component of the leaf of $\mathcal{F}/U$ containing the point $\phi^{-1}(0,y)$. A *basic element* $k \in C_c^\infty(V/\mathcal{F})$ will be a kernel that has support in the "amalganated product" $(U_0, \phi_0) \times_{\mathcal{F}} (U_1, \phi_1)$ of two foliation charts. More precisely, suppose points $z_0, z_1 \in V$ are given which lie in a common leaf L of $S\mathcal{F}$. Let (U_0, ϕ_0) be a foliation chart about z_0, and (U_1, ϕ_1) a chart about z_1. *In addition*, suppose that there is given a family of paths

$$\gamma : [0,1] \times (-1,1)^n \to V$$

satisfying:

(3.1) for each $y \in (-1,1)^n$, $\gamma_y : [0,1] \to V$ is contained in the leaf of $S\mathcal{F}$ through $\phi_0^{-1}(0,y)$ and $\gamma_y(1) \in \phi_1^{-1}(\{0\} \times (-1,1)^n)$.

(3.2) $$\gamma_0(t) = z_t \quad \text{for } t = 0, 1.$$

Then

$$(U_0, \phi_0) \times_{S\mathcal{F}} (U_1, \phi_1) = \{(x, x', y) | (x,y) \in U_0, x' \in P_{\phi_1}(\gamma_y(1))\}.$$

Then a basic kernel k is a complex-valued smooth function with compact support in the variables (x, x', y). Every element of $C_c^\infty(V/\mathcal{F})$ can then be, *by definition*, written as a finite sum

$$k = \sum_{i=1}^{N} k_i$$

for some N, where each k_i is a basic kernel. Given two basic kernels k and k' with domains $(U_0, \phi_0) \times_{\mathcal{F}} (U_1, \phi_1)$ and $(U_0', \phi_0') \times_{\mathcal{F}} (U_1', \phi_1')$, such that $U_1 \cap U_0' \neq \emptyset$, we define their convolution to be

$$k * k'(x, x', y) = \int k(x, z, y) \times k(z, x', y) \times \text{Vol}_L(z). \quad z \in P_\phi(y). \tag{3.3}$$

If $U_1 \cap U_0' \neq \emptyset$, then we set $k * k' = 0$. The $*$-operation on k is defined by

$$k^*(x, x', y) = \overline{k(x', x, y)}. \tag{3.4}$$

Proposition 3.1. The flow $\{f_t\}$ induces a $*$-automorphism of the $*$-algebra $C_c^\infty(V/\mathcal{F})$.

Proof. The flow $\{f_t\}$ induces a product flow on $V \times V$, which respects the equivalence relation on $V \times V$ defined by $\mathcal{F}$. It therefore maps basic kernels onto basic kernels, where we define

$$Cf_t(k)(x, x, y) = k(f_t(x), f_t(x'), f_t(y)).$$

The flow leaves Vol_L invariant, so the convolution product is invariant under Cf_t. It is clear that $Cf_t(k^*) = Cf_t(k)^*$. □

The C^*-algebra of the foliation $S\mathcal{F}$ is the algebra $C^*(V/\mathcal{F})$ obtained by completing $C_c^\infty(V.\mathcal{F})$ with respect to the norm induced by the representations of the smooth kernels on the leaves of $\mathcal{F}$. The von Neuman algebra of $S\mathcal{F}$, $W^*(V/\mathcal{F})$, is obtained by considering the weak-closure of $C^*(V/\mathcal{F})$ with respect to the standard Lebesgue measure on V. For details, see [Co1], [FS].

Corollary 3.2. The geodesic flow induces isometric $*$-automorphisms:

$$C^* f_t : C^*(V\mathcal{F}) \to C^*(V/\mathcal{F})$$
$$W^* f_t : W^*(V/\mathcal{F}) \to W^*(V/\mathcal{F}).$$

The flow on $C_c^\infty(V/\mathcal{F})$ contains possibly more information than the induced automorphisms of $C^*(V/\mathcal{F})$. The spectrum $\sigma(\mathcal{F})$ of $C^*(V/\mathcal{F})$ detects the "points" of the "space" $V/S\mathcal{F}$, which for an ergodic foliation will often reduce to a singleton [FS]. But for the algebra $C_c^\infty(V/\mathcal{F})$, its cyclic cohomology $H^*(C_c^\infty(V/\mathcal{F}))$ represents the "de Rham cohomology" of $M/\mathcal{F}$, and this can be highly non-trivial for all foliations. The continuous family of automorphisms $H_\lambda^*(Cf_t)$ will be constant in t, so Cf_t induces the identity on cyclic cohomology. None-the-less, the action of the flow on cyclic cocycles will be highly non-trivial, and can be used to produce *local* (cyclic) *indices* along periodic orbits of $\{f_t\}$. These indices are related via a Lefschetz formula to zeta functions for the foliation geodesic flow [H11], (cf. [Fr]).

§4. Topological and metric entropies for foliations

In this section, we will describe the definitions and give some of the properties of the various "entropies" associated to a foliation $\mathcal{F}$ on a (possibly non-compact) manifold M without boundary. The key to defining these entropies is to use the leafwise geodesic flow $\{f_t\}$ on V. The resulting toplogoical entropy $h(M/\mathcal{F}, f_t)$ and metric entropies $h_\nu(M/\mathcal{F}, f_t)$ will depend only upon $\mathcal{F}$ and f_t up to homeomorphism, and the choice of a measure ν, but not upon other choices. Moreover, the topological entropy dominates the metric entropies (corresponding to Goodwyn's theorem for the case of flows), and the supremum of the metric entropies equals the topological entropy (the foliation version of the Dinaberg-Goodman theorem for flows). We can draw a number of conclusions about $h(M/\mathcal{F}, f_t)$ from these basic results.

A definition of topological entropy for a foliation $\mathcal{F}$ was given by Ghys-Langevin-Walczak for compact manifolds M in [GLW]. We denote that entropy by $h_{GLW}(M/\mathcal{F})$. Parts of the present work was motivated by discussions with J. Heitsch about this entropy, and attempts to gain a better understanding of it. These authors use the Bowen approach via (n, ϵ)-separated sets, applied to a compact transversal space to $\mathcal{F}$. Here, "n" represents the distance between two points x and y on the same leaf, measured by counting the least number of flow boxes crossed by a leafwise path from x to y. This definition of entropy depends strongly upon the choice of an open cover for M by foliation charts. In order to eliminate this ambiguity, we define the distance between x and y to be the length of the shortest geodesic (in a given homotopy class) from x to y. The resulting entropy is precisely $h(M/\mathcal{F}, f_t)$, where one can show the answer only depends upon the flow f_t and the topological structure of $\mathcal{F}$. It follows that $h(M/\mathcal{F}, f_t)$ and $h_{GLW}(M/\mathcal{F})$ either simultaneously vanish or are positive. Thus, the results stated in this section solve a number of the questions posed in [GLW].

Let $h(V, f_t)$ denote the ususal topological entropy for the flow $\{f_t\}$. For M compact, and hence V compact, this is defined using either the open cover definition of Adler-Konheim-McAndrew, or the Bowen-Dinaberg definition; but for M open we use the latter definition.

Theorem 4.1. Let $\mathcal{F}$ be a C^1-foliation on a smooth manifold without boundary M, such that the leaves of $\mathcal{F}$ are C^2-submanifolds of M. Let $\{f_t\}$ be a flow on the sphere bundle V over M, where f_t corresponds to the leafwise geodesic flow for some Riemannian metric on TM and V is homeomorphic to the unit leafwise tangent bundle $T^1\mathcal{F}$. For M open, we also assume as given data, a quasi-isometry class $[g]$ of Riemannian metrics on TM. Then

(4.1.1) For each closed, $\mathcal{F}$-saturated subset $\Omega \subset M$, there is a well-defined toplogical entropy for $\mathcal{F}/\Omega$, denoted by $h(\Omega/\mathcal{F}, f_t)$, or $h(\Omega/\mathcal{F}, g_t, [g])$ for Ω not compact.

(4.1.2) Let $\phi : V \to V'$ be a homeomorphism such that ϕ maps the leaves of $S\mathcal{F}$ onto the leaves of a foliation $S\mathcal{F}'$ lifted from $\mathcal{F}'$ on M', ϕ conjugates the flow $\{f_t\}$ into the corresponding flow $\{f'_t\}$, and ϕ is a quasi-isometry with respect to the metrics induced on V and V' by g and g'. Then $h(M/\mathcal{F}, f_t) = h(M'/\mathcal{F}', f'_t)$.

(4.1.3) For each closed $\mathcal{F}$-satuared set $\Omega \subset M$, define

$$h(\mathcal{F}/\Omega) = h(T^1\Omega, f_t) - h(\Omega/\mathcal{F}, f_t).$$

Then $h(\mathcal{F}/\Omega)$ is non-negative, and is called the *leaf entropy* for $\mathcal{F}$ in Ω. □

As described above, given a compact transversal $K \subset M$ to $\mathcal{F}$, we define the topological entropy of $\mathcal{F}$ relative to K, a non-negative number $h(M/\mathcal{F}, f_t, [g]; K)$, using Bowen-Dinaberg approach and geodesic distance along leaves, and some choice of quasi-isometry class of metric $[g]$ on a family of complete transversals to $\mathcal{F}$. For M compact, one can give a corresponding definition $h(M/\mathcal{F}, f_t; K)$ where K is a *complete* transversal to $\mathcal{F}$, using the minimum number of open sets needed for an ϵ-cover in the d_n-metrics of Bowen, again measuring the transverse distortion of the given metric g in a geodesic distance n along leaves. Note that for M compact, the choice of $[g]$ does not effect the resulting entropy. We then set

$$h(M/\mathcal{F}, f_t, [g]) = \sup_K h(M/\mathcal{F}, f_t, [g]; K). \tag{4.2}$$

As natural as the above definition of foliation entropy appears, there is a much more flexible definition which is also quite natural. For the flow $\{f_t\}$ on V, one can choose a complete set of transversals, denoted by $\{S_i\}$ which lie in flowboxes both for the flow $\{f_t\}$ *and* the foliation $S\mathcal{F}$, and each transversal S_i, of dimension $(2m+n-2)$, has a codimension n foliation $\mathcal{F}_i = S\mathcal{F}|S_i$, with the quotient $S_i/\mathcal{F}_i$ diffeomorphic to one of the transversals for $\mathcal{F}$ on M. The discrete groupoid of $\{f_t\}$ induced on the transversal spaces $\{S_i\}$ preserves this foliation, and thus one can repeat the constructions of Adler-Konheim-McAndrew and Bowen-Dinaberg to define entropy for $\{f_t\}$ which are relative to the foliations to the S_i. That is, the transversal distances between points will be measured by projecting them to the T_i and measuring there. One can show this yields precisely the same entropies as defined on M. This definition, using $\{f_t\}$ "acting" on the quotients $\{S_i/\mathcal{F}_i\}$, makes the inequality, for M compact,

$$h(V, f_t) \geq h(M/\mathcal{F}, f_t)$$

obvious, so that the leaf entropy $h(\mathcal{F}, f_t)$ is always non-negative. For M open, a similar estimate holds once a choice $[g]$ has been made.

The standard result for flows, that the supremum in (4.2) can be taken using arbitrarily small compact transversals, also holds for foliations. Thus, the entropy

$h(M/\mathcal{F}, f_t; [g])$ measures the largest transverse exponential growth of $\mathcal{F}$ that can be implemented in a transversal of some arbitrarily small size. In contrast, the transvese linear holonomy cocylce of $\mathcal{F}$ measures the *infinitesimal* transverse exponential growth of $\mathcal{F}$, which should be clearly related to the entropy by a mean-value-theorem, but this requires a transverse invariant measure to obtain "generic" points. We discuss next the existence of such measures.

Recall that a transverse measure μ for a foliation $\mathcal{F}$ on M assigns to each compact transverse submanifold $T \subset M$ to $\mathcal{F}$ a non-negative real number $\mu(T)$. The measure is $\mathcal{F}$-invariant if given two compact transversals T and T' related by the property that each can be decomposed into disjoint, measurable pieces so that the pieces of each related by "sliding" them along the leaves of $\mathcal{F}$", then $\mu(T) = \mu(T')$. A general foliation need not possess any transverse $\mathcal{F}$-invariant measure, except the trivial one $\mu = 0$. Observe that the transverse space for $S\mathcal{F}$ on V is equivalent, as a groupoid under leaf holonomy, with the transverse space of $\mathcal{F}$ on M, and thus both either have non-trivial $\mathcal{F}$-invariant measures, or both do not. However, for M compact the foliation $S\mathcal{F}$ *always* admits transverse $\{f_t\}$-invariant measures; for given a transverse measure μ, the pull-back $f_t^*(\mu)$ is again transverse to $S\mathcal{F}$, and by the choice of a complete transversal to $S\mathcal{F}$, one can suitably renormalize each $f_t^*(\mu)$ so that the new measures $\{\nu_t\}$ have a convergent mean in t, which converges to a transverse measure, ν, invariant under $\{f_t\}$. If μ is $\mathcal{F}$-invariant, then ν will simply be a scalar multiple of μ. We thus have:

Proposition 4.2. Let M be a compact manifold. Each transverse measure μ for $S\mathcal{F}$ on V and choice of complete transversal $T \subset V$ yields a transverse measure ν for $S\mathcal{F}$ which is $\{f_t\}$-invariant, and assigns mass one to the transversal T. □

Given the existence of $\{f_t\}$-invariant measures for $S\mathcal{F}$, it is then straightforward to modify the definition of topological entropy for $\mathcal{F}$, in terms of the flow $\{f_t\}$, to obtain definitions of metric entropy.

Theorem 4.3. Let M be a compact manifold, with foliation $\mathcal{F}$. For each $\{f_t\}$-invariant transverse measure ν for $S\mathcal{F}$, there is a well-defined metric entropy

$h_\nu(M/\mathcal{F}, f_t)$. □

Ledrappier and Young have previously defined the metric entropy of a diffeomorphism relative to an invariant partition (§9, [LY] and remarked that this could possibly have other applications. By extending their construction to flows, one obtains an entropy *on leaves* complementary to the result above, giving a metric entropy model of the foliation leafwise topological entropy.

Fix a complete transversal $T \subset V$ to $S\mathcal{F}$, then introduce the cone $\mathcal{M}(f_t, T)$ of $\{f_t\}$-invariant transverse measures for $S\mathcal{F}$ such that T has total mass one. Let $\mathcal{M}_e(f_t, T)$ denote the subset of ergodic measures. Both sets are non-empty by Proposition 4.2. The following extensions of the upper estimation of metric entropy by Goodwyn and the supremum theorem of Dinaberg-Goodman hold:

Theorem 4.4. Let M be a compact manifold and $\mathcal{F}$ a C^1-foliation of M with C^2 leaves. For each complete transversal $T \subset M$ to $\mathcal{F}$ and $\nu \in \mathcal{M}(f_t, T)$,

$$h_\nu(M/\mathcal{F}, f_t) \leq h(M/\mathcal{F}, f_t). \tag{4.3}$$

Moreover,

$$\underset{\nu \in M(f_t, T)}{\text{supremum}}\ h_\nu(M/\mathcal{F}, f_t) = h(M/\mathcal{F}, f_t). \tag{4.4}$$

□

The proof of (4.3) uses the "open-covers" definition of topological entropy, and follows the method of proof given in §7.3 of Walters [Walt] for this estimate with flows. The proof of (4.4) uses the Bowen-Dinaberg definition of topological entropy and the construction of elements of $\mathcal{M}(f_t, T)$ from long pieces of orbits of f_t, and again the proof of (4.4) for flows given by Walters adapts easily to foliations.

The supremum principle (4.4) can be used to prove a number of qualitative properties of the topological entropy. For example, we say a leaf L of is *non-wandering* if for each point $x \in L$ and each open ball $B(x, \epsilon)$ about x in M, the intersection $L \cap B(x, \epsilon)$ contains an infinite number of connected components. The union of the wandering leaves of $\mathcal{F}$ is denoted by $\Omega(\mathcal{F})$, and its complement consists of the *proper* leaves. The set of proper leaves is Lebesgue measurable, so the same holds for $\Omega(\mathcal{F})$. Note that $\Omega(S\mathcal{F}) = \pi^{-1}(\Omega(\mathcal{F}))$. A transverse measure ν for $S\mathcal{F}$ is non-wandering if $\nu(\Omega(\mathcal{F}) \cap T) = \nu(T)$ for all transversals T to $S\mathcal{F}$. Denote by $\mathcal{M}_{nw}(f_t, T)$ the subset of $\mathcal{M}(f_t, T)$ of non-wandering measures. One then deduces from (4.4) the

Corollary 4.5. Let M be compact. Then

$$h(M/\mathcal{F}, f_t) = \underset{\nu \in \mathcal{M}_{nw}(f_t, T)}{\text{supremum}} \; h_\nu(M/\mathcal{F}, f_t).$$

□

If every leaf of $\mathcal{F}$ is proper, then we have $\Omega(S\mathcal{F}) = \emptyset$, so in particular:

Corollary 4.6. Let M be compact and suppose that every leaf of $\mathcal{F}$ is a proper. Then for all leafwise geodesic flows $\{f_t\}$, the topological entropy $h(M/\mathcal{F}, f_t)$ is zero. □

We conclude this discussion with a remark on other definitions of metric entropy. As mentioned in the Introduction, Statistical Mechanics provides physically-motivated definitions of metric entropy for amenable group actions, but these vanish for C^1-diffeomorphisms acting on a compact manifold, and detect fundamentally different aspects of the dynamic system than the entropies of this section. For an $\mathbf{R}$-action on a C^*-algebra or W^*-algebra, equipped with an invariant weight, Connes [Co3] has defined a metric entropy. Applying his construction to the flow $\{C^* f_t\}$ on $C^*(V/\mathcal{F})$ in §3, when $\mathcal{F}$ has a transverse invariant measure, we obtain a metric entropy for $\mathcal{F}$. The relation between this entropy and ours is not clear, but the Connes entropy seems to agree rather with the entropy to Statistical Mechanics (cf. [CS]).

§5. Lyapunov theory for foliations

Let Γ be a finitely generated group, (X, ν) a probability space and $\rho : \Gamma \times X \to X$ an ergodic action which preserves the measure ν. A *cocyle* ϕ over ρ into a Lie group G is a "homomorphism" $\phi : \Gamma \times X \to G$; i.e., ϕ satisfies

$$\phi(\gamma_1, x) \cdot \phi(\gamma_2, \gamma_1 x) = \phi(\gamma_2 \circ \gamma_1, x). \tag{5.1}$$

For ϕ a measurable function and $G \subset GL(N, \mathbf{R})$ a linear group, the problem of constructing Lyapunov exponents for ϕ is to show there is an actual homomorphism $\rho_1 : \Gamma \to G$ so that for all $\epsilon > 0$ and generating sets $\Gamma_0 \subset \Gamma$, there is a coboundary $F : X \to G$ such that for all $\gamma \in \Gamma_0$,

$$\ln \|\phi(\gamma, x) \cdot F(\gamma x)^{-1} \cdot \rho_1(\gamma)^{-1} \cdot F(x)\|_{GL(N,\mathbf{R})} < \epsilon, \quad \text{a.e. } x \in X.$$

For $\Gamma = Z$ and $G = GL(N, \mathbf{R})$, the Oseledec Theorem asserts that such a ρ_1 always exists; but for other groups Γ the existence of such ρ_1 is often unknown (cf. [H5]).

For a foliation $\mathcal{F}$ on M, a cocycle ϕ over $\mathcal{F}$ will be a homomorphism

$$\phi : \Gamma(\mathcal{F}) \to G$$

where $\Gamma(\mathcal{F})$ is the universal topological groupoid of $\mathcal{F}$. An element consists of two points $x, y \in M$ on the same leaf and a homotopy class of paths from x to y *in the leaf of* x. Composition is given by

$$(x, y, [c_1]) * (y, z, [c_2]) = (x, z, [c_1 * c_2]).$$

For a foliation, the problem of defining Lyapunov exponents for a given cocycle ϕ looks even more hopeless than in the group action case. However, if one views the data $\rho_1 : \Gamma \to GL(N, \mathbf{R})$ above as simply giving a sequence of elements in $GL(N, \mathbf{R})$ as one goes off to infinity in Γ, then the geodesic flow for $\mathcal{F}$ can be used to provide a very satisfactory solution. Moreover, this solution can be applied to any group action via the suspension construction, whenever Γ is given as the fundamental group of a compact manifold, B. If B has, moreover, a "canonical" Riemannian metric and hence geodesic flow, then the resulting exponents will also be canonical. We begin with an elementary observation.

Proposition 5.1. Let $\mathcal{F}$ be a foliation on M and $\phi : \Gamma(\mathcal{F}) \to G$ a cocycle over $\mathcal{F}$. Then there is a canonical lift of ϕ to a cocycle $S\phi$ over $S\mathcal{F}$, and this induces a cocycle over the flow $\{f_t\}$:

$$F\phi : \mathbf{R} \times V \to G.$$

If ϕ_1 and ϕ_2 are cohomologous, then $F\phi_1$ and $F\phi_2$ will also be cohomologous. □

For $G = GL(N, \mathbf{R})$ and for each ergodic probability measure ν^* for $\{f_t\}$ acting on V, we can apply the usual Oseledec Theorem (cf. [R]) to obtain the Lyapunov exponents for the cocycle $F\phi$ over the flow $\{f_t\}$ with respect to the measure ν^*. The flow f_t is by diffeomorphisms, so we obtain:

Theorem 5.2. Let $\phi : \Gamma(\mathcal{F}) \to GL(N, \mathbf{R})$ be a measurable cocycle over $\mathcal{F}$ and $\nu \in \mathcal{M}(f_t; T)$ an ergodic $\{f_t\}$-invariant transverse measure on $S\mathcal{F}$. Then for some $1 \leq s \leq N$, there exists exponents $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ and a filtration of $\mathbf{R}^N$, depending measurably on x,

$$E_1(x) \oplus \ldots \oplus E_s(x) \cong \mathbf{R}^N$$

so that for ν-a.e. $x \in V$,

$$\lim_{t\to\infty} \frac{1}{t} \ln \|F\phi(x,t)v\| = \lambda_i \quad \text{for all} \quad 0 \neq v \in E_i(x). \tag{5.2}$$

□

The exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ and the measurable partition $\{E_i(x)\}$ will depend upon the choice of ν, in general, so a precise notation would better be $\{\lambda_i(\nu)\}$ and $\{E_i^\nu(x)\}$; we adopt this notation only when it is necessary to emphasize the dependence on ν.

There is little necessary to say for the proof of this theorem, except that given the measure ν, one uses the invariant Liouville measure on leaves to obtain an invariant measure, ν^*, on V for $\{f_t\}$, and one applies the usual Oseledec Theorem to an ergodic summand of ν^*.

For the case of a group action $\Gamma \times X \to X$, where X is a compact manifold and Γ acts via C^1-actions, $\mathcal{F}$ will be the foliation on the quotient $(\widetilde{B} \times X)/\Gamma$ obtained from the foliation of $\widetilde{B} \times X$ with leaves $\{\widetilde{B} \times pt\}$. A leafwise geodesic in $\mathcal{F}$ then corresponds precisely to a path in the universal cover $\widetilde{B}$, so can truly be thought of as geometrically going to infinity in Γ. The exponents $\{\lambda_1, \ldots, \lambda_s\}$ are then the "principal values" of the sought after homomorphism ρ_1 restricted to this path, and these exist even when ρ_1 cannot.

Every C^1-foliation $\mathcal{F}$ of codimension n has a natural cocycle $Dh : \Gamma(\mathcal{F}) \to GL(n, \mathbf{R})$, called the linear holonomy cocycle, obtained by expression the natural linear parallelism along leaves in terms of an orthonormal measurable framing of the normal bundle Q of $\mathcal{F}$. Applying Theorem 5.2 with $\phi = Dh$, we conclude that for each $\nu \in \mathcal{M}_e(f_t, T)$, there exists a canonical set of exponents reflecting the infinitesimal hyperbolic transverse behavior of $\mathcal{F}$ along ν-a.e. leafwise geodesic in the essential support of ν. These exponents yield very strong qualitative behaviour about $\mathcal{F}$. For example, there is a foliation version of the Margulis upper estimate for metric entropy:

Theorem 5.3. Let M be a compact manifold and $\mathcal{F}$ a C^1-foliation of codimension n. For each ergodic f_t-invariant measure $\nu \in \mathcal{M}(f_t, T)$, there are Lyapunov exponents $\{\lambda_1(\nu), \ldots, \lambda_s(\nu)\}$ for $s \leq n$ and

$$h_\nu(M/\mathcal{F}, f_t) \leq \sum_{\lambda_i(\nu)>0} \lambda_i(\nu). \tag{5.3}$$

□

Strelcyn first proved a version of this type of result, and Theorem 5.3 can be thought of as a specialization of the results in Appendix, Chapter V of [KS]. We thank F. Ledrappier for pointing this out.

Corollary 5.4. Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If $h(M/\mathcal{F}, f_t) > 0$, then there exists an ergodic, non-wandering measure $\nu \in \mathcal{M}(f_t, T)$ such that some $\lambda_i(\nu) > 0$. □

Let G be a normed group, and $\mathcal{F}$ a foliation equipped with a Riemannian metric on leaves. Then the growth rate of a cocycle ϕ over $\mathcal{F}$ is defined by:

$$gr(\phi) = \lim_{\substack{x,y\in M \\ |[c]|\neq 0}} \sup \frac{1}{|[c]|} \ln \|\phi(x, y, [c])\| \tag{5.4}$$

where $|[c]|$ is the infimum of the lengths of all C^1 curves in the homotopy class [c]. For $K \subset M$ a saturated subset, we also define

$$gr(\phi, K) = \lim_{\substack{x,y\in K \\ |[c]|\neq 0}} \sup \frac{1}{|[c]|} \ln \|\phi(x, y, [c])\|. \tag{5.4'}$$

Corollary 5.5. Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If $h(M/\mathcal{F}, f_t) > 0$, then $gr(Dh) > 0$. □

One of the standard problems in the ergodic theory for flows is to obtain a converse to Corollary 5.5. That is, what hypotheses on an invariant measure ν and the Lyapunov exponents $\{\lambda_1(\nu), \ldots, \lambda_s(\nu)\}$ for Dh are sufficient to imply $h_\nu(M/\mathcal{F}, f_t) > 0$? To require that the exponents all be non-zero is certainly not sufficient (cf. Introduction to [K]). However, with some extra hypotheses on the measure, ν, for example, converses do exist and this brings us to the role of Sacksteder's Theorem in this paper, as discussed in the next section. We close this section with a result that can be easily shown using the subadditivity of the norm $\ln \|\cdot\|$ and weak $*$ compactness of transverse measures.

Proposition 5.6. Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Assume that the growth type of the linear holonomy is positive, $gr(Dh) > 0$. Then there exists an ergodic $\nu \in \mathcal{M}_e(f_t, T)$ for which $\lambda_i(\nu) > 0$ for some i. □

In the conclusion of this proposition, there is nothing to a priori prevent the support of the measure ν from being a closed orbit of the flow $\{f_t\}$, or more generally to be wandering.

§6. Application of Lyapunov exponents

We give some applications now of the Lyapunov exponent theory to the study of foliation dynamics, which are based upon the following notion and subsequent theorem. For $(x, v) \in V$, denote by $\gamma^+(x, v) = \{\pi(f_t(x, v)) | 0 \leq t < \infty\}$ the geodesic ray in the leaf L_x through x, starting at x with velocity v. Then a *stable transverse manifold* along $\gamma^+(x, v)$ for the flow $\{f_t\}$ is an immersion of a strip, for some $\epsilon > 0$ and $1 \leq p \leq n$, with uniform subexponential estimates on the derivatives:

$$\Gamma_{x,v} : [0, \infty) \times (-\epsilon, \epsilon)^p \to V \tag{6.1}$$

such that

$$\Gamma_{x,v} \text{ is uniformly transverse to } S\mathcal{F} \tag{6.1.1}$$

$$\pi \circ \Gamma_{x,v}(t,0) = \pi(f_t(x,v)), \quad 0 \leq t < \infty. \tag{6.1.2}$$

$$f_t(\Gamma_{x,v}(s,z)) = \Gamma_{x,v}(s+t+\epsilon(s,t,z), G(x,t,z)) \tag{6.1.3}$$

where $\epsilon(s,t,0) = 0$, and $G : [0,\infty) \times [0,\infty) \times (-\epsilon,\epsilon)^p \to (-\epsilon,\epsilon)^p$ are functions defined by (6.1.3). Finally, there exist constants $c_0 > 0$ and $c_1 < 0$ so that

$$\|G(x,t,x)\| \leq c_0 \cdot \exp(t \cdot c_1) \tag{6.1.4}$$

The stable manifold theorem of Pesin [Pe], [R], [FHY] along with standard methods of transversality combine to yield:

Theorem 6.2. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation of codimension n, for some $0 < \alpha \leq 1$. Suppose $\{f_t\}$ admits an ergodic measure $\nu \in \mathcal{M}(f_t, T)$ for which $\lambda_1(\nu) < 0$. Then for ν-a.e. point $(x,v) \in V$, there is a stable transverse manifold along $\gamma^+(x,v)$, where $p = \dim E_1(x,v)$ and for all $\delta > 0$, we can take $c_1 = \lambda_1(\nu) + \delta$. □

Combining Theorem 6.2 with Corollary 5.4, the observation that the flow $\{f_t\}$ is reversible and that ν-a.e. the exponents for the reverse flow are the opposites of those for $\{f_t\}$, we obtain:

Theorem 6.3. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation of codimension n on a compact manifold M. If $h(M/\mathcal{F}, f_t) > 0$, then there exists a non-wandering ergodic measure $\nu \in \mathcal{M}(f_t, T)$ so that for ν-a.e. $(x,v) \in V$, there is a stable transverse manifold along $\gamma^+(x,v)$. □

A foliation $\mathcal{F}$ is said to be *distal* if for all transversals T to $\mathcal{F}$ and pairs $x, y \in T$, there is a constant $c = c(x,y) > 0$ so that if $h : U \to V$ is an element of the holonomy of $\mathcal{F}$ restricted to T, for $U, V \subset T$ with $x, y \in U$, then $\text{dist}_T(h(x), h(y)) \geq c$. That is, along any path in a leaf of $\mathcal{F}$,the holonomy along this path does not attract any nearby leaves. For example, if $\Gamma \times X \to X$ is a C^1-action of a fundamental group $\Gamma = \pi_1(B)$ on a manifold X, then the suspension foliation $\mathcal{F}$ on $M = (\tilde{B} \times X)/\Gamma$ is distal precisely when Γ acts distally on X; i.e., for all $x, y \in X$, there is $c(x,y) > 0$ so that $\text{dist}(\gamma.x, \gamma.y) > c(x,y)$ for all $\gamma \in \Gamma$. It is an unsettled question whether a distal foliation must have all secondary characteristic classes zero (cf. [H6], [H4]). However, by the above theorem, it is obvious that:

Corollary 6.4. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation on a compact manifold M. If $\mathcal{F}$ is distal, then $h(M/\mathcal{F}, f_t) = 0$ for all geodesic flows $\{f_t\}$. Consequently, for every $\nu \in \mathcal{M}(f_t, T)$, $h_\nu(M/\mathcal{F}, f_t) = 0$ also. □

Our second application of Lyapunov exponents is to prove a result relating entropy with resilient leaves for $C^{1+\alpha}$-foliations of codimension one. The following result was first proved in [GLW] for C^2-foliations, using the structure theory of codimension one foliations. On the other hand, the proof given below is very natural, as it is the specialization to codimension one of a more general result described in the next section.

Theorem 6.5. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation of codimension one on a compact manifold M, for some $0 < \alpha \leq 1$. If $h(M/\mathcal{F}, f_t) > 0$, then $\mathcal{F}$ has a resilient leaf with linearly contracting holonomy. □

The proof of this involves one new notion, that of sliding a non-closed geodesic arc along a leaf to produce elements of holonomy for a given fixed transversal. To wit, by the hypothesis, there is a non-wandering ergodic ν and a stable transverse manifold $\Gamma_{(x,v)}$ for ν-a.e. $(x, v) \in V$. As $n = 1$, we must have $p = n$ so that $\Gamma_{x,v}$ is a full transversal to $S\mathcal{F}$. Because the measure ν is non-wandering, there must exist such an (x_0, v_0) for which $\gamma_0^+ = \gamma^+(x_0, v_0)$ intersects some flow box (U, ϕ) infinitely often, with an accumulation point in the interior of U. For t_0 sufficiently large, we then set $y_0 = \pi(f_{t_0}(x_0, v))$, choose the transversal $T_0 = \{\Gamma_0(0, z) | -\epsilon < z < \epsilon\}$ and by (6.1.4) we can assume that y_0 is on some plaque $P_\phi(y)$, for $y = \Gamma_0(0, z)$, some $|z| < \epsilon/2$ and that $c_0 \cdot \exp(t_0 \cdot c_1)$ is much smaller than ϵ. This gives the situation of figure 6.1. The leaves of $\mathcal{F} \cap U$ can be thought of as horizontal slices in the figure, so that by sliding γ_0^+ along the plaque $P_\phi(y)$, we obtain $h_0 : T_0 \to T_0$ which is a linear contraction. The map h_0 will have a fixed-point $z_0 \in T_0$ with linearly contracting holonomy, and thus the leaf L' through z_0 will have a holonomy loop with the same property. However, nothing so far prevents L' from being a wandering leaf, even compact, and so the positivity of $h_\nu(M/\mathcal{F}, f_t)$ enters. The map $\Gamma_{x,v}$ depends measurably on the choice of $(x, v) \in V$, so there is a set of positive ν-measure, say

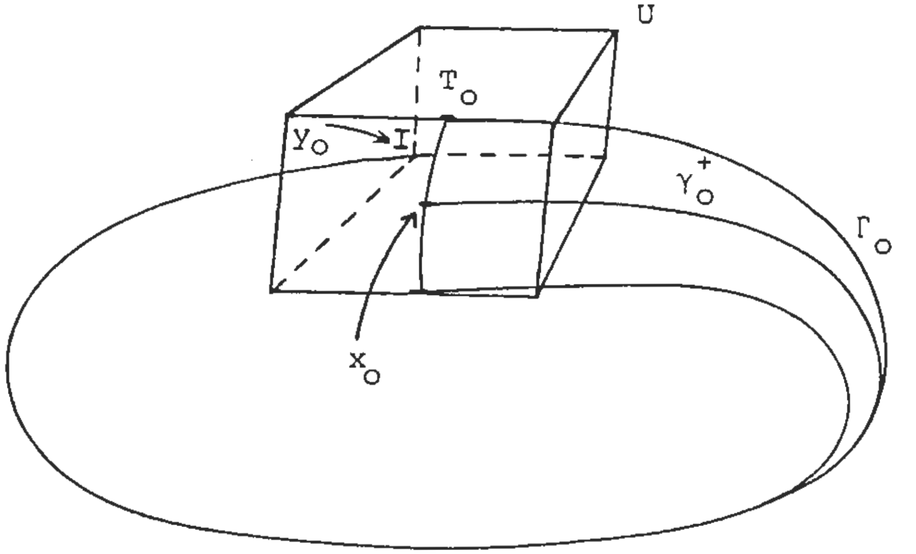

Figure 6.1

$X(\delta)$, for which every $(x, v) \in X(\delta)$ has the image of $\Gamma_{x,v}(0, (-\epsilon, \epsilon))$ a transverse arc of length at least $\delta > 0$. Choose a point $(x_1, v_1) \in X(\delta)$ which is generic for ν, and thus the segment $\gamma^+(x_1, v_1) = \gamma_1^+$ is not proper; i.e. is recurrent. Then construct h_1 as above for this γ_1^+, where t_1 is chosen large enough that z_1 is within $\delta/10$ of x_1. Choose a second generic point (x_2, v_2) within $\delta/2$ distance of (x_1, v_1), and let h_2 be the holonomy about z_2. Then z_1 will be in the domain of h_2, and z_2 will be in the domain of h_1. It follows that the pseudogroup on a common transversal T_3 containing both domains of h_1 and h_2 will generate an exceptional minimal set with the orbits of z_1 and z_2 being resilient. In fact, as the conjugates of h_1 or h_2 are dense in the exceptional minimal set, there will be infinite number of periodic orbits for the holonomy, and they grow exponentially fast in number as a function of their word length. □

The proof of Theorem 6.5 is of course very similar to standard arguments of the dynamics of hyperbolic flows (cf. [K]). In the above, the leaves of the foliation $\mathcal{F}$ are being used in place of a strong unstable foliation for $\{f_t\}$. If $\{f_t\}$ were also hyperbolic in the leaves of $\mathcal{F}$ (e.g., there was a metric of negative curvature on the leaves) then the orbits of $\{f_t\}$ could be closed up to give closed leafwise geodesics with transverse hyperbolicity. As this is not so important to us, we can close up the orbits using only the slide-along-leaves method, which produces non-geodesic closed loops.

The above theorem suggests that the proof of Sacksteder's Theorem can be viewed as a two-step process: First, given a $C^{1+\alpha}$-foliation with an exceptional minimal set K, so that $K \cap T$ is a Cantor set for each transversal T, the "geometry" of the Cantor sets $K \cap T$ can be used to obtain $h(K/\mathcal{F}, f_T) > 0$. Then the above theorem, using Lyapunov exponents and Pesin stable manifold theory, yields the rich dynamical behavior of $\mathcal{F}$ in K expected. These two steps cn be seen in all proofs of the Sacksteder Theorem, but with the C^2 hypothesis usually imposed, the uniformity of the estimates makes the geometry much simpler.

§7. Sufficient conditions for positive foliation entropy

In this section, we explore a number of sufficient conditions for a foliation to have positive entropy. These will, in general, require $\mathcal{F}$ to be $C^{1+\alpha}$, and the linear holonomy of $\mathcal{F}$ to have hyperbolicity on some non-wandering set. We begin with codimension-one, and first recall a result whose proof is elementary (cf. [GLW]).

Proposition 7.1. Let $\mathcal{F}$ be a C^1-foliation of codimension-one on a compact manifold M. Suppose $\mathcal{F}$ has a resilient leaf, L (not necessarily linearly contracting). Then $h(K/\mathcal{F}, f_t) > 0$, where K is the closure of L in M. □

Corollary 7.2. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation of codimension-one on a compact manifold M. If $\mathcal{F}$ has a resilient leaf L, then $\mathcal{F}$ also has a linearly contracting resilient leaf L'. □

Next, from the proof of Theorem 6.5 we can extract the following result.

Proposition 7.3. Let $\mathcal{F}$ be codimension-one, $C^{1+\alpha}$-foliation on compact manifold M. Suppose there exists a non-wandering ergodic measure $\nu \in \mathcal{M}(f_t, T)$ with $\lambda_1(\nu) < 0$, *and* there exists a point $(x, v) \in V$ *which* is generic for ν and the cocycle FDh, and such that the closure $\overline{\gamma^+(x,v)} \subset \Omega(\mathcal{F})$. Then $\mathcal{F}$ has a linearly contracting resilient leaf, and in particular, $h(M/\mathcal{F}, f_t) > 0$. □

Let $K \subset M$ be an exceptional minimal set for $\mathcal{F}$. Then each $L \subset K$ has closure all of K, hence $K \subset \Omega(\mathcal{F})$ and given a measure $\nu \in \mathcal{M}(f_t, T)$ supported on K, for every generic point (x, v) of ν, we have $\gamma^+(x, v) \subset K \subset \Omega(\mathcal{F})$. This is the geometric basis for our main result in codimension-one:

Theorem 7.4. Let $\mathcal{F}$ be a codimension-one, $C^{1+\alpha}$-foliation of a compact M. Let K be an exceptional minimal set for $\mathcal{F}$, and suppose $gr(Dh, K) > 0$. Then $h(K/\mathcal{F}, f_t) > 0$, and $\mathcal{F}$ has a linearly contracting resilient leaf in K.

Proof. The norm $\ln \| \ \|$ is additive on $GL(1, \mathbf{R})$. The hypothesis $gr(Dh, K) > 0$ implies there are arbitrarily long geodesic arcs in K along which Dh grows at an exponential rate. For a complete transversal T to $\mathcal{F}$, the cocycle Dh restricted to T is (cohomologous to a) continuous cocycle, so by the weak-* compactness of probability measures on T and the Birkhoff Ergodic Theorem, there is a limit ergodic measure $\nu \in \mathcal{M}(f_t, T)$ supported on K with $\lambda_1(\nu) > 0$. By the remark preceding the proposition, we can now apply Proposition 7.3 to conclude $h(K/\mathcal{F}, f_t) > 0$. □

The above theorem can be considered as an extension of Sacksteder's Theorem from C^2- to $C^{1+\alpha}$-foliations. For any exceptional minimal set K, the geometry of the Cantor set $K \cap T$ and the Mean-Value-Theorem imply that Dh is unbounded on K. For the Denjoy counterexample, Dh has subexponential growth on the exceptional set K, so the question becomes what additional geometric hypotheses will suffice to force Dh to have some (non-uniform) exponential growth: i.e., when does there exist $C < 0$, sequences $\{x_i\} \subset K \cap T$ and holonomy elements $\{\gamma_i : U_i \to V_i\}$ where $x_i \in U_i \subset T$ such that the derivatives satisfy

$$|\gamma_i'(x_i)| \leq \exp . C \cdot |\gamma_i|$$

where $|\gamma_i|$ is the length of the leafwise geodesic from x_i to $\gamma_i(x_i)$. Given such a sequence of partial geometric contractions in K, the compactness of K, continuity of the cocycle Dh, ergodicity and the Pesin regularity theory for stable manifolds force K to be "geometric"; i.e., to be the closure of a leaf with linearly contracting holonomy. It is exactly this sequence of ideas that this paper formalizes, both to help gain insight into the meaning of the Sacksteder Theorem, and to explore possible extensions to higher codimension, which we turn to now.

There are basically two types of hypotheses which are sufficient to guarantee $h(M/\mathcal{F}, f_t)$ is positive for $\mathcal{F}$ of codimension greater than one. The first involves the notion of completely (transversally) hyperbolic measures and is a direct generalization of Proposition 7.3, and extends to foliations the ideas of Katok for flows from [K]. The second approach, and one which is uniquely applicable to the foliation context, is to require the linear holonomy cocycle Dh be *essentially hyperbolic*. This condition replaces the hypothesis that FDh have hyperbolic behavior on a measure which is well-distributed with respect to the hyperbolicity, as seen in codimension one above, with enough generators of the infinitesimal holonomy of $\mathcal{F}$ to force the partially hyperbolic measures to have intersecting stable manifolds, and hence again to generate homoclinic orbits and positive entropy. These ideas are discussed in [H8].

An ergodic measure $\nu \in (f_t, T)$ is *transversally hyperbolic* if all exponents $\lambda_i(\nu) \neq 0$. Following Katok, we say an invariant measure ν^* for $\{f_t\}$ is *hyperbolic* if all of the exponents for the flow $\{f_t\}$ on V are distinct from 0. The following result is the foliated version of a result in [K]:

Theroem 7.5. Let $\mathcal{F}$ be $C^{1+\alpha}$-foliation of codimension n. Suppose there exists a non-wandering, transversally hyperbolic ergodic measure $\nu \in \mathcal{M}(f_t, T)$. Then $h(M/\mathcal{F}, f_t) > 0$. Moreover, there exists a collection of closed loops $\{\gamma_i\}$ in the leaves of $\mathcal{F}$ such that for each i, the holonomy h_i along γ_i has hyperbolic linear part, and the closure of the saturations of the γ_i contains support ν:

$$\text{support}(\nu) \subset \overline{\cup_i \text{Sat}_{SF}(\gamma_i)}.$$

□

A foliation $\mathcal{F}$ is an $SL(n, \mathbf{R})$-foliation if there is a closed n-form $d\overline{y}$ on M whose kernel is precisely the tangent distribution to $\mathcal{F}$. For an $SL(n, \mathbf{R})$-foliation, the cocycle Dh can be chosen to take values in $SL(n, \mathbf{R})$.

Corollary 7.6. Let $\mathcal{F}$ be a $C^{1+\alpha}$, $SL(2, R)$-foliation on a compact manifold M. If there exists a non-wandering, ergodic measure $\nu \in \mathcal{M}(f_t, T)$ with $\lambda_1(\nu) < 0$, then the conclusions of Theorem 7.7 hold for $\mathcal{F}$ and ν. □

The *algebraic hull* of the cocyle Dh is the smallest algebraicly closed subgroup $H \subset GL(n, \mathbf{R})$ for which Dh is cohomologous to a cocycle with values in H. As we can allow cohomologies which are either measurable or continuous, there are two possible algebraic hulls which result, denoted by H_m and H_c, respectively. It is a theorem of Zimmer that the algebraic hull exists [Z]. Note that $H_m \subset H_c$, and the equality can be strict. A subgroup $H \subset GL(n, \mathbf{R})$ is amenable if, when equipped with the induced topology, it is amenable as a topological group. The maximal amenable subgroups of $GL(n, \mathbf{R})$ have been classified by Moore [Mo1] (see also section 4 in [HK1]). We say that Dh is *(measurably) essentially hyperbolic* if the algebraic hull H_c (or H_m) is not amenable.

Recall that for a point $x \in M$, the fundamental group of the leaf L_x through x can be identified with $\pi_1(L_x, x) \simeq \{(x, x, [c]) \in \Gamma(\mathcal{F})\}$. The restriction of Dh to this subgroup is called the linear holonomy homomorphism for L_x based at x. We say L_x has *hyperbolic linear holonomy* if the algebraic hull of the image of

$$Dh : \pi_1(L_x, x) \to GL(n, \mathbf{R})$$

is not amenable. Our last theorem is then an extension Theorem 1 of [H2] to this context.

Theorem 7.7. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliatiion of a (possibly open) manifold M. If $\mathcal{F}$ has a leaf with hyperbolic linear holonomy, then $h(M/\mathcal{F}, f_t) > 0$. □

§8. Problems, questions and final comments

We gather here some of the problems, questions or conjectures which arose from the text. These fall naturally into two classes: Those asking whether some property or area of research for flows can be extended to the context of the geodesic flow of a foliation (and of which we give only a small sampling of the possible questions), and those problems based on relating the ergodic theory of the geodesic flow with other areas in the study of foliations, especially the continuation of the program developed in [D], [CC5], [H1], [HH], [H1], [H2], [H4], [HK1].

For a classical dynamical system, a basic problem is to find sufficient conditions for the flow to be ergodic. As discussed in the talks by Katok and also Burns-Gerber at this conference, this was one of the original motivations for the development of the Pesin Theory, to be applied to the Billiards Problem. For foliations, we can pose the more modest:

Problem 8.1. Given an ergodic, $\mathcal{F}$-invariant transverse measure μ for $\mathcal{F}$ on M, find conditions on the linear holonomy cocycle Dh which are sufficient to imply μ is absolutely continuous with respect to Lebesgue measure on transversals. What hypotheses on Dh are sufficient to imply that M has a countable decomposition into ergodic, $\mathcal{F}$-saturated measurable sets?

The Smale program for the study of differentiable dynamics [Sm] focused attention on the non-wandering set Ω of a flow, and especially on transversally hyperbolic behavior along orbits in Ω.

Problem 8.2. Carry over to $\Omega(\mathcal{F})$ the ideas associated to the non-wandering set of a flow. Can sufficient hyperbolicity hypotheses on Dh near $\Omega(\mathcal{F})$ be formulated to deduce a structure theory for foliations, parallel to the theory for Axiom A flows?

The tests for positive entropy developed in section 7 relied on the construction of hyperbolic closed loops in the leaf holonomy. The Bowen-Dinaburg definition of topological entropy of $\mathcal{F}$ suggests that the entropy can be estimated by counting the growth of closed elements in holonomy, and for hyperbolic measures, a more precise estimate along the lines found by Katok [K] may be possible:

Problem 8.3. Find a formula estimating $h_\nu(M/\mathcal{F}, f_t)$ with the growth rate in the number of closed loops with hyperbolic holonomy, for ν a hyperbolic non-wandering measure.

The estimate of foliation entropy from above, (5.3) is valid for C^1-foliations. Estimates of *flow* topological entropy from below by exponents are much more delicate and require a $C^{1+\alpha}$ hypothesis, along with regularity assumptions on the invariant measure ν^*. F. Ledrappier and L.-S. Young have characterized those measures for which the entropy estimate (5.3) is equality [LY]. In a second paper, they also established that for any ergodic measure ν^* with exponents $\{\lambda_i(\nu^*)\}$, there is a very precise formula

$$h_{\nu^*}(f_t) = \sum_{i=1}^{s} \lambda_i(v^*) \cdot \gamma_i(\nu^*) \tag{8.1}$$

where $\gamma_i(\nu^*)$ is the "Hausdorff dimension of ν^* in the direction of the subspace $E_i^{\nu^*}$", so that $0 \leq \gamma_i(\nu^*) \leq \dim E_i^{\nu^*}$.

Question 8.4. Can the Ledrappier-Young formula (8.1) be generalized to the metric entropies of foliations? That is, given an ergodic $\nu \in \mathcal{M}(f_t, T)$, is there a well-defined notion of transverse Hausdorff dimension for ν, say $\gamma_i(\nu)$, so that (8.1) holds?

One expects that an affirmative answer to Question 8.4 would give revealing insight into foliation structure, and that the dimension functions $\gamma_i : \mathcal{M}_e(f_t, T) \to [0, \infty)$ contain a great deal of information. It is also interesting to note that as the authors introduce in §9 of [LY,II] an "entropy for a flow conditioned to a foliation", a solution to the above question may simply involve "reconditioning" the techniques used by Ledrappier and Young.

We turn now to questions arising from open problems in foliation theory. First, we gave in §7 a $C^{1+\alpha}$-version of the Sacksteder Theorem, with an hypothesis on the linear holonomy cocycle on the exceptional set.

Conjecture 8.5. Let $\mathcal{F}$ be a $C^{1+\alpha}$-foliation of codimension one on a compact manifold. If K is an exceptional minimal set for $\mathcal{F}$, find geometric hypotheses on the shape of K, or on $\mathcal{F}$ (besides being C^2), which imply the hypotheses of Theorem 7.4 hold.

Proposition 7.1 was observed by Ghys, Langevin, Walczak who combined it with Duminy's Theorem to obtain that $h(M/\mathcal{F}, f_t) = 0$ implies $GV(\mathcal{F}) = 0$. This kind of problem, to show an ergodic hypothesis implies some secondary class vanishes, started with subexponential growth implies $GV(\mathcal{F}) = 0$. Using the entropy invariants, one can formulate a large number of similar questions. The basic idea is that if some entropy vanishes, then the normal linear holonomy cocycle cannot be "too hyperbolic", so the appropriate Weil measure (or Godbillon measure for codimension-one) must vanish. This will imply corresponding secondary classes vanish, as discussed in [HH].

Problem 8.6. Suppose $\mathcal{F}$ is a $C^{1+\alpha}$-foliation on a compact manifold. If $h(M/\mathcal{F}, f_t) = 0$, show that all of the Weil measures of $\mathcal{F}$ must vanish.

Problem 8.7. Suppose $\mathcal{F}$ is a $C^{1+\alpha}$-foliation on a compact manifold. Suppose that Dh is measurably essentially hyperbolic, then show $h(M/\mathcal{F}, f_t) \neq 0$.

Note that by the results of [HK1], a positive solution to 8.7 would imply a positive solution to 8.6 while we feel reasonably certain that a result such as that asked for in Problem 8.7 must hold, the only evidence in codimension greater than one is Theorem 7.7. There is a related, less concrete problem to mention which is however much more certain to have positive solutions. Recall that in the paper [H], the author showed that subexponential growth of the Radon-Nikodyn cocycle for a foliation of subexponential growth implies the Godbillon measure is zero. The hypothesis on the cocycle is equivalent to the sum of the Lyapunov exponents vanishing for a.e. $\nu \in \mathcal{M}(f_t, T)$.

Problem 8.8. Use the Lyapunov exponent theory of section 5 and Pesin theory of section 6 to find a natural class of hyperfinite foliations for which the Godbillon measure must vanish. This "class" of foliations should contain those with subexponential growth, but must be considerably larger; for example, can a general criterion be developed for all amenable foliations?

The original proof that topological entropy dominates metric entropy by Goodwyn [Go] used a technique of representing the given flow into factor flows on infinite product spaces. If V has a covering by N elements, then for $X = \Pi_{-\infty}^{\infty} X_i$, $X_i \simeq [0,1]^N$ thre is a flow on X and a map $V \to X$ preserving flows, so that the entropies are estimated by entropies of an appropriate flow on X. We call such a factor flow $S(\mathcal{F})$.

Problem 8.9. Given a foliation $\mathcal{F}$ on compact M, what dynamical properties of $\mathcal{F}$ can be derived from the Kakutani-equivalence class (the K-class) of the derived flow $S(\mathcal{F})$?

Question 8.10. Let $W^*(V/\mathcal{F})$ be the von Neumann algebra of $S\mathcal{F}$ described in section 3. How does the Murray-von Neumann type and algebraic structure of this algebra (as surveyed in Moore [Mo2]) relate to:

(8.10.1) The Hausdorff dimension functions γ_i on $\mathcal{M}(f_t, T)$?

(8.10.2) The K-class of the factor flow $S(\mathcal{F})$?

(8.10.3) The values of the metric and topological entropies for $\mathcal{F}$?

Appendix
A brief history of secondary classes of foliations and dynamics.

(A.1) The Anosov ancestry and Plante's work.

The study of dynamics of flows and of foliations have both been profoundly influenced by the special examples of Anosov flows [A] and their weak (strong)-stable (unstable) foliations, and in particular those Anosov flows arising as the geodesic flow for a compact manifold with strictly negative curvature. The strong stable or unstable foliation of an Anosov flow has polynomial growth, while the weak stable or unstable foliation has leaves of exponential growth. Plante studied how the growth types of these foliations were related to the topology of the ambient manifold in a number of papers (e.g. [P1]). For the geodesic flow on a surface, the strong stable foliation is given by the horocycle flow, which is uniquely ergodic [BM] so that each leaf of this foliation defines an asymptotic measure which is flow invariant. In related developments, Plante studied the existence of transverse invariant measures for foliations, showed how subexponential growth of some leaf of a foliation suffices to construct a transverse invariant measure [P2]. The measures obtained represent generalized Poincaré Recurrence Cycles (cf. [RS]) but their geometric description has remained effectively out of reach, except in the case of codimension-one, C^2-foliations (cf. [CC], [Hec]). Following the work of Plante, the existence of subexponential growth leaves for a foliation has been associated with some form of regular recurrent, *non-hyperbolic* behavior in the dynamics of the foliation. An extension of foliation cycles to more general contexts was developed by Sullivan in [Su].

(A.2) The Codimension-one Theory.

The secondary classes of foliations were introduced by a number of researchers in the period 1971–72 (cf. the discussion in [L2]) but the work which put the theory into development was a note by Godbillon and Vey [GV], where they introduced the class associated to this work, and also gave Roussarie's calculation that the Godbillon-Vey class was non-zero for the weak-stable foliation of the geodesic flow for a compact surface with constant negative curvature. (The necessity that the curvature be constant was removed 15 years later [HK2]). Later examples of Thurston [T], Heitsch [He] and Rasmussen [Ra] all continued the theme that a foliation with non-trivial Godbillon-Vey classes was built around an Anosov geodesic flow, or which otherwise has transversally hyperbolic behavior as in the examples of Kamber and Tondeur [KT2] and Yamato [Y]. In the period 1974–75, Moussu-Pelletier [MP] and Sullivan [Sc] formulated a question which these examples all support: Must a codimension-one C^2-foliation on a compact manifold with non-trivial Godbillon-Vey class have leaves with exponential growth?

Paraphrased, this asks whether the Godbillon-Vey class measures some type of transverse hyperbolicity in a foliation, and if all leaves have subexponential growth, then there is not enough of this quality present to give non-trivial classes. Essentially, this paraphrasal can be shown to be true for all codimensions.

The first progress on the M-P-S Question was obtained by M. Herman as a consequence of his study of commuting diffeomorphisms of the circle [Her 1]. This method was extended by Wallet [W], so that one knew that a codimension one, C^2-foliation $\mathcal{F}$ on a circle bundle over a surface has $GV(\mathcal{F}) = 0$ if *no leaf of* $\mathcal{F}$ *has holonomy.* Equivalently, $\mathcal{F}$ is obtained as the suspension of a group of commuting diffeomorphism of the circle. This result was extended by Morita and Tsuboi [MT] to show $GV(\mathcal{F}) = 0$ for any foliation whose leaves have no holonomy. Note that it is exactly here that Sacksteder's Theorem enters for the first time, for a corollary of his theorem is that such a foliation has a good transverse invariant measure, and the vanishing of $GV(\mathcal{F})$ is a consequence of analyzing the "regularity" of this measure (cf. [Her 1]).

The next progress was made by Mizutani, Morita, Tsuboi [MMT], and also by Cantwell, Conlon [CC2], who showed that if $\mathcal{F}$ has *almost no holonomy* (i.e., the only leaves with holonomy are compact) then $GV(\mathcal{F}) = 0$. Perhaps surprisingly, it was these two works which represented the "crack in the dam", for they observed (implicitly) that the Godbillon-Vey invaraint could be localized to open foliation-saturated sets. The development of this last idea went through successive, more explicit elaborations in the works of Cantwell-Conlon [CC3], Nishimori [N], Tsuchiya [Ts] Duminy-Sergiescu [DS], culminating in Duminy's unpublished note [D], where he introduced the Godbillon-Vey measure on $\mathcal{B}_o(\mathcal{F})$, the Borel algebra generated by the open saturated sets of a foliation of codimension one. Duminy's note was also distinguished for a second, crucial factor: he essentially used ergodic theory techniques to estimate the values of the Godbillon-Vey measure. The closest analogue, in the ergodic theory of flows, to Duminy's method is the upper estimation of topological entropy by the Lyapunov exponents. What Duminy established is the result:

Theorem. (Duminy) Let $\mathcal{F}$ be a C^2-foliation of codimension-one on a compact manifold. If $GV(\mathcal{F}) \neq 0$, then $\mathcal{F}$ must have a linearly contracting resilient leaf. □

That the conclusion of Sacksteder's Theorem should arise here is not surprising, for the method of proof invokes the Poincaré-Bendixson Theory of Cantwell-Conlon and Hector. The hypothesis that no resilient leaf exists implies that no exceptional minimal sets exist, and from this starting point Duminy deduces the result. A very good recent treatment of the theorem has been published by Cantwell-Conlon [CC5], who also showed that the theorem holds for M open as well. (The fact that the methods also can be made to work for M open is analogous to the fact that topological entropy can be defined for M open by restricting to compact transversals). A resilient leaf must have exponential growth; in fact, L. Conlon has pointed out that a linearly contracting resilient leaf in a foliation forces the foliation to have an *open* set of leaves with exponential growth. Thus, Duminy settled in a spectacular way the M-P-S Question.

(A3) Theory in higher codimensions-examples.

The Poincaré-Bendixson Theory for codimension-one foliations depends at many stages on the well-ordering of the line, so the methods are unapplicable to the study of dynamics of foliations in higher codimension. Another phenomenon to be accounted for when $n > 1$ is the super-abundance of examples arising from homotopy theoretic constructions. Let us briefly describe this new situation. The classification of foliations up to concordance of foliated microbundles was effected by A. Haefliger who introduced the classifying spaces $B\Gamma_n^{(r)}$, where $r \geq 0$ indicate the degree of differentiablity (which for $r \geq 1$ can include a modulus of continuity) and n is the codimension. A foliation $\mathcal{F}$ on M determines a map $h_{\mathcal{F}} : M \to B\Gamma_n^{(r)}$. Conversely, a map $h : M \to B\Gamma_n^{(r)}$ determines a foliated microbundle over M. For M open without boundary, the data h plus an appropriate splitting of the tangent bundle into $TM = F \oplus Q$ suffices to determine a foliation $\mathcal{F}$ on M such that $h_{\mathcal{F}}$ and h are homotopic. This is a consequence of the Gromov-Phillips immersions theory. For M compact, exactly the same conclusions hold, except now $\mathcal{F}$ is constructed using Thurston's extremely powerful but totally abstract realization theorems. For a discussion, see Chapter 4 of Lawson [L2]. Making these constructions concrete, even for codimension-one foliations of a 3-manifold, is a truly daunting task. One can view D. Gabai's constructions of foliations on 3-manifolds as having implemented some of the Thurston ideas but the topology of the ambient manifold quickly enters, and the complexity of making the foliation constructed explicit becomes formidable [G]. None-the-less, the wealth of information that Gabai has shown can be obtained indicates that Thurstons' methods are a rich vein.

The theorems cited above reduce the existence of foliations on a compact manifold M to problems about the homotopy classes of maps of M into $B\Gamma_n^{(r)}$. For $r \geq 2$, there is a fairly extensive knowledge about non-trivial elements in $\pi_*(B\Gamma_n^{(r)})$ (cf. [H3]). These results are all consequences of properties of the secondary characteristic homomorphism

$$\Delta(\mathcal{F}) : H^*(WO_n) \to H^*_{deR}(M)$$

where the algebra $H^*(WO_n)$ consists of Pontragin classes (of Q) up to degree $2n$, and for degrees $2m+1$ to (n^2+n) consists of exotic secondary invariants (cf. Lawson [L2]). For degrees greater than one, there are dual homotopy invariants, and also tertiary exotic classes which measure non-triviality of homotopy classes of maps $M \to B\Gamma_n^{(r)}$ (cf. [HL]). The conclusion of these results if that for a given manifold of dimension $n+m$, where $m > n$, if some splitting condition holds on TM, then the manifold often has uncountably many foliations, all with distinct secondary or tertiary data, and thus are to be thought of as dynamically fundamentally different. The problem is then how to quantify this last statement. One of the tests that a good theorey of dynamics for codimension $n \geq 1$ must pass, is that it should explain dynamically how these abstractly constructed foliations differ. Since one can not realistically hope to decode the Thurston construction into explicit steps, the other option which suggests itself is to study how the secondary invariants might depend upon dynamical quantities, and to let this aspect of the problem guide the development, hoping that a broad-based theory results.

(A4) Theory in higher codimension – some results.

One of the consequences of Plante's work on the existence of transverse invariant measures is the existence of a Ruelle-Sullivan current associated to the measure [RS], or equivalently an asymptotic homology class when there is a leaf of subexponential growth. The author observed in 1982 that not only was there an asymptotic homology class from the measure, but there were also exotic asymptotic classes, given by a map

$$\chi_\mu : H^*(g\ell_n, 0_n) \to H_c^*(M; \mathbf{R})$$

where μ denotes the invariant measure, the LHS is the relative Lie algebra cohomology of the pair and the RHS compactly supported cohomology of M. The value $\chi_\mu(1) = \langle c_\mu \rangle^*$ is the Poincaré dual of the asymptotic homology class $\langle c_\mu \rangle$. By varying the measure μ, one begins to formulate a dependence of χ_μ on the ergodic structure of $\mathcal{F}$. With the appearance of Duminy's work, it became clear his Godbillon measure generalized into a framework incorporating the maps χ_μ above, the secondary classes of foliations, the leaf classes of foliations and most other secondary data for $\mathcal{F}$. This was carried out in the papers [HH], [HK1], [H4], where the Weil measures were defined and basic properties outlined. The technical idea for the Weil measures, to isolate in the construction of secondary classes the role played by the transgression factor, has in fact been implicit in earlier works which studied secondary classes via a spectral sequence associated to filtrations of WO_n.

This kind of construction was most clearly stated by Kamber-Tondeur (Chapter 5, [KT1]). However, these formal constructions had to wait until Duminy's observation that they could be made *effective* by using two sets of connection data, one to calculate the Chern data (and which will be held fixed) and the other to calculate the transgression data. It was this freedom-of-choice for the second connection that Duminy used to prove his theorem in codimension-one. The final technical result in this direction is that the value of the transgression opeartors, i.e. the Weil measures, is a function only of the *measurable* cohomology class of the linear holonomy cocycle. For the Godbillon measure, this is shown in [H4]; for general Weil measures associated to cocycles with values in $GL(n, \mathbf{R})$, this is shown in [HK1]. Thus these Weil measures are exactly the sort of dynamical invariants of foliations or pseudogroups envisioned by Mackey [M].

In addition to Duminy's Theorem, the Weil measure techniques have been applied to obtain two results which are valid for any codimension. The first is a solution of the M-P-S Question for all codimension, found in Spring 1984.

Theorem. [H4]. Let $\mathcal{F}$ be a C^2-foliation of codimension n on a manifold M without boundary. Suppose that there is a Riemannian metric on M for which a.e. leaf of $\mathcal{F}$ has subexponential growth. Then the map $\Delta(\mathcal{F})$ is zero in degree $(2n+1)$. That is,all Godbillon-Vey classes of $\mathcal{F}$ vanish. Moreover, all generalized Godbillon-Vey classes in degrees $> 2n+1$ also vanish (cf. section 2, [HK1]).

The next result is a tremendously broader statement involving not growth of leaves, but a purely measure-theoretic property of the foliation. We say that $\mathcal{F}$ is *amenable* if the equivalence relation induced on M by $\mathcal{F}$ is amenable. In Fall, 1983, the author and A. Katok proved the following:

Theorem. [HK1]. Let $\mathcal{F}$ be a C^2-foliation of codimension n on a manifold M without boundary. If $\mathcal{F}$ is amenable, then the map $\Delta(\mathcal{F})$ vanishes on all residual secondary classes of degree $> 2m+1$.

For the motivation and history of this particular result, the reader is referred to the Introduction of [HK1].

Both of the above theorems are proved by establishing that the linear holonomy cocycle has non-hyperbolic behavior, and then using ideas from geometry and Lyapunov adapted metric techniques to prove that the behavior is uniform enough to force the appropriate Weil measures to vanish. As such, they do not begin to address the question of what *exactly* does it mean in terms of the dynamics of $\mathcal{F}$ to have non-trivial secondary classes. To date, we only know that the linear holonomy cocycle or the leaves of $\mathcal{F}$ have some hyperbolicity. In order to begin the quantification of this hyperbolicity, we introduce the Lyapunov exponents of a foliation and the metric entropies. However, the reader will clearly see in problems 8.4 to 8.10 the expressed hope that a more precise theory of dynamics of foliations will result from the methods of this paper.

This survey will conclude with three further notes. First, as an appplication of the cyclic cohomology theory of foliations, Connes showed in [Co4] that if the von Neumann algebra $W^*(M/\mathcal{F})$ of a foliation has Murray-von Neumann type III_0 and the flow of weights has no invariant measure, then $GV(\mathcal{F}) = 0$. This result should have a direct dynamical systems proof. It also illustrates that the von Neumann algebra $W^*(M/\mathcal{F})$ is a very rich invariant of $\mathcal{F}$, and another test of a good theory of foliation dynamics would be to illucidate how the internal properties of $W^*(M/\mathcal{F})$ reflect the global dynamics of $\mathcal{F}$.

The second result to mention is the observation by Ghys that Proposition 7.1 combined with Duminy's theorem implies that if $h(M/\mathcal{F}, f_t) = 0$, then $GV(\mathcal{F}) = 0$ in codimension-one. Such a result indicates that the topological entropy is measuring some of the same hyperbolicity which goes into making $GV(\mathcal{F}) \neq 0$. It would be very interesting to have a direct proof of this remark, which could be generalized to arbitrary codimension. The implication that $GV(\mathcal{F}) \neq 0$ forces $h(M/\mathcal{F}, f_t) > 0$ suggests that there may be further dynamical implications of non-vanishing secondary classes, which the various foliation entropies and Hausdorff dimension functions on $\mathcal{M}(f_t, T)$ can begin to quantify.

Finally, the "naive distortion lemma" that is the key to the proof of Sacksteder's theorem has a second derivative counterpart based on the Schwartzian derivative. This "sophisticated distortion lemma" is equally the key for the study of a number of properties of C^2-dimension-one dynamical systems: period doubling maps, Feigenbaum eigenvalues and the asymptotic behavior of C^2-maps (cf. [La]). Sullivan has suggested that for C^2-foliations of codimension-one (equivalently, compactly generated C^2-pseudogroups on the line), the Schwartzian distortion lemma may have applications to studying the dynamics of the foliation near exceptional minimal sets and resilient leaves. This is exactly the area where the conventional Poincarè-Bendixson analysis breaks off, so that the ideas used in the study of period-doubling maps could provide the key to new classification theorems for hyperbolic C^2-foliations in codimension-one, giving another direction in which the Sacksteder theorem can be extended.

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