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Cyclic cocycles, renormalization and eta-invariants

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1. Introduction

In this monograph, we will put into a unified context three of the real-valued invariants that are associated to a self-adjoint pseudo-differential operator (ψDO) on a manifold:

a) The relative eta-invariant of Atiyah-Patodi-Singer [6] for a first order, elliptic differential operator coupled to a trivialized flat bundle.
b) The odd analogue of the Breuer index for a self-adjoint, leafwise elliptic ψDO along the leaves of a measured foliation, as defined by Connes [27, 28].
c) The odd analogue of the distributional G-index of a self-adjoint differential operator, transversally elliptic for a smooth G-action, based on the transverse index theory of Atiyah [1] an Singer [81, 82].

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The unification of these three classes of real-valued invariants is a consequence of the authors' study of a long-standing problem posed by I. M. Singer (cf. page 92, [6] and page 135, [82]):

Problem 1.1. Construct the relative eta-invariant as a Breuer index in an appropriate von Neumann algebra.

As shown in Sect. 4 below, there are appropriate choices of foliated manifold and leafwise elliptic data so that the *topological* formulas for the *analytic* invariants of a) and b) above coincide. Pursuit of an analytic proof of this coincidence led the authors to a solution of Problem 1.1, and the work in this monograph.

At first, it would appear that the transverse (analytic) index invariants of c) have no relation to the index invariants of a) or b). However, one of the main themes developed in this monograph is that both of the latter invariants can be transformed into transverse analytic invariants, and in this context the problem of Singer has a solution. The first point is that the relative eta-invariant can be reformulated as a distribution on the algebra $R_\infty(U_N)$ of smooth central functions, where $\pi_1(M)$ is represented on the unitary group U_N. Secondly, Cheeger-Gromov estimates for the absolute eta-invariant and the technique of asymptotic cycles for the dual space $\hat{U}_N$ are used to show that renormalization of an appropriate transverse "Toeplitz" index problem yields a distributional eta-invariant. Finally, a different analysis of the renormalization procedure applied to the transverse index problem, based on the Fubini principle and the Weyl asymptotic formula, shows that the transverse index renormalizes to the Breuer index of a leafwise elliptic operator for a flat bundle foliation. Thus, we equate the two "longitudinal" invariants a) and b) by equating them to renormalized transverse invariants. This use of the transverse index context to prove equality of two different invariants can be viewed as a more sophisticated version of the method introduced in [23, 24] to prove the index theorem for almost periodic operators.

To state our main theorem and its corollary, we introduce some notation which is further explained in later sections:

M	denotes a compact, odd-dimensional Riemannian manifold of dimension m without boundary.
D	is a geometric operator, operating on sections $C^\infty(E)$ of a smooth bundle $E \to M$ of Clifford modules.
G	is a connected compact Lie group.
$\alpha : \Gamma \to G$	is a representation of the fundamental group $\Gamma = \pi_1(M)$.
$\varrho : G \to U_N$	is a finite-dimensional unitary representation of G.
$E(\varrho \circ \alpha) \to M$	is the flat Hermitian $\mathbf{C}^N$-bundle associated to $\varrho \circ \alpha$.
V	denotes the principal (flat) G-bundle associated to $\varrho \circ \alpha$, with flat connection ∇^α.
$\mathcal{F}_\alpha$	is the G-invariant foliation of V whose tangential distribution consists of the horizontal spaces of ∇^α. The leaves of $\mathcal{F}_\alpha$ are coverings of M associated to the kernel of α.
$\Theta : V \to M \times G$	is a topological trivialization of the G-bundle V.

$u: V \to U_N$	is the composition of $V \xrightarrow{\Theta} M \times G \xrightarrow{\varrho} U_N$.
$\nabla^{\bar{\alpha}}$	is the Hermitian flat connection on $\varepsilon^N \equiv M \times \mathbf{C}^N \to M$ obtained by pushing forward ∇^{α} under Θ.
$D_0 = D \otimes I_N$	is the product extension of D to $C^\infty(E \otimes \varepsilon^N)$.
$D_1 = D \otimes \nabla^{\bar{\alpha}}$	is the twisted extension of D to $C^\infty(E \otimes \varepsilon^N)$.
$D_t = t \cdot D_1 + (1-t) D_0$	
$\eta(D, \alpha, \Theta) = \int_0^1 \dot{\eta}(D_t) dt$	
D_α	is the *leafwise* lift of $D \otimes I_N$ from M to compactly supported smooth sections of the lift of $E \otimes \varepsilon^N$ to V then restricted to leaves.
$c_\Phi \in H_\lambda^{m+2}(C^\infty(V))$	is the odd degree cyclic cocycle constructed from the "phase" Φ of the leafwise essentially self-adjoint operator D_α.
$\hat{c} \in H_\lambda^{m+2}(C^\infty(V))$	is the odd degree cyclic cocycle constructed by renormalizing the transverse cocycle constructed from D_α viewed as a transversally elliptic operator for the G-action on V.

After this lengthy list of notations, we can now state our main result.

Theorem 1.2. *The longitudinal cocycle c_Φ and the transverse cocycle $\hat{c}$ are identically equal as cocycles over $C^\infty(V)$. Moreover, when evaluated on the unitary $[u] \in H_{\text{odd}}^\lambda(C^\infty(V))$,*

(1.1) $$c_\Phi([u]) = \hat{c}([u]) = -\eta(D, \alpha, \Theta). \qquad \square$$

As is explained in Appendix B, the left-hand side $c_\Phi([u])$ of (1.1) can be identified with the Breuer index of a family $\{T_u^+\}$ of leafwise Toeplitz operators obtained by compressing the multiplier operator of u to the positive ranges of the leafwise operators D_α. The Breuer index, $\text{Ind}_\mu(T_u^+)$, is calculated by applying the von Neumann trace, tr_μ, on the foliation von Neumann algebra $W^*(V/\mathscr{F}_\alpha, \mu)$ obtained from the Haar measure μ on G (cf. [26]), to the operator

$$\left\{(1^+ - T_{u^*}T_u)^{\frac{m+1}{2}} - (1^+ - T_u T_{u^*})^{\frac{m+1}{2}}\right\}.$$

Thus, we obtain an analytic proof of

Corollary 1.3.

(1.2) $$-\eta(D, \alpha, \Theta) = \text{Ind}_\mu(T_u) = \text{Tr}_\mu\left\{(1^+ - T_{u^*}T_u)^{\frac{m+1}{2}} - (1^+ - T_u T_{u^*})^{\frac{m+1}{2}}\right\} \qquad \square$$

We give next an overview of the contents of the rest of this monograph, which will serve also as a more detailed overview of the proof of Theorem 1.2.

Section 2 gives the details and sets notation for the basic geometric constructions used throughout the monograph. The material is standard for the fields from which it is drawn, but for the reader's convenience we gather it together here, and give where possible additional insights into their use later in the monograph. The reader is encouraged to consult the basic references [28], [15] and [63].

Section 3 constructs the longitudinal cyclic cocycle for D_α. In one sense, the material in this section is also standard, for this section is basically an ellaboration upon Sects. 2 and 7 of Connes' fundamental paper [30] in the measured foliation context. With this said, the casual reader will probably not recognize immediately this correspondence. Thus, Sect. 3 gives full details on the construction of the longitudinal cocycle. It begins by introducing the class of longitudinal (or leafwise) differential operators, then discusses the construction of ψDO parametrices with controlled supports for these operators. This material is based in part on Chapter VII of Moore and Schochet [67]. After these preparations, we obtain (bounded) pre-Fredholm modules from D_α, then use either the 2×2 trick (even case) or the 4×4 trick (odd case) to convert the pre-Fredholm module to a Fredholm module, in the sense of (Appendix 2, [30].) The 4×4-trick is the "Clifford version" of the 2×2 trick, and we are indebted to *A*. Connes for providing the preliminary version [29] in which the details were worked out in full. Given the Fredholm module of D_α, the longitudinal cocycle is defined via a simple formula.

Section 4 is incidental to the proof of Theorem 1.2, but we include this material as it has strong related interest. We first discuss the topological formula for the measured foliation index of the Toeplitz operators $\{T_u\}$. This relies on some calculations relegated to Appendix B and Connes' foliation index theorem. The second result of this section shows that the topological index formula for $\{T_u\}$ agrees with the topological formula for the relative eta-invariant. The proof of this uses some observations about the construction of transgression forms for flat bundles. J. Cheeger pointed out that Theorem 4.7 below is a generalization of (Theorem 8.15, [20]).

Section 5 introduces the concept of a *sharp parametrix* for a transverse operator. This idea was developed in conversations with John Roe, and plays a key role in renormalization. In the Connes construction of a transverse Chern character, a key step is the construction of a transverse parametrix which is a ψDO on the ambient manifold V. In contrast, the sharp parametrix is only a ψDO when restricted to the leaves of an auxillary transverse foliation. Moreover, the transverse sharp cocycle requires for its construction two transverse Riemannian foliations, and a leafwise elliptic ψDO along one of the foliations. Renormalization requires that the other foliation be *taut* in the sense of [59], [80].

Given a sharp parametrix, the construction of the transverse sharp cocycle follows the algebraic formalism developed by Connes for the even case. Both the sharp transverse cocycle and the Connes transverse cocycle, denoted respectively by $c^{\#}$ and $c^{\pitchfork}$, are cyclic cocycles for the convolution algebra $C_c^\infty(\mathscr{F}_\pi)$ of smooth kernels along the leaves of the foliation $\mathscr{F}_\pi$ to which the given operator is transverse. The last result of Sect. 5 shows that these two cocycles are cohomologous.

Section 6 gives the fundamental new material of this monograph. We show that the sharp transverse cocycle behaves well with respect to a carefully chosen sequence of approximate units in $C_c^\infty(\mathscr{F}_\pi)$. The renormalization procedure lifts the cocycle $c^{\#}$ from the convolution algebra $C_c^\infty(\mathscr{F}_\pi)$ to the commutative algebra $C^\infty(V)$. We develop a criterion, which unfortunately is fairly narrow, for when a general cocycle over $C_c^\infty(\mathscr{F}_\pi)$ can be renormalized. A key result of this section is that the Weyl asymptotic formula implies that renormalization of $c^{\#}$ transforms it into the longitudinal cocycle c_Φ of Sect. 3. This yields the relation described previously between the longitudinal index invariants b) and the transverse index invariants c).

Section 7 reformulates the eta-invariant of D with twisted coefficients as a distribution on the central functions $R_\infty(G)$. The basic observation is that the map $\varrho \mapsto \eta(D \otimes \overline{V^{\varrho\circ\alpha}})$ extends to $R_\infty(G)$ as a consequence of an estimate of Cheeger and Gromov [17, 18] and Ramachandran [71]. The flat bundle techniques of Sect. 4 reappear in this section, when we show that the eta-distribution can be equally defined using the operator D_α, considered as an operator on V. This reformulation of the eta-distribution sets the stage for the proof of Theorem 1.2 in Sect. 8. The distributional eta-invariant was first introduced for finite groups by Atiyah, Patodi and Singer [5]. (The corresponding G-index theorem for manifolds with boundary was proven by Donnelly in [36]). The extension of the finite group case to the non-trivial, compact connected group case given in this section requires foliation methods.

The final Sect. 8 shows that the distributional relative-eta-invariant applied to $[\varrho] \in R_\infty(G)$ agrees up to sign with the renormalized sharp transverse cocycle applied to $[u] \in H^\lambda_{\text{odd}}(C^\infty(V))$, which is the pull-back via Θ of the odd class $[\varrho] \in H^\lambda_{\text{odd}}(C^\infty(G))$. The proof of this uses two observations. First, we introduce the *Fölner condition*, which states that the approximate units in $C_c^\infty(G)$ obtained from the heat kernel of the invariant Laplacian define a sequence of "Fölner sets" in the dual $\hat{G}$. We then observe that the renormalization procedure is equivalent to the convergence to Plancherel measure of the measures determined by this Fölner sequence on $L^2(\hat{G})$. This is the group-theoretic interpretation of the Weyl asymptotic formula for $L^2(G)$, using the Peter-Weyl theorem. The second key step is to use the Kasparov formulation of the odd index theorem to prove that before renormalization, the values of the sharp transverse cocycle and a spectral flow distribution agree. The argument is concluded by invoking the estimate of Cheeger and Gromov from Sect. 7 to show that the renormalized spectral flow converges to the eta-distribution. We have then established the correspondence

$$\begin{aligned} & c_\Phi([u]) \leftrightarrow \hat{c}([u]) \\ (1.3) \qquad & \leftrightarrow \text{renormalized } c^\#([u]) \\ & \leftrightarrow \text{renormalized spectral flow} \\ & \leftrightarrow \text{eta distribution } ([\varrho]) \end{aligned}$$

which proves the equalities of Theorem 1.2.

We conclude this monograph with four Appendices. The first, A, elaborates on a remark of A. Connes that Theorem 1.2 justifies calling a certain K-homology class in $K^1{}_{\text{an}}(B\bar{U})$ the "KK-eta-invariant". This is also discussed in (§4, [9]). Appendix B calculates $c_\Phi([u])$ as a longitudinal index. The material of this appendix is developed more fully in [45]. Appendix C contains a technical result giving estimates of the eigenvalues of $D \otimes V^{\varrho\circ\alpha}$ which are uniform in the dimension N of the representation ϱ. Finally, in Appendix D we discuss the development in index theory in the 1970's which played a role in the present work.

A preliminary version of the results of this monograph was announced in a plenary address by the second author at the conference on "Operator Algebras" at the Mathematical Sciences Research Institute, Berkeley, in June 1985. The M.S.R.I. preprint [42] announced the main theorem with the additional hypothesis that Γ is amenable. The special case of this program for $G = U(1)$ was developed in [43], where the Fourier Transform on $L^2(S^1)$ was used to simplify many of the technical arguments used in this monograph. The group structure on $\hat{G}$ enables one to give a more traditional presentation of the Fölner set ideas of Sect. 8. Other aspects of this program have appared in [39, 40, 41], [53, 54, 55], [60] and an announcement of Corollary 1.3 was made in [44].

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2. Basic foliation geometry

Let $\mathscr{F}$ denote a C^∞-foliation of a connected manifold V without boundary. Let m denote the dimension of the leaves, and n the codimension of $\mathscr{F}$. Let $I=(-1,1)$ be the open interval, and $p_1: I^{m+n}\to I^m$, respectively $p_2: I^{m+n}\to I^n$, be the projection onto the first (respectively second) factor in the product $I^{m+n}=I^m\times I^n$. A *foliation chart* for $\mathscr{F}$ is a coordinate chart, $\phi: U\to I^{m+n}$, with $U\subset V$ open, such that the level sets

(2.1) $$P(y)=(p_2\circ\phi)^{-1}(y)\,; \qquad y\in I^n$$

are the connected components of the restrictions of the leaves of $\mathscr{F}$ to U. The sets $P(y)$ are called the *plaques* of $\mathscr{F}$ in U.

A covering of V by foliation charts $\{(U_j,\phi_j)|j\in\mathfrak{J}\}$ is said to be *good* if the covering is locally finite and each non-empty intersection

$$U_{j_1\dots j_p}=U_{j_1}\cap\dots\cap U_{j_p}$$

is a contractible space.

A Riemannian metric, g, on TV determines an orthogonal decomposition $TV\cong F\oplus Q$, where $F=T\mathscr{F}$ is the sub-bundle of vectors tangent to leaves of $\mathscr{F}$. For each leaf $L\subseteq V$, g restricts to a Riemannian metric, g_L, on $TL=FL$. The inner product on the bundle F is denoted by g_F. We make the basic assumption that there is a metric g for which:

(2.2) Each leaf L is a complete Riemannian manifold for the metric g_L.

(2.3) There is a positive lower bound on the injectivity radius of V. That is, there exists a constant $c_0>0$ so that for all $x\in V$, the exponential map $\exp: T_xV\to V$ is an injective diffeomorphism on the ball $B(0,c_0)\subset T_xV$ of radius c_0 about the origin.

(2.4) V and all leaves L of $\mathscr{F}$ have uniformly bounded geometry (cf. §4, [19] or §2, [74]).

A foliation chart $\phi: U \to I^{m+n}$ is *regular* if ϕ has an extension $\tilde{\phi}: \tilde{U} \to (-1-\varepsilon, 1+\varepsilon)^{m+n}$, $\varepsilon > 0$ where $U \subset \tilde{U}$, $\tilde{\phi}|U = \phi$ and the level sets $(p_2 \circ \tilde{\phi})^{-1}(y)$ are connected subsets of leaves of $\mathcal{F}$.

A cover $\{U_j | j \in \mathfrak{J}\}$ of V has *Lebesgue number* $c_1 > 0$ if for any set $X \subset V$ of diameter less than c_1, there is an index $j \in \mathfrak{J}$ for which $X \subset U_j$.

We say that a covering $\{(U_i, \phi_i) | i \in \mathfrak{J}\}$ of V by foliation charts is c_1*-regular* if the covering is good, each chart $\phi_j: U_j \to I^{m+n}$ is regular, and there is a Lebesgue number $c_1 > 0$ for the cover.

Lemma 2.1. *With the hypotheses* (2.2), (2.3) *and* (2.4) *on* $(V, \mathcal{F}, g)$, *there exists* c_1*-regular foliation covers for some* $c_1 > 0$.

Proof. Set $c_1 = c_0/4(m+n)$. For each $x \in V$, there exists a regular foliation chart $\phi_x: U_x \to I^{m+n}$ such that U_x contains the ball of radius $2c_1$ about x, and for each $y \in I^n$ the plaque $P_x(y)$ is *convex* (in the leaf metric) and contains the leafwise ball of radius $2c_1$ about $\phi^{-1}(0, y) \in U_x$. Choose a countable subset of points $\{x_j | j \in \mathfrak{J}\} \subset V$ such that every point $y \in V$ is within distance c_1 of some x_j, and for $j \neq j'$ we have $\text{dist}_V(x_j, x_{j'}) > \frac{1}{2}c_1$. (Such a set is called a c_1*-net* in V.) For each j set $U_j = U_{x_j}$, $\phi_j = \phi_{x_j}$. The collection $\{U_j | j \in \mathfrak{J}\}$ is good, as the plaques in each U_j are convex, so all multiple intersections will be convex, hence contractible. Let $X \subset V$ be a subset of diameter less than c_1. For $y \in X$, there exists $j \in \mathfrak{J}$ so that $\text{dist}_V(y, x_j) < c_1$, and hence $X \subset B(x_j, 2c_1) \subset U_j$. □

A transversal to $\mathcal{F}$ is a compact n-manifold T, possibly with boundary, and an immersion $f: T \to V$ whose image is everywhere transverse to the leaves of $\mathcal{F}$. Two transversals, (T, f_0) and (T, f_1) are said to be *holonomy related* if there exists a smooth vector field on V, everywhere tangent to the leaves of $\mathcal{F}$ and with a time $t = 1$ flow $h_1: V \to V$ such that $h_1 \circ f_0 = f_1$.

A transverse measure, μ, for $\mathcal{F}$ is a countably additive measure defined on the Borel subsets of the transversals to $\mathcal{F}$. Given a transversal $f: T \to V$ and Borel subset $E \subset T$, the value of $\mu(E, f)$ depends only on the image $f(E)$ of E and we write $\mu(f(E))$ for this value. The measure μ is *finite* if $\mu(T, f)$ is finite for all transversals. We say that μ is *holonomy invariant* if given any two holonomy related transversals (T, f_0) and (T, f_1), and a Borel subset $E \subset T$, then $\mu(f_0(E)) = \mu(f_1(E))$. (This definition of holonomy invariance is equivalent to the original definition of Plante [70]).

For each positive integer N, let $\mathcal{A}_N \equiv M(N, C^\infty(V))$ denote the algebra of $N \times N$ matrices with entries in the complex-valued smooth functions on V. There is an isomorphism of this algebra with the smooth maps from V to $M(N)$, where $M(N)$ denotes the $N \times N$ complex matrices. A self-adjoint idempotent, $e \in \mathcal{A}_N$, determines a complex vector bundle $E_e \to V$, where the fiber over $x \in V$ is the subspace of $\mathbf{C}^N$ spanned by vectors in the range of the projection $e(x) \in M(N)$, viewing E_e as a sub-bundle of $V \times \mathbf{C}^N$. Conversely, each complex vector bundle $E \to V$ determines a self-adjoint idempotent $\tilde{e}_E: V \to M(N)$. This correspondence implements, on the positive elements, the isomorphism of K-groups $K_0(C^0(V)) \cong K^0(V)$.

Let $U_N \subset M(N)$ denote the $N \times N$-unitary subgroup, such that for $A \in U_N$ then $A \cdot A^* = \text{Id}$, where A^* is the conjugate transpose of A. Each element $[u] \in K^1(V)$ is represented for some N by a smooth map $u: V \to U_N$, which is well-defined up to homotopy. The map u is equivalent to a unitary element $\tilde{u} \in U_N(C^\infty(V))$, and thus determines a class $[\tilde{u}] \in K_1(C^0(V))$. The correspondence $[u] \to [\tilde{u}]$ implements the isomorphism $K^1(V) \cong K_1(C^0(V))$, (cf. [61]).

We next briefly describe the convolution algebras associated to a foliation. Extensive treatments are given in [27, 28] and (Chapter 6, [67]).

The *graph* $\mathscr{G}_{\mathscr{F}}$ of $\mathscr{F}$ is the set of triples (x, y, γ_{xy}) where $x, y \in V$ are on the same leaf of $\mathscr{F}$, and γ_{xy} is a leafwise path between them, which is well-defined up to holonomy along the path. A *basic open set* U_γ in $\mathscr{G}_{\mathscr{F}}$ consist of data

$$(x, y, z, \gamma)\,; \qquad (x, y, z) \in I^m \times I^m \times I^n$$

and γ is a continuous family of leafwise paths such that:

(2.5) There are foliation charts (U_0, ϕ_0) and (U_1, ϕ_1)

(2.6) $\gamma: I^m \times I^m \times I^n \times [0,1] \to V$ such that $\gamma_{(x,y,z)}: [0,1] \to V$ is a leafwise path from $\phi_0^{-1}(x,z)$ to $\phi_1^{-1}(y,z)$, and $\gamma_{(x,y,z)}$ depends continuously on (x,y,z).

The leafwise metrics g_L and the foliation charts of (2.5) determine Riemannian metrics $g_{i,y}$ on the plaques $P_i(y)$, and hence metrics on the coordinates $I^m \times \{y\}$. Let $dv(g_{i,y})$ denote the smooth volume form on $I^m \times \{y\}$ determined by this metric.

For a compactly supported continuous function $k = k(x,y,z)$ on $I^m \times I^m \times I^n$, we define on a basic open set U_γ a *basic kernel* k_γ by

$$k_\gamma(x, y, z, \gamma) \equiv k(x, y, z)\,.$$

There is an involution on these kernels:

$$k_\gamma^*(x, y, z, \gamma) = \overline{k(y, x, z)}\,. \tag{2.7}$$

The algebra $C_c^\infty(V/\mathscr{F})$ consists of formal finite sums of basic kernels and their products. The sum is subject to the relation that if k_{γ_1} and k_{γ_2} are basic kernels on the same open set U_γ, then $k_{\gamma_1} + k_{\gamma_2}$ is the kernel defined by the sum of functions $k_1 + k_2$ on $I^m \times I^m \times I^n$. The product is defined by the convolution rule for basic kernels: let k_{γ_1} on U_{γ_1}, k_{γ_2} on U_{γ_2} be given, and suppose that the chart $\phi_1^1: U_1^1 \to I^{m+n}$ for U_γ is the same as $\phi_0^2: U_0^2 \to I^{m+n}$ for U_{γ_2}. We then define as basic open set $U_{\gamma_1 * \gamma_2}$ with first open set (U_0^1, ϕ_0^1), second open set (U_1^2, ϕ_1^2) and with $\gamma_1 * \gamma_2$ the concantenation of the paths γ_1 and γ_2. Then for $(x,y,z) \in I^m \times I^m \times I^n$,

$$k_{\gamma_1} \circ k_{\gamma_2}(x, y, z, \gamma_1 * \gamma_2) = \int_{w \in I^m} k_1(x, w, z) k_2(w, y, z)\, dv(g_{1,z}^1)\,. \tag{2.8}$$

If the sets U_1^1 and U_0^2 are disjoint, then the product, $k_{\gamma_1} \circ k_{\gamma_2}$ is defined to be identically zero. The general product of two elements in $C_c^\infty(V/\mathscr{F})$ can be reduced to a sum of basic products satisfying one of these two cases.

If $\pi: V \to M$ is a fibration whose fibers define the foliation $\mathscr{F} = \mathscr{F}_\pi$ on V, then we let $C_c^\infty(\pi) = C_c^\infty(V/\mathscr{F}_\pi)$. The graph is the fiber pull-back

$$\begin{array}{ccc} \mathscr{G}_{\mathscr{F}} & \xrightarrow{\ r\ } & V \\ s\downarrow & & \downarrow \pi \\ V & \xrightarrow{\ \pi\ } & M \end{array} \tag{2.9}$$

The leaves of $\mathscr{F}_\pi$ have trivial holonomy, so that points of $\mathscr{G}_{\mathscr{F}}$ are described by pairs

$$\mathscr{G}_{\mathscr{F}} = \{(z, z') \in V \times V \mid \pi(z) = \pi(z')\}\,.$$

The source and range maps satisfy

$$s(z,z')=z\,; \qquad r(z,z')=z'\,.$$

An element of $C_c^\infty(\pi)$ is a compactly supported smooth function k on the closed subset $\mathcal{G}_{\mathcal{F}} \subset V\times V$. We use the notation $k=k(z,z')$, and observe that the product of two elements is

$$k_1 \circ k_2(z,z'')=\int_{z'\in\pi^{-1}(z)} k_1(z,z')k_2(z',z'')dv_\pi(z')$$

where dv_π is the Riemannian volume element along the fibers of π.

A *diagonal basic open set* in $\mathcal{G}_{\mathcal{F}}$ is a basic open set U_γ where $U=U_0=U_1$, $\phi=\phi_0=\phi_1$ and $\gamma_{(x,y,z)}$ is the geodesic path in the plaque $P(z)$ between x and y, for $(x,y,z)\in I^m\times I^m\times I^n$. We adopt the notation U_ϕ^* for these sets, as they are determined by the chart (U,ϕ). For k on $I^m\times I^m\times I^n$, the diagonal kernel it determines is denoted $k_\phi^*: U_\phi^*\to \mathbf{C}$. A transverse invariant measure μ for $\mathcal{F}$ defines a linear functional on diagonal kernels,

$$\mathrm{Tr}_\mu(k_\phi^*)=\int_{I^n}\left\{\int_{I^m} k(x,x,z)dv(g_z)\right\}d\mu^*(z)\,. \tag{2.10}$$

The measure $d\mu^*$ is the induced measure on the transversals, $x\in I^m$, $\phi(x,\ )^{-1}: I^n\to V$, where $d\mu^*$ is independent of x by the holonomy invariance of μ.

Lemma 2.2. (cf. Corollary 6.28, [67]). *Let k_{γ_1} and k_{γ_2} be basic kernels with the compositions $k_{\gamma_1}\circ k_{\gamma_2}$ both defined and diagonal. Then*

$$\mathrm{Tr}_\mu(k_{\gamma_1}\circ k_{\gamma_2})=\mathrm{Tr}_\mu(k_{\gamma_2}\circ k_{\gamma_1})\,. \quad \square \tag{2.11}$$

Corollary 2.3. *The map Tr_μ extends to a linear functional on $C_c^\infty(V/\mathcal{F})$ which satisfies* (2.11) *for arbitrary pairs in this algebra.* $\square$

For the foliation $\mathcal{F}_\pi$, the choice of a Borel, locally finite measure μ_M on M determines a holonomy invariant transverse measure for $\mathcal{F}_\pi$, denoted by μ_π. Given a transversal (T,f), we define

$$\mu_\pi(E,f)=\mu_M\{\pi\circ f(E)\}\,.$$

The resulting trace on $C_c^\infty(\pi)$ will be denoted by Tr_π, where the measure μ_M will be assumed given so there is no ambiguity of notation.

For each leaf $L\subset V$, there is a $*$-representation of the algebra $C_c^\infty(V/\mathcal{F})$ on the Hilbert space of the leaf (for the Riemannian volume on L.) The C^*-algebra of $\mathcal{F}$, denoted by $C^*(V/\mathcal{F})$, is defined to be Banach $*$-algebra completion of $C_c^\infty(V/\mathcal{F})$ with respect to the *uniform* family of semi-norms determined by the leaf representations.

Given the trace functional Tr_μ on $C_c^\infty(V/\mathcal{F})$, we can also define a completion with respect to the norm:

$$\|k\|_\mu^2=\mathrm{Tr}_\pi(k^*\circ k)\,. \tag{2.12}$$

The yields the *von Neumann algebra* of $\mathcal{F}$, $W^*(V/\mathcal{F},\mu)$, which can also be obtained as the completion of $C^*(V,\mathcal{F})$ with respect to the norm (2.12). (See Chapter VI, [67] for more details, and also [26].)

A special class of foliations plays a particularly important role in the development of this paper. Let M denote a compact closed manifold with fundamental group $\Gamma=\pi_1(M,x_0)$ which acts on the universal cover $\tilde{M}$ via deck transformations (on the right). Let X be a complete Riemannian manifold with bounded geometry. The isometry group of X will be denoted by G, which acts on X on the left. Associated to each representation $\alpha:\Gamma\to G$, there is a fiber-product manifold, V, with a pair of transverse foliations $\mathscr{F}_\alpha$ and $\mathscr{F}_\pi$. On the product $\tilde{M}\times X$, there are foliations:

(2.13) $$\mathscr{F}_h \text{ with leaves } \{\tilde{M}\times\{y\}\,|\,y\in X\}$$

(2.14) $$\mathscr{F}_v \text{ with leaves } \{\{\tilde{x}\}\times X\,|\,\tilde{x}\in\tilde{M}\}.$$

We obtain from α a diagonal left action of Γ on $\tilde{M}\times X$, where $\gamma\cdot(\tilde{x},y)=(\tilde{x}\cdot\gamma^{-1},\alpha(\gamma)\cdot y)$. Both foliations $\mathscr{F}_h$ and $\mathscr{F}_v$ are Γ-invariant, so descend to foliations denoted by $\mathscr{F}_\alpha$, and $\mathscr{F}_\pi$ on the quotient

$$V=V_\alpha\equiv\tilde{M}\times_\Gamma X.$$

There is a fibration map $\pi:V\to M$ with fibers diffeomorphic to X, and $\mathscr{F}_\pi$ is the foliation by fibers of π.

Each leaf of $\mathscr{F}_\alpha$ is transverse to the fibers of π, so that π restricted to a leaf is a covering map of the base M. For $y\in\pi^{-1}(x_0)$, the leaf L_y of $\mathscr{F}_\alpha$ through y is the covering associated to the subgroup

(2.15) $$\Gamma_y=\{\gamma\in\Gamma\,|\,\alpha(\gamma)(y)=y\}.$$

Given an isometric identification, $X\cong\pi^{-1}(x_0)$, the foliation $\mathscr{F}_\alpha$ determines the representation α up to conjugation in G. (cf. Chap. 5, [15]). Thus, we say that $(V,\mathscr{F}_\alpha)$ is a *geometric model* for the conjugacy class of α.

The Riemannian volume dv_X on X is invariant under G, so the transverse measure to $\mathscr{F}_h$ on $\tilde{M}\times X$ determined by dv_X descends to a transverse invariant measure $d\mu$ for $\mathscr{F}_\alpha$.

The foliations $\mathscr{F}_\alpha$ and $\mathscr{F}_\pi$ are transverse, so lead to a natural class of Riemannian metrics on TV. Fix a Riemannian metric g_M on TM, and let g_α be its lift to a leafwise metric on $\mathscr{F}_\alpha$ via the covering property of the leaves. The Riemannian metric on $\tilde{M}\times TX$ descends to a fiber wise metric g_π on the leaves of $\mathscr{F}_\pi$. We then obtain a metric g on TV by declaring that the distributions $F_\alpha=T\mathscr{F}_\alpha$ and $F_\pi=T\mathscr{F}_\pi$ are orthogonal. Note that g is *projectable* with respect to both foliations, so the pairs $(\mathscr{F}_\alpha,g_\pi)$ and $(\mathscr{F}_\pi,g_\alpha)$ are both Riemannian foliations of V in the sense of [72].

A G-trivialization of $(V,\mathscr{F}_\alpha)$ is an isomorphism Θ of principal G-bundles,

$$P(\alpha)\equiv\tilde{M}\times_\Gamma G\equiv M\times G.$$

We introduce the notation $\bar{\alpha}=(\alpha,\Theta)$. Since $V=P(\alpha)\times_G X$, the map Θ induces a trivialization

$$\Theta_X:V\to M\times X$$

of the bundle $V\to M$. The image of $\mathscr{F}_\pi$ under Θ_X is the product foliation with leaves X. The image of $\mathscr{F}_\alpha$ under Θ_X is a foliation of $M\times X$, denoted by $\mathscr{F}_{\bar{\alpha}}$, whose leaves are everywhere transverse to the X-factor. However, the leaves of $\mathscr{F}_{\bar{\alpha}}$ are not, in general, strictly horizontal and exhibit complicated dynamics. The existence of a G-trivialization, Θ, is a topological property of α (cf. Appendix A).

$\mathscr{F}_\alpha$ admits foliation charts of a special type which are called *cylindrical*. Let $x \in M$ and $U_0 \subset M$ be a contractible open neighborhood of x. The image foliation $\mathscr{F}_{\bar{\alpha}}$ restricted to $U_0 \times X$ has a trivialization

$$H_{\bar{\alpha}}: U_0 \times X \to U_0 \times X,$$

which satisfies

(2.16.1) $\quad H_{\bar{\alpha}}$ covers the identity map on U_0

(2.16.2) $\quad H_{\bar{\alpha}}: \{x\} \times X \to \{x\} \times X$ is the identity

(2.16.3) $\quad$ For each $y \in X$, $H_{\bar{\alpha}}^{-1}(U_0 \times \{y\})$ is a plaque of $\mathscr{F}_{\bar{\alpha}} | U_0 \times X$.

The construction of $H_{\bar{\alpha}}$ is standard, and based on the remark that the restriction $\mathscr{F}_{\bar{\alpha}} | U_0 \times X$ is a transverse foliation without holonomy, so admits a global trivialization over the base U_0.

A *cylindrical foliation chart* (U, ϕ) for $\mathscr{F}_\alpha$ is obtained by choosing a chart $\phi_0 : U_0 \to I^m$ on M, and a coordinate chart $\phi_1 : U_1 \to I^n$ for X. Then let

$$\tilde{U} = H_{\bar{\alpha}}^{-1}(U_0 \times U_1)$$

$$U = \Theta_X^{-1}(\tilde{U})$$

and set ϕ to be the composition

(2.17) $$U \xrightarrow{\Theta_X} \tilde{U} \xrightarrow{H_{\bar{\alpha}}} U_0 \times U_1 \xrightarrow{(\phi_0, \phi_1)} I^m \times I^n.$$

Observe that a cylindrical foliation chart for $\mathscr{F}_\alpha$ can also be considered as a foliation chart for $\mathscr{F}_\pi$ by reversing the factors $I^m \times I^n$ in the range. This follows from (2.16.1) which implies that the preimages $\phi^{-1}(\{x\} \times I^n)$ are plaques for $\mathscr{F}_\pi | U$. For a pair of foliations $(\mathscr{F}_\alpha, \mathscr{F}_\pi)$ as above on V, the proof of Lemma 2.1 goes through with cyclindrical foliation charts (U_x, ϕ_x) which will then yield charts regular with respect to both $\mathscr{F}_\alpha$ and $\mathscr{F}_\pi$. Naturally, such a cover for V will then be called *c_1-biregular* for the pair $(\mathscr{F}_\alpha, \mathscr{F}_\pi)$.

A special case of the above construction is to take $X = G$, where G is a connected Lie group acting on itself on the left, with a left-invariant Riemannian metric, g_G. In addition, we require that

(2.18.1) g_G is right H-invariant, for $H \subset G$ a fixed maximal compact subgroup.

(2.18.2) G is unimodular with the volume form of g_G right-G-invariant.

Condition (2.18.2) implies that the suspension foliation $(\mathscr{F}_\alpha, g_\pi)$ is *taut* in the sense of [59], [80].

We conclude with one last observation about the geometry of $\mathscr{F}_\alpha$ for $X = G$. The right and left actions of G on itself commute, so there is a well-defined right G-action on the quotient V of $\tilde{M} \times G$ by the left Γ-action. The right G-orbits are the leaves of $\mathscr{F}_\pi$. Moreover, the right G-action preserves the leaves of $\mathscr{F}_\alpha$, and as the action is transitive, all of the isotropy groups Γ_y are the same, and isomorphic to the kernel of α.

3. The longitudinal cyclic Chern character

Let $\mathscr{F}$ be a given foliation of a connected manifold V with a holonomy-invariant, transverse measure μ. In this section, we construct a Chern character for leafwise-elliptic pseudo-differential operators on $\mathscr{F}$, with values in the cyclic cohomology of the algebra $C_c^\infty(V)$ of compactly supported smooth functions on V. Our method uses the construction of Connes (§ 7, [30]), which for graded operators yields even degree cocycles. These pair with the projective e_E associated to vector bundle $E \to V$ to give the foliation μ-index of the leafwise operator coupled to E. For self-adjoint ungraded operators, the Chern character yields an odd degree cocycle, which pairs with unitary-valued maps $u: V \to U_N$ to give the foliation μ-index of the Toeplitz operator associated to the compression of multiplication by u to the positive ranges of the leafwise operators (cf. Appendix B).

Fix a Riemannian metric, g, on TV with corresponding distance function $d_g: V \times V \to [0, \infty)$. Choose, and fix, a good covering of V by foliation charts $\{(U_j, \phi_j) j \in \mathfrak{I}\}$ for which the cover $\{U_j\}$ has Lebesgue number $c_1 > 0$. Choose also a partition-of-unity (p.o.u.) $\{\lambda_j | j \in \mathfrak{I}\}$ subordinate to this cover.

Let $C^\infty(F)$ denote the $\mathscr{F}$-tangential vector fields on V. Thus, $\vec{X} \in C^\infty(F)$ is a smooth vector field on V which is everywhere tangent to the leaves of $\mathscr{F}$. Let $E \to V$ be a smooth Hermitian vector bundle with connexion ∇^E compatible with the inner product. Each $\vec{X} \in C^\infty(F)$ defines a first order differential operator

$$\nabla^E_{\vec{X}}: C^\infty(E) \to C^\infty(E).$$

Let $M(E) \to V$ denote the bundle of fiberwise endomorphisms of E; so if E has rank k, then the fibers of $M(E)$ are isomorphic to the $k \times k$ complex matrices, $M(k)$. A section $\varphi \in C^\infty(M(E))$ defines a linear operator, M_φ, on $C^\infty(E)$ via pointwise multiplication.

A *leafwise differential operator* on $C^\infty(E)$ is defined to be any operator obtained by taking finite sums of products of operators $\nabla^E_{\vec{X}}$ for $\vec{X} \in C^\infty(F)$, and M_φ for $\varphi \in C^\infty(M(E))$. Let $\mathscr{D}(E, \mathscr{F})$ denote the resulting algebra of differential operators. This algebra is independent of the choice of connexion on E, since two connexions differ by a 1-form with values in $M(E)$. Each element $D \in \mathscr{D}(E, \mathscr{F})$ can be written in the form

$$(3.1) \qquad \begin{cases} D = \sum_I M_{\varphi_I} \cdot \nabla^E_{\vec{X}_I} \\ I = (i_1, \dots, i_s) \text{ some } s \geqq 0 \\ \nabla^E_{\vec{X}_I} = \nabla^E_{\vec{X}_{i_s}} \circ \dots \circ \nabla^E_{\vec{X}_{i_1}} \\ \qquad \text{each } \vec{X}_{i_j} \in C^\infty(F). \end{cases}$$

We say that D has *order p* if D can be written as a sum with all I appearing in (3.1) having $s \leqq p$, and p is the least integer with this property.

For each leaf $L \subset V$ of $\mathscr{F}$, let $E_L \to L$ denote the pull-back bundle of E under the inclusion. Each section $\vec{X} \in C^\infty(F)$ restricts to a vector field $\vec{X}_L$ along L, so $\nabla^E_{\vec{X}}$ also restricts to an operator on $C^\infty(E_L)$. As multiplication operators obviously restrict to submanifolds, each $D \in \mathscr{D}(E, \mathscr{F})$ has a restriction to a *leafwise operator*

$$(3.2) \qquad D_L; C^\infty(E_L) \to C^\infty(E_L).$$

The bundles E_L inherit an Hermitian inner product, and TL has a Riemannian metric obtained from g on TV, so there are well-defined Sobolev spaces of sections of E for each $s \in \mathbf{R}$, which are denoted $W^s(E_L)$. Evidently, if $D \in \mathscr{D}(E, \mathscr{F})$ has order p, then for each leaf L we have a bounded operator

$$(3.3) \qquad D_L : W^s(E_L) \to W^{s-p}(E_L), \quad s \in \mathbf{R}.$$

(cf. page 218, [67]). For the special case $s=0$, we abuse notation by suppressing E and denote

$$\mathscr{H}_L \equiv W^0(E_L)$$

for the Hilbert space of L^2-sections of E_L.

The operator D is *leafwise elliptic* if each restriction D_L is elliptic of order p. That is, the symbol of D_L, given by a fiberwise map

$$\sigma_{D_L} : T^*L \to M(E_L)$$

is required to be invertible on non-zero elements $\xi \in T^*L$. We often abuse notation by writing σ_D for σ_{D_L}. Typical examples of leafwise-elliptic, first-order operators are obtained by making a leafwise construction of the *geometric operators* of Gromov-Lawson ([49], see also [74]) where E will be a Clifford module over the leafwise Clifford bundle $C(F) \to V$, with connexion ∇^C defined only along leaves of $\mathscr{F}$. (For details on this construction, see Chap. 4, [56]).

It is necessary to also introduce the algebra $\mathscr{D}_\psi(E, \mathscr{F})$ of leafwise pseudo-differential operators acting on $C^\infty(E)$, and its filtration into subspaces $\mathscr{D}^p_\psi(E, \mathscr{F})$ of operators of order p, for all $p \in \mathbf{R}$. This algebra is constructed in detail in (Sect. A, Chap. 7 of [67]). An element $P \in \mathscr{D}^p_\psi(E, \mathscr{F})$ is determined by its restrictions to the leaves of $\mathscr{F}$. For each leaf L, the restriction

$$(3.4) \qquad P_L : C^\infty(E_L) \to C^\infty(E_L)$$

is a pseudo-differential operator of order $\leqq p$, with distributional kernel k_{P_L} supported in a uniform ε-neighborhood of the diagonal $\Delta_L \subset L \times L$, for some $\varepsilon > 0$ independent of L. Moreover, for each smooth function λ with support in a coordinate neighborhood U_j, the kernel for $M_\lambda \circ P \circ M_\lambda$ restricted to a plaque $P_j(y) \subset U_j$ varies continuously (in the uniform topology on distributions) as a function of the transversal parameter $y \in I^n$. Let $C^{\infty,0}_c(E)$ denote the space of sections of E which are compactly supported in V, smooth along leaves, and their restrictions to leaves vary continuously in the transversal parameter. Then P defines a map $P : C^{\infty,0}_c(E) \to C^{\infty,0}_c(E)$.

An element $P \in \mathscr{D}^p_\psi(E, \mathscr{F})$ is said to have exactly order p if some restriction P_L has exactly order p, and no restriction P_L has larger order. For such P, the leafwise elliptic estimate implies that each P_L defines a bounded operator

$$(3.5) \qquad P_L : W^s(E_L) \to W^{s-p}(E_L).$$

We introduce a subspace of the algebra $\mathscr{D}_\psi(E, \mathscr{F})$ which satisfies additional regularity.

Definition 3.1. For each $\delta > 0$ and integer $l \geqq 0$, let $\mathscr{D}^p_\psi(E, \mathscr{F}, \delta, l)$ denote the subspace consisting of $P \in \mathscr{D}^p_\psi(E, \mathscr{F})$ which satisfy

(3.6SA) For each leaf L, P_L is a symmetric operator for the L^2-inner product on $C^\infty_c(E_L)$, and P_L admits a unique closure to a self-adjoint (possibly unbounded) operator on $\mathscr{H}_L$. That is, P_L with domain $C^\infty_c(E_L)$ is *essentially self-adjoint*.

(3.6δ) The distributional kernel k_{P_L} on $L \times L$ has support contained in a δ-neighborhood of the leaf diagonal Δ_L, for all L.

(3.6l) For each $j \in \mathfrak{J}$, the distributional kernel of the restriction to a plaque $P_j(y) \subset U_j$ of the compression $M_{\lambda_j} \circ P \circ M_{\lambda_j}$ depends C^l on $y \in I^n$, where we use the uniform topology on distributions.

Note that standard local theory of pseudo-differential operators implies that each kernel k_{P_L} is smooth off of the diagonal Δ_L, and the strength of the singularity along Δ_L is precisely given by the symbol σ_{P_L}. The reader is referred to Taylor [87] as a basic reference for the theory of pseudo-differential operators used here.

As an example, if V is a compact manifold, then all leaves of $\mathscr{F}$ are complete in the induced Riemannian metric space structure, so all of the leafwise geometric operators associated to this metric are in $\mathscr{D}^1_\psi(E, \mathscr{F}, \delta, \infty)$ for all $\delta > 0$ by results of (§ 1, [49]). For an arbitrary $P \in \mathscr{D}^p_\psi(E, \mathscr{F})$, even if P_L is symmetric, the condition that P_L be essentially self-adjoint is an additional regularity hypothesis.

Observe that $P \in \mathscr{D}^p_\psi(E, \mathscr{F}, \delta, l)$ defines an operator from $C^{l,\infty}_c(E)$ to itself by (3.6δ) and (3.6l).

One of our motivations for introducting the leafwise pseudo-differential operators is the following result, whose proof is a minor modification of that given for (Proposition 7.12, [67]), and so is omitted.

Proposition 3.2. *Let $P \in \mathscr{D}^p_\psi(E, \mathscr{F}, \delta, l)$ be leafwise elliptic. Then there exists a leafwise parametrix $Q \in \mathscr{D}^{-p}_\psi(E, \mathscr{F}, \delta, l)$ such that*

$$\begin{cases} PQ = \mathrm{Id} - S_0 \\ QP = \mathrm{Id} - S_1 \end{cases} \tag{3.7}$$

where $S_0, S_1 \in \mathscr{D}^{-1}_\psi(E, \mathscr{F}, 2\delta, l)$. □

A fiberwise, self-adjoint involution $\varepsilon \in M(E)$ induces an orthogonal decomposition, for each leaf L,

$$\mathscr{H}_L = \mathscr{H}^+_L \oplus \mathscr{H}^-_L \tag{3.8}$$

into its $\{\pm 1\}$ eigenspaces. An operator $P \in \mathscr{D}_\psi(E, \mathscr{F})$ is ε-graded if $\varepsilon P + P\varepsilon = 0$, and the pair (P, ε) is called a *graded leafwise operator*. The operator P decomposes into

$$P = \begin{bmatrix} 0 & P^- \\ P^+ & 0 \end{bmatrix}$$

with respect to the decomposition (3.8). If P has a self-adjoint closure, then the leafwise restrictions

$$P^\pm_L : \mathscr{H}^\pm_L \to \mathscr{H}^\mp_L$$

have unique closures which are adjoints of each other. If P is elliptic with parametrix Q, then without loss we can assume that (Q, ε) is also a graded operator. For the purpose of establishing notation, we record the identities

$$\begin{cases} P^- Q^+ = \mathrm{Id}^+ - S^+_0 \\ P^+ Q^- = \mathrm{Id}^- - S^-_0 \\ Q^- P^+ = \mathrm{Id}^+ - S^+_1 \\ Q^+ P^- = \mathrm{Id}^- - S^-_1 \end{cases} \tag{3.9}$$

For $\delta < \delta'$ there is an obvious inclusion

$$\mathscr{D}_\psi(E, \mathscr{F}, \delta, l) \subseteq \mathscr{D}_\psi(E, \mathscr{F}, \delta', l). \tag{3.10}$$

There is a retract to this map, constructed using cut-off functions.

Lemma 3.3. *For $\delta < \delta'$ there is a continuous linear map*

$$\lambda(\delta, \delta') : \mathscr{D}_\psi(E, \mathscr{F}, \delta', l) \to \mathscr{D}_\psi(E, \mathscr{F}, \delta, l) \tag{3.11}$$

which is the identity on the image of (3.10) *for* $\delta/2$.

Proof. Let $\varphi(\delta)$ be a monotone decreasing smooth map from $[0, \infty)$ to $[0,1]$ satisfying

$$\varphi(\delta)(r) = \begin{cases} 1 & \text{if } r \leqq \delta/2 \\ 0 & \text{if } r \geqq \delta \end{cases}.$$

For each leaf L, let $d_L : L \times L \to [0, \infty)$ be the leafwise Riemannian distance function. The compositions $\varphi(\delta) \circ d_L$ is continuous on $L \times L$ with support in a δ-uniform neighborhood of the diagonal Δ_L, and is identically 1 on a $\delta/2$-uniform neighborhood. If δ is less than the injectivity radius of L, then $\varphi(\delta) \circ d_L$ will also be smooth, and bounded geometry of the leaves implies that there are uniform estimates (over $L \times L$) on all of its derivatives. If δ exceeds the injectivity radius, then bounded geometry of the leaves implies there exists a smooth approximation to $\varphi(\delta) \circ d_L$ with all of the above properties. Moreover, these approximations can be chosen to depend C^∞ on the transverse parameter. So without loss, we can assume $\varphi(\delta) \circ d_L$ has uniform smooth estimates.

Define the retraction $\lambda(\delta, \delta')$ on P by letting $P' = \lambda(\delta, \delta')(P)$ be the operator with distributional kernel on $L \times L$,

$$k_{P'_L} = \varphi(\delta) \circ d_L \cdot k_{P_L}. \tag{3.12}$$

Since the kernel $k_{P'_L}$ is obtained by multiplying by a function which is identically equal to one on a neighborhood of the diagonal, P' is again pseudo-differential. Clearly, the condition on δ-support is satisfied, and continuity of $\lambda(\delta, \delta')$ follows from the uniform estimates on the multipliers. □

This lemma is an elementary technical device that is quite useful in constructing cyclic cocycles. We give two immediate applications of it.

Corollary 3.4. *Let $P \in \mathscr{D}_\psi^p(E, \mathscr{F}, \delta, l)$ be leafwise elliptic. For all integers $d > 0$, there exists $Q_d \in \mathscr{D}_\psi^{-p}(E, \mathscr{F}, \delta, l)$ such that*

$$\begin{cases} P \circ Q_d = \mathrm{Id} - S_{d,0} \\ Q_d \circ P = \mathrm{Id} - S_{d,1} \end{cases} \tag{3.13}$$

where $S_{d,0}$ and $S_{d,1} \in \mathscr{D}_\psi^{-d}(E, \mathscr{F}, 2\delta, l)$. Moreover, if P is ε-graded, then Q_d can be chosen ε-graded, with $S_{d,0}$ and $S_{d,1}$ commuting with ε.

Proof. Let Q be chosen as in Proposition 3.2. Introduce

$$R_d = \mathrm{Id} + S_0 + S_0^2 + \ldots + S_0^{d-1}$$

$$\tilde{Q}_d = Q \circ R_d$$

$$Q_d = \lambda(\delta, d \cdot \delta)(Q \circ R_d).$$

Then $P\tilde{Q}_d = \mathrm{Id} - S_0^d$, and from (3.7) we obtain $S_0 P = P S_1$ so that

$$\begin{aligned}\tilde{Q}_d P &= Q(1+S_0+\ldots+S_0^{d-1})\cdot P\\ &= QP(1+S_1+\ldots+S_1^{d-1})\\ &= \mathrm{Id} - S_1^d.\end{aligned}$$

The multiplicative property of symbols implies that $S_0^d,\ S_1^d \in \mathscr{D}_\psi^{-d}(E, \mathscr{F}, (d+1)\delta, l)$.

The difference $(Q_d - \tilde{Q}_d)$ is represented by a smooth kernel on $L \times L$, since Q_d is pseudodifferential hence has smooth kernel off the diagonal. Therefore, both $P \circ (Q_d - \tilde{Q}_d)$ and $(Q_d - \tilde{Q}_d) \circ P$ are represented by leafwise smoothing operators, so

$$S_{d,0} = \mathrm{Id} - P \circ Q_d = S_0^d + \text{smoothing}$$

$$S_{d,1} = \mathrm{Id} - Q_d \circ P = S_1^d + \text{smoothing}$$

and have support in a 2δ-uniform neighborhood of the diagonal.

If (P, ε) is a graded operator, then we replace Q_d with $\frac{1}{2}(Q_d - \varepsilon Q_d \varepsilon)$ to obtain a graded parametrix. □

For a self-adjoint leafwise operator P without grading, the construction of a Chern character is based upon introducing its phase, which will be a self-adjoint involution. More precisely, since the leaves of $\mathscr{F}$ are generally open manifolds, we must first seek an element $\Phi \in \mathscr{D}_\psi^0(E, \mathscr{F}, \delta, l)$ which represents this phase up to "lower order approximations", satisfying $\Phi^2 = \mathrm{Id} + S$ where S has negative order. This is provided by the next two results.

Proposition 3.5. *Let $P \in \mathscr{D}_\psi^p(E, \mathscr{F}, \delta, l)$ be leafwise elliptic. Then there exists*

$$|P|^{-1} \in \mathscr{D}_\psi^{-p}(E, \mathscr{F}, \delta, l)$$

$$\Phi \in \mathscr{D}_\psi^0(E, \mathscr{F}, \delta, l)$$

such that for all leaves L of $\mathscr{F}$ and non-zero $\xi \in T^*L$:

(3.14) *$\sigma_{|P|^{-1}}(\xi)$ is positive definite.*

(3.15) *$\sigma_\Phi(\xi)$ is a self-adjoint involution.*

(3.16) *$\sigma_P(\xi)$, $\sigma_{|P|^{-1}}(\xi)$ and $\sigma_\Phi(\xi)$ pairwise commute, and*

$$\sigma_\Phi(\xi) = \sigma_{|P|^{-1}}(\xi) \cdot \sigma_P(\xi).$$

Proof. For each $\xi \in T^*L$, the fiberwise endomorphisms $\sigma_P(\xi)$ are symmetric, so admit a fiberwise factorization

$$\sigma_P(\xi) = Ph(\xi) \cdot Sc(\xi) \tag{3.17}$$

such that:

all of the endomorphisms in (3.17) commute and are symmetric

$Sc(\xi)$ is a positive-definite matrix and as a function of ξ is homogeneous of order p

$Ph(\xi)$ is an involution and homogeneous in ξ of order zero.

The space of positive-definite symmetric matrices is a cone in each fiber of $M(E)$, so we can choose the factorization (3.17) to be continuous on $T^*\mathscr{F} \subset T^*V$, smooth along the leaves of $\mathscr{F}$ and transversally of class C^l.

Given the hypothesis that the leaves of $\mathscr{F}$ have uniformly bounded geometry, the methods of (§ VIII, [27]), or (Proposition 7.16, [67]) yield operators

$$|\tilde{P}|^{-1} \in \mathscr{D}_{\psi}^{-p}(E, \mathscr{F}, \delta', l)$$
$$\tilde{\Phi} \in \mathscr{D}_{\psi}^{0}(E, \mathscr{F}, \delta'', l)$$

for some δ', $\delta'' > 0$, and such that for $0 \neq \xi \in T^*L$,

$$\sigma_{|\tilde{P}|^{-1}}(\xi) = Sc(\xi)$$
$$\sigma_{\tilde{\Phi}}(\xi) = Ph(\xi).$$

Then apply Lemma 3.3 to reduce the support of these operators to δ. □

Corollary 3.6. $\Phi^2 = \mathrm{Id} - R$, *where* $R \in \mathscr{D}_{\psi}^{-1}(E, \mathscr{F}, 2\delta, l)$.

Proof. From (3.15), the principal symbol of Φ^2 is $\sigma_{\Phi}^2 = \mathrm{Id}$. □

Next, we replace the phase Φ above by a perturbation which is an exact involution. This is accomplished via the "4×4 trick", which results from applying the usual "2×2 trick" to the two-dimensional complex Clifford algebra $C(2)$ (cf. page 27, [29]; §7 and Appendix 2 of [30]). For L a leaf of $\mathscr{F}$, define $\mathscr{H}_L^4 = (\mathscr{H}_L) \otimes_{\mathbf{C}} \mathbf{C}^4$.

A uniformly bounded, complex valued function $f \in C_b(V)$ acts as a multiplication operator on each $\mathscr{H}_L^4$ via restriction of f to $L \subset V$, then extending to a function $f: L \to M(\mathbf{C}^4)$ via embedding into the upper left diagonal entry. For $\tilde{h} = (h_1, h_2, h_3, h_4)^t \in \mathscr{H}_L^4$,

$$f \cdot \tilde{h} = (f \cdot h_1, 0, 0, 0)^t. \tag{3.18}$$

Introduce operators on $\mathscr{H}_L$

$$\begin{cases} x = \frac{1}{2}(3\Phi - \Phi^3) \\ y = \mathrm{Id} - \Phi^2 = R \\ z = \frac{1}{2}(\Phi^3 - \Phi) = -\frac{1}{2}\Phi \circ R \end{cases} \tag{3.19}$$

so that $x = \Phi - z$, and by Corollary 3.6

$$y, z \in \mathscr{D}_{\psi}^{-1}(E, \mathscr{F}, 3\delta, l).$$

Introduce the exact involution

$$\Pi = \begin{bmatrix} x & y & -z & 0 \\ y & -x & 0 & -z \\ z & 0 & -x & -y \\ 0 & z & -y & x \end{bmatrix} \tag{3.20}$$

in $\mathscr{D}_{\psi}^{0}(E^4, \mathscr{F}, 3\delta, l)$, where $E^k = E \otimes \mathbf{C}^k = E \oplus \ldots \oplus E$, k-copies.

For $f \in C_b^{\infty}(V)$, we have

$$f \cdot \Pi = \begin{bmatrix} fx & fy & -fz & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3.21)
$$\Pi \cdot f = \begin{bmatrix} xf & 0 & 0 & 0 \\ yf & 0 & 0 & 0 \\ zf & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the commutator $[\Pi, f] = \Pi f - f\Pi$ has all entries in $\mathcal{D}_\psi^{-1}(E, \mathcal{F}, 3\delta, l)$ since the symbol of Φ commutes with the scalar f.

For m the dimension of the leaves, fix an odd integer $q \geqq m+1$. Let $c_1 > 0$ be the Lebesgue number of the fixed cover of V, and let $\delta = (6q+12)^{-1} \cdot c_1$. For a $(q+1)$-tuple of functions $f_0, f_1, \ldots, f_q \in C_b^\infty(V)$, define a leafwise operator

(3.22)
$$\tilde{c}_\Pi(f_0, \ldots, f_q) = -(-4)^{-\left(\frac{q+1}{2}\right)} \Pi \cdot [\Pi, f_0] \ldots [\Pi, f_q].$$

Proposition 3.7. *For $q \geqq m$, the leafwise operators $\tilde{c}_\Pi(f_0, \ldots, f_q)$ are represented by trace-class continuous kernels on leaves, whose supports are contained in $\frac{1}{2} \cdot c_1$-uniform neighborhoods of the leaf diagonals. Moreover, the kernels depend C^l on the parametrization of the leaves by a local transversal.*

Proof. It was observed above that each commutator $[\Pi, f_j]$ is a leafwise pseudo-differential operator of order -1, with 3δ-supported distributional kernel. Therefore, the product (3.22) is pseudo-differential with order $-(p+1) < -m$, so by the Sobolev lemma it is represented by a continuous kernel. There are $(q+2)$-kernels appearing in (3.22), so the product has $(q+2)3\delta$-support. Finally, each operator Π depends on C^l on the transversal parameter, so the same holds for the product. □

Definition 3.8. Let $q > m$ be an odd integer, μ a transverse invariant measure for $\mathcal{F}$, and $P \in \mathcal{D}_\psi^p(E, \mathcal{F}, \delta, 0)$ a leafwise elliptic operator with pseudo-differential phase Φ. The cyclic Chern character for the data $(V, \mathcal{F}, \mu, \Phi)$ is the $(q+1)$-multilinear functional on $C_c^\infty(V)$,

(3.23)
$$c_\Phi(f_0, \ldots, f_q) = \operatorname{Tr}_\mu(\tilde{c}_\Pi(f_0, \ldots, f_q)); \quad f_j \in C_c^\infty(V).$$

There is a natural extension of c_Φ to the algebra of matrix-valued smooth functions

$$\mathcal{A}_N = M(N, C_c^\infty(V)) \cong \{m : V \to M(N)\}.$$

The operator P has a canonical product extension to $C_c^\infty(E \otimes \mathbf{C}^N)$. Each $m \in M(N, C_c^\infty(V))$ acts as a multiplier on this space, so we repeat the above construction, using the fiberwise trace on $M(E \otimes \mathbf{C}^N)$ to extend Tr_μ to kernels with values in these endomorphisms. (In the notation of (page 103, [31]) we are forming the cocycle product $c_\Phi \# \operatorname{Tr}$).

Proposition 3.9. *For each integer $N > 0$, the function c_Φ is a cyclic q-cocycle over the algebra $\mathcal{A}_N$.*

Proof. The proposition is an algebraic consequence of two basic properties:

(3.24)
$$\Pi^2 = \mathrm{Id}$$

(3.25)
$$\operatorname{Tr}_\mu(S \circ T) = \operatorname{Tr}_\mu(T \circ S)$$

whenever

$$S \in \mathscr{D}_{\psi}^{-s}(E^{4N}, \mathscr{F}, \varepsilon, 0)$$
$$T \in \mathscr{D}_{\psi}^{-t}(E^{4N}, \mathscr{F}, \varepsilon, 0)$$

with $s+t>m$.

First, (3.25) implies that for $m_i \in \mathscr{A}_N$

$$\Pi \circ [\Pi, m_i] = -[\Pi, m_i] \circ \Pi \tag{3.26}$$

which combined with (3.26) yields the cyclic property of c_{Φ}:

$$c_{\Phi}(m_1, \ldots, m_q, m_0) = (-1)^q c_{\Phi}(m_0, m_1, \ldots, m_q). \tag{3.27}$$

The bounded linear operator $m \to [\Pi, m]$ is a derivation on the algebra $\mathscr{A}_N$. We leave it to the reader to check that this observation, along with (3.26) and (3.27) implies that the cocycle condition $\delta c_{\Phi} = 0$, where:

$$\begin{cases} \delta c_{\Phi}(m_0, \ldots, m_{q+1}) = \sum_{j=0}^{q} (-1)^j c_{\Phi}(m_0, \ldots, m_j m_{j+1}, \ldots, m_{q+1}) \\ \qquad + (-1)^{q+1} c_{\Phi}(m_{q+1} m_0, m_1, \ldots, m_q) \\ \qquad \text{for} \quad m_0, \ldots, m_{q+1} \in \mathscr{A}_N. \quad \square \end{cases} \tag{3.28}$$

The cyclic cohomology of the algebra $C_c^{\infty}(V)$ is the quotient space

$$H_{\lambda}^q(C_c^{\infty}(V)) \equiv Z_{\lambda}^q(C_c^{\infty}(V)) / B_{\lambda}^q(C_c^{\infty}(V)) \tag{3.29}$$

where

$$Z_{\lambda}^q(C_c^{\infty}(V)) = \text{closed cyclic } q\text{-cocycles}$$
$$B_{\lambda}^q(C_c^{\infty}(V)) = \text{image under } \delta \text{ of cyclic}$$
$$q\text{-linear functionals}.$$

The formulas (3.28) and (3.29) can be applied to any algebra $\mathscr{B}$ and integer q to yield the cyclic groups $H_{\lambda}^q(\mathscr{B})$. A basic observation of Connes (Corollary 24, page 113, [31]) is that the inclusion $C_c^{\infty}(V) = \mathscr{A}_1 \subset \mathscr{A}_N$ into the "upper-left-corner", induces a canonical isomorphism of cyclic cohomology. Then the extension $c_{\Phi} \# \mathrm{Tr}$ to $\mathscr{A}_N$ and the cocycle c_{Φ} determine the same cohomology class under this isomorphism.

The cyclic Chern character of the data $(V, \mathscr{F}, \mu, P)$ is the cohomology class

$$\mathrm{ch}_{\mu}(P) = [c_{\Phi}] \in H_{\lambda}^q(\mathscr{A}_N), \tag{3.30}$$

which by the previous remarks is independent of N (up to isomorphism).

Proposition 3.10. *Let $\{P_t \mid 0 \leqq t \leqq 1\}$ be a continuous family of leafwise ϕDO's which satisfy conditions* (3.6). *For $q > m$ odd, the cyclic cohomology classes $\mathrm{ch}_{\mu}(P_t) \in H_{\lambda}^q(\mathscr{A}_N)$ are independent of t.*

Combining Proposition 3.10 with the Poincaré Duality principle (cf. Theorem 2 and Lemma 4.10, [12]) we obtain:

Corollary 3.11. *For $q > m$ odd, there is a well-defined homology Chern character*

$$\mathrm{ch}_{\mu}: K^0(S^* \mathscr{F}) \to H_{\lambda}^q(\mathscr{A}_N). \tag{3.31}$$

Remark 3.12. The Chern character (3.31) is the foliated version of the odd analogue of the even Chern character for compact manifolds constructed by Connes (§6, [30]). There is a topological formula for (3.31), discussed in the next section, which shows that the deRham character determined by ch_μ takes values which are exotic classes of the pair $(\mathscr{F}, \mu)$ in the sense of the μ-classes of [52]. It is also shown in the next section that for special suspension foliations, the values of ch_μ agree with Cheeger-Simons transgressed classes, hence are secondary in the customary sense. In this regards, the Chern character (3.31) exhibits new geometric information not present in the previously considered cases (cf. [11], [30]). □

Proof of (3.10). Let $\{\Phi_t | 0 \leqq t \leqq 1\}$ be a continuous family of leafwise elliptic operators in $\mathscr{D}^0_\psi(E^N, \mathscr{F}, \delta, 0)$ constructed from $\{P_t | 0 \leqq t \leqq 1\}$ via the process of Proposition 3.5, such that $\Phi_t^2 = \mathrm{Id} - R_t$ with each R_t of order -1 or less. It will suffice to show that $ch_\mu(P_0) = ch_\mu(P_1)$. The space of ψDO's with self-adjoint principal symbol is locally smoothly contractible, so the dependence of P_t on t can be regularized to obtain a path from P_0 to P_1 in this class which is smooth in t. Correspondingly, we can choose a smooth family $\{\Phi_t | 0 \leqq t \leqq 1\}$ and hence obtain a smooth family of involutions $\{\Pi_t | 0 \leqq t \leqq 1\}$. Then set

$$L_t = \frac{d}{dt}(\Pi_t) \in \mathscr{D}^0_\psi(E^{4N}, \mathscr{F}, 3\delta, 0). \tag{3.32}$$

Note that for fixed t and $f \in \mathscr{A}_N$

$$[\Pi_t, f] \in \mathscr{D}^{-1}_\psi(E^{4N}, \mathscr{F}, 3\delta, 0). \tag{3.33}$$

Thus for each $0 \leqq t \leqq 1$ there is a quasi-homomorphism ϱ_+ constructed from Π_t as in the proof of (Proposition 4b, page 72, [30]). Moreover, the operators L_t are bounded so that for $q > m$ the same proof applies to show that $\frac{d}{dt}(c_{\Phi_t})$ is exact for each t. It follows that the cohomology class $ch_\mu(P_t)$ is independent of t. □

Proof of (3.11). We define the map ch_μ on vector bundles, then extend it formally to $K^0(S^*\mathscr{F})$. Let $\xi \to S^*\mathscr{F}$ be a complex vector bundle of fiber dimension k. Choose an inverse bundle η and an identification $\xi \oplus \eta \cong \varepsilon^N$ for some $N > 0$. Introduce a fiberwise involution on ε^N by declaring ξ to be the bundle of positive fiberwise eigenvectors, and η the bundle of negative eigenvectors. There is a leafwise ψDO, $\Phi(\xi, \eta) \in \mathscr{D}^0_\psi(\varepsilon^N, \mathscr{F}, \delta, 0)$ whose principal symbol over $S^*\mathscr{F}$ consists of this involution, constructed as in (Proposition 7.16, [67]). Define $ch_\mu[\xi]$ to be the cohomology class of the cocycle $c_{\Phi(\xi,\eta)}$.

We must show that

(3.34) $ch_\mu[\xi]$ is unchanged when ξ is replaced by $\varepsilon^l \oplus \xi$ for $l > 0$

(3.35) $ch_\mu[\xi]$ is unchanged when η is replaced by $\eta \oplus \varepsilon^l$ for $l > 0$

(3.36) $ch_\mu[\xi]$ is independent of the choice of isomorphism $\xi \oplus \eta \cong \varepsilon^N$.

It will then follow that ch_μ formally extends to all of $K^0(S^*\mathscr{F})$.

For the proof of (3.34), note that we assume given an identification $\xi \oplus \eta \cong \varepsilon^N$, and then use the natural extension to $(\varepsilon^l \oplus \xi) \oplus \eta \cong \varepsilon^{l+N}$. Let $\pi_l : \varepsilon^{l+N} \to \varepsilon^{l+N}$ be the fiberwise projection,

$$\varepsilon^{l+N} \cong \varepsilon^l \oplus \varepsilon^N \to \varepsilon^l \oplus \{0\} \subset \varepsilon^l \oplus \varepsilon^N \cong \varepsilon^{l+N}. \tag{3.37}$$

Extend $\Phi(\xi, \eta)$ to an operator on sections of $\varepsilon^{l+N} \cong \varepsilon^l \oplus \varepsilon^N$ by letting it act trivially on the first summand. We can then take

(3.38) $$\Phi(\varepsilon^l \oplus \xi, \eta) = \pi_l \oplus \Phi(\xi, \eta).$$

For this choice, the commutators in the formula (3.22) used to define $c_{\Phi(\varepsilon^l \oplus \xi, \eta)}$ decompose into $l \times l$ and $N \times N$ blocks along the diagonal. The operator π_l commutes with sections of ε^l, so applying the trace yields an equality of cocycles

(3.39) $$c_{\Phi(\varepsilon^l \oplus \xi, \eta)} = c_{\Phi(\xi, \eta)}.$$

The cohomology class of c_Φ depends only on the homotopy class of the principal symbol of Φ by Proposition 3.10, so we obtain

$$\begin{aligned} \mathrm{ch}_\mu[\varepsilon^l \oplus \xi] &= [c_{\Phi(\varepsilon^l \oplus \xi, \eta)}] \\ &= [c_{\Phi(\xi, \eta)}] \\ &= \mathrm{ch}_\mu[\xi]. \end{aligned}$$

The obvious modification of the above proof also proves (3.36).

Property (3.36) follows from (3.35), Proposition 3.10 and some homotopy theory. Let two isomorphisms be given,

$$I_0, I_1 : \xi \oplus \eta \cong \varepsilon^N.$$

For each $l > 0$ we extend these two isomorphisms

(3.40) $$I_{0,l}, I_{1,l} : \xi \oplus (\eta \oplus \varepsilon^l) \cong \varepsilon^N \oplus \varepsilon^l \cong \varepsilon^{N+l}.$$

Lemma 3.13. *For l sufficiently large, the isomorphisms $I_{0,l}$ and $I_{1,l}$ are smoothly isotopic.*

Proof. The identifications (3.40) are equivalent to specifying smooth maps

$$\hat{I}_{0,l}, \hat{I}_{1,l} : V \to G(N+l, k)$$

where $G(N+l, k)$ is the Grassmannian of complex k-planes in $\mathbf{C}^{N+l}$. For each l there is a "stabilization" induced inclusion $G(N+l, k) \subset G(N+l+1, k)$ and it is a classical fact that $\varinjlim G(N+l, k) = G(\infty, k)$ has the homotopy type of BU. Hence, the smooth homotopy classes of $\hat{I}_{0,l}$ and $\hat{I}_{1,l}$ for l large are determined by the bundle isomorphism class of ξ. □

To conclude the proof of (3.36) and of Corollary 3.11, note that for Φ_0, Φ_1 the involutions obtained from the two given identifications, we can stabilize then in the η-bundle without changing the cyclic cohomology classes of the cocycles c_{Φ_0}, c_{Φ_1}. Then by Lemma 3.13 the symbols of Φ_0 and Φ_1 are homotopic, hence by Proposition 3.10 the cocycles are cohomologous. □

For a graded leafwise operator (P, ε) there is an analogous construction of an even degree, cyclic Chern character. Choose a parametrix, via Corollary 3.4,

$$Q \in \mathscr{D}_\psi^{-p}(E, \mathscr{F}, \delta, \mathbb{Q})$$

so that $PQ = I - S_0$, $QP = I - S_1$ where S_0, S_1 have order $-(m+1+|p|)$, hence are represented by leafwise smooth kernels. For the even case, a "2 x 2-trick" is used to make a compact perturbation of P and Q so that they are exactly inverses (cf. Appendix II, [30]).

For each leaf L, let $\mathscr{H}_1 = \mathscr{H}_L^+ \oplus \mathscr{H}_L^-$ be the graded Hilbert space associated to ε. Define a new Hilbert space $\tilde{\mathscr{H}}_L = \mathscr{H}_L \otimes_{\mathbf{C}} \mathbf{C}^2$, with grading $\tilde{\varepsilon}$ which on a vector $\tilde{h} = (h_1, h_2)$ acts via $\tilde{\varepsilon}(h_1, h_2) = (\varepsilon h_1, -\varepsilon h_2)$. This decomposes $\tilde{\mathscr{H}}_L$ into a direct sum

$$\text{(3.41)} \qquad \begin{cases} \tilde{\mathscr{H}}_L^+ = \mathscr{H}_L^+ \oplus \mathscr{H}_L^- \\ \tilde{\mathscr{H}}_L^- = \mathscr{H}_L^- \oplus \mathscr{H}_L^+ \end{cases}$$

The module action of $C_b(V)$ on $\tilde{\mathscr{H}}_L$ is given by $f \cdot (h_1, h_2) = (f \cdot h_1, 0)$.

Extend the operators P and Q to

$$\text{(3.42)} \qquad \tilde{P} = \begin{bmatrix} P & S_0 \\ S_1 & -(S_1 + \mathrm{Id}) Q \end{bmatrix}$$

$$\text{(3.43)} \qquad \tilde{Q} = \begin{bmatrix} (\mathrm{Id} + S_1) & S_1 \\ S_0 & -P \end{bmatrix}$$

Lemma 3.13. *Let $P \in \mathscr{D}_\psi^p(E, \mathscr{F}, \delta, 0)$ with $p \geqq 0$.*
a) $\tilde{P} \circ \tilde{Q} = \tilde{Q} \circ \tilde{P} = \mathrm{Id}$ *on* $\tilde{\mathscr{H}}_L$.
b) *For any smooth function $f\colon V \to \mathbf{C}$, the leafwise operator*

$$\tilde{Q}[\tilde{P}, f] \in \mathscr{D}_\psi^{-1}(E^2, \mathscr{F}, 6\delta, 0).$$

Proof. a) follows from the relations (3.13) and the identities $PS_0 = S_1 P$ and $QS_0 = S_1 Q$.

For b), we must show that the order is -1. Explicit calculation gives

$$\text{(3.44)} \qquad \tilde{Q}[\tilde{P}, f] = \begin{bmatrix} (I + S_1) Q [P, f] + S_1^2 f & (I + S_1) Q f S_0 \\ S_0 [P, f] - P S_1 f & S_0 f S_0 \end{bmatrix}$$

in which each entry has order at most $-(m+1)$ except for the term $Q[P, f]$ of order -1, since $[P, f]$ has order $(p-1)$. □

Corollary 3.14. *Let $q = 2l > m$. For all $(q+1)$-tuples $f_0, \ldots, f_q \in C_c^\infty(V)$, the leafwise operator*

$$\text{(3.45)} \qquad \tilde{c}_{P,Q}(f_0, \ldots, f_q) = (2\pi i)^l \cdot l! \cdot \tilde{Q}[\tilde{P}, f_0] \ldots \tilde{Q}[\tilde{P}, f_q]$$

is represented by a leafwise continuous kernel supported in a $6(q+1)\delta$-uniform neighborhood of the diagonal. □

The Chern character for compact manifolds in (§6, [30]) extends to a foliation Chern character using the operators $\tilde{P}$ and $\tilde{Q}$ defined above.

Definition 3.15. Let $q > m$ be an even integer, μ a transverse invariant measure for $\mathscr{F}$ and (P, ε) a graded leafwise elliptic operator, with parametrix Q as above. The *cyclic Chern cocycle* for the data $(V, \mathscr{F}, \mu, P, Q, \varepsilon)$ is the $(q+1)$-multilinear functional on $C_c^\infty(V)$

$$\text{(3.46)} \qquad c_P(f_0, \ldots, f_q) = \mathrm{Tr}_\mu(\tilde{\varepsilon} \cdot \tilde{c}_{P,Q}(f_0, \ldots, f_q)).$$

As in the odd case, c_P admits a canonical extension, $c_P \# \mathrm{Tr}$, to the algebra $\mathscr{A}_N$. By abuse of notation, we call all of these extensions c_P.

Proposition 3.16. *For $(V, \mathscr{F}, \mu, P, Q, \varepsilon)$ as above,*
a) *The linear functional c_P is cyclic and closed for the coboundary* (3.29).

b) *For all* $N \geqq 1$, *the cohomology class*

$$\mathrm{ch}_\mu(P, \varepsilon) = [c_P] \in H^q_\lambda(\mathscr{A}_N)$$

is constant when (P, ε) *is continuously deformed through self-adjoint, leafwise elliptic graded pseudo-differential operators of order* p. □

The proof of part a) is an easy calculation, while that of b) is essentially identical to methods of (§6, [30]), so is omitted.

4. The longitudinal de Rham Chern character and Cheeger-Simons classes

We apply the foliation index theorem [27] to obtain a de Rham homology interpretation of the odd cyclic Chern character for the data $(V, \mathscr{F}, \mu, P)$. In the special case of the geometric models $(V, \mathscr{F}_\alpha)$ of a group representation α, we prove that the secondary contribution to the longitudinal index formula agrees with a Cheeger-Simons transgressed class for the flat unitary bundle associated to α and ϱ.

We begin by recalling a theorem of Connes that interprets cyclic cocycles over $C_c^\infty(V)$ as de Rham currents on V. Let $\Omega_c^q(V)$ denote the Frechet space of compactly-supported smooth q-forms on V, and $C_q^{(\infty)}(V)$ its topological dual space of complex-valued q-currents on V. Also, let δ be the transpose of exterior differentiation, $X_q^{(\infty)}(V)$ the closed subspace of closed q-currents and $H_q^{(\infty)}(V)$ the quotient homology group. The homology de Rham theorem (cf. [73]) identifies $H_q^{(\infty)}(V)$ with the locally-finite singular chain homology $H_q^{lf}(V; \mathbf{C})$. In addition, a closed q-current canonically defines a linear map on $H_c^q(V)$ and there is induced an isomorphism

$$H_q^{(\infty)}(V) \to \mathrm{Hom}\,(H_c^q(V), \mathbf{C})\,. \tag{4.1}$$

For V oriented, this latter group is identified with $H_{\mathrm{deR}}^{m+n-q}(V)$ by Poincaré duality. For $[c] \in H_q^{(\infty)}(V)$ and $[\omega] \in H_c^q(V, \mathbf{C})$, we denote the map (4.1) by

$$[c] \mapsto \langle \cdot, [c] \rangle$$

where $\langle \cdot, [c] \rangle ([\omega]) = \langle [\omega], [c] \rangle$.

Theorem 4.1. (Theorem 46, [31]). *For* $q \geqq 0$ *and* $N \geqq 1$ *there are isomorphisms*

$$H^q_\lambda(\mathscr{A}_N) \cong Z_q^{(\infty)}(V) \oplus H_{q-2}^{(\infty)}(V) \oplus H_{q-4}^{(\infty)}(V) \oplus \dots \tag{4.2}$$

Moreover, the induced mapping

$$H^q_\lambda(\mathscr{A}_N) \to Z_q^{(\infty)}(V)$$

is canonical, while the maps to the other summands depend upon choices (*cf. Remark* 47a), [31]).

Corollary 4.2. *The cyclic Chern character of* (3.31) *yields a total de Rham homology Chern character,*

$$\mathrm{ch}_*(P, \mu) = \sum_{l=0}^{\infty} \mathrm{ch}_{q-2l}(P, \mu)\,. \tag{4.3}$$

Here, $q>m$ is even or odd corresponding to whether P is graded or ungraded, and each

(4.4) $$\mathrm{ch}_{q-2l}(P,\mu)\in H^{(\infty)}_{q-2l}(V)$$

is the homology class determined by the isomorphism (4.2).

The odd topological Chern character

(4.5) $$\mathrm{ch}^*: K^1_c(V)\to H^{\mathrm{odd}}_c(V)$$

associates to a smooth map $u: V\to U_N$ the cohomology class $u^*([\mathrm{T}\widetilde{\mathrm{ch}}])$. Here, $\mathrm{T}\widetilde{\mathrm{ch}}$ denotes the differential form on U_N that transgresses the universal Chern form $\widetilde{\mathrm{ch}}$ on BU_N, and $[\mathrm{T}\widetilde{\mathrm{ch}}]$ is its cohomology class. The map (4.5) is an isomorphism after tensoring with $\mathbf{C}$, so that the odd homology Chern character (4.3) is determined by the pairings

(4.6) $$\langle \mathrm{ch}^*[u], \mathrm{ch}_*(P,\mu)\rangle\in\mathbf{R}$$

for $[u]\in K^1_c(V)$. As remarked in Sect. 3, $u\in\mathscr{A}_N$ via the inclusion $U_N\subset M(N)$. Let $u^*: V\to U_N$ be the conjugate transpose map. For q odd, we define

$$c_\Phi(u)=c_\Phi(u,u^*,\dots,u,u^*).$$

The relationships between the pairing (4.6), the measured foliation index theorem of Connes, and the Kasparov pairings of $[u]$ and $[\Phi]$ are summarized by the next theorem, whose proof is given in Appendix B.

Theorem 4.3. *Let $q>m$ be odd, $u: V\to U_N$ smooth and oriented. Then*

(4.7) $$c_\Phi(u)=\langle \mathrm{ch}^*[u], \mathrm{ch}_*(P,\mu)\rangle$$

(4.8) $$=\mathrm{Tr}_\mu([u]\boxtimes[\Phi^+])$$

(4.9) $$=(-1)^m\langle \psi^{-1}(\mathrm{ch}^*(\xi))\cup \mathrm{Td}(F)\cup \mathrm{ch}^*[u], [C_\mu]\rangle.$$

The equality (4.7) is a consequence of the naturality of the isomorphism (4.2) with respect to the pairing of cyclic homology and cyclic cohomology [64].

The equality (4.8) involves the Kasparov outer product $[u]\boxtimes[\Phi^+]$ of the KK-class of the unitary $[u]$ with the KK-class of the involution over $C^*(V,\mathscr{F})$ determined by the leafwise operator Φ. This is explained in detail in Appendix B.

The equality (4.9) is the odd version of the foliation index theorem. Here, ξ denotes the bundle over $S^*\mathscr{F}$ of positive eigenvectors for the symbol of Φ, and ψ^{-1} is the inverse Thom isomorphism. $\mathrm{Td}(F)$ is the Todd class of the complexified foliation tangent bundle $F\otimes\mathbf{C}$. Finally, $[C_\mu]$ is the Ruelle-Sullivan de Rham homology class associated to the transverse measure μ. The derivation of (4.9) from (4.8) is given in Appendix B.

When the normal bundle to $\mathscr{F}$ is also oriented, so that V has an orientation, the class $[C_\mu]$ has a Poincaré dual, denoted by $[d\mu]$. (If μ is defined by integration against a transverse closed q-form ω, then $[d\mu]=[\omega]$.) In this case, we can rewrite (4.9) as

(4.9′) $$c_\Phi(u)=(-1)^m\langle \psi^{-1}(\mathrm{ch}^*(\xi))\cup \mathrm{Td}(F)\cup \mathrm{ch}^*[u]\cup[d\mu], [V]\rangle.$$

For a suspension foliation $\mathscr{F}_\alpha$ on $V=\tilde{M}\times_\Gamma G$ associated to $\alpha:\Gamma\to G$, there is a second interpretation of the term $\mathrm{ch}^*[u]\cup[d\mu]$ which appears in (4.9′), as a Cheeger-Chern-Simons class for the representation α. Assume a trivialization $\Theta: V\to M\times G$ is fixed and a representation $\varrho: G\to U_N$ is given. Define $u=u(\varrho)$ to be the composition

$$V \xrightarrow{\Theta} M \times G \xrightarrow{\pi_2} G \xrightarrow{\varrho} U_N . \tag{4.10}$$

Let $\theta = A^{-1}dA$ denote the left-invariant Maurer-Cartan matrix-valued form on U_N. For any base space, B, we also let θ denote the product connection on the product bundle $B \times U_N \to B$ with horizontal spaces $\{B \times \{A\} \mid A \in U_N\}$. The homomorphisms ϱ and u induce gauge automorphisms of $G \times U_N \to G$ and $V \times U_N \to V$ by

$$\tilde{\varrho}(g, A) = (g, \varrho(g)^{-1}A) ; \quad (g, A) \in G \times U_N \tag{4.11}$$

$$\tilde{u}(x, A) = (x, u(x)^{-1}A) ; \quad (x, A) \in V \times U_N . \tag{4.12}$$

The product connexions θ pull-back to twisted connections denoted by

$$\theta^\varrho = \tilde{\varrho}^*(\theta) = \theta + \varrho \cdot d\varrho^* \tag{4.13}$$

$$\theta^u = \tilde{u}^*(\theta) = \theta + u \cdot du^* . \tag{4.14}$$

Introduce the composition

$$\Theta_2 = \pi_2 \circ \Theta : V \to M \times G \to G .$$

Then $u = \varrho \circ \Theta_2$ and $\Theta_2^*(\theta^\varrho) = \theta^u$.

Lemma 4.4. *The horizontal spaces of θ^ϱ are the graphs $\{(g, \varrho(g) \cdot A) \mid g \in G\}$ for $A \in U_N$. The horizontal spaces of θ^u are the graphs $\{(x, u(x)A) \mid x \in V\}$, for $A \in U_N$.*

Proof. $\tilde{\varrho}$ and $\tilde{u}$ maps these graphs into the level sets $G \times \{A\}$ and $V \times \{A\}$, which are horizontal for θ. □

Let B denote a smooth manifold, and assume that the product bundle $B \times U_N$ has a given flat connection θ^1. Extend θ^1 and the trivial connection θ to connection forms on $B \times \mathbf{R} \times U_N$, then define a family of connections:

$$\theta^t = (1-t)\theta^1 + t \cdot \theta . \tag{4.15}$$

The associated curvature 2-form on $B \times \mathbf{R}$

$$\Omega^t = d(\theta^t) + \tfrac{1}{2}[\theta^t, \theta^t] \tag{4.16}$$

takes values in $\mathfrak{u}(N) \subset M(N)$, and we set

$$\mathrm{ch}(\Omega^t) = \sum_{l=0}^{\infty} \mathrm{Tr}_{M(N)} \left(\frac{-1}{2\pi} \cdot \Omega^t \right)^l . \tag{4.17}$$

Let $i(\partial/\partial t)$ denote the contraction operator, from forms on $B \times \mathbf{R}$ to forms on B, against the parameter vector field $\partial/\partial t$. The Cheeger-Chern-Simons form of the data (θ^1, θ) is the closed, odd-degree form on B (cf. [20])

$$\mathrm{Tch}(\theta^1, \theta) = \int_0^1 \{ i(\partial/\partial t)\, \mathrm{ch}(\Omega^t) \} \, dt . \tag{4.18}$$

Note that interchanging θ^1 and θ is equivalent to reversing the t-parameter, so that

$$\mathrm{Tch}(\theta^1, \theta) = -\mathrm{Tch}(\theta, \theta^1) . \tag{4.19}$$

For θ^{Id} the connection on $U_N \times U_N \to U_N$ obtained from the identity map $\mathrm{Id}: U_N \to U_N$, we obtain a closed form $\mathrm{Tch}(\theta^{\mathrm{Id}}, \theta)$ on U_N. The basic property of this form, proved by H. Cartan (based on a letter of Weil [16]) is that:

Proposition 4.5. *The cohomology class* $[\mathrm{Tch}(\theta^{\mathrm{Id}}, \theta)] = [\mathrm{T\tilde{ch}}]$ *in* $H^*(U_N)$. □

By the functoriality of the construction (4.18), we obtain as a corollary the identification

Corollary 4.6. *For* $u: V \to U_N$ *as above,*

(4.20) $$\mathrm{ch}^*[u] = [\mathrm{Tch}(\theta^u, \theta)]$$

in $H^*(V)$.

Proof. $\mathrm{ch}^*[u] = u^*[\mathrm{T\tilde{ch}}] = u^*[\mathrm{Tch}(\theta^{\mathrm{Id}}, \theta)] = [\mathrm{Tch}(\theta^u, \theta)]$. □

We now apply the above discussion to the particular case of the flat bundle

(4.21) $$P(\varrho \circ \alpha) = \tilde{M} \times_{\Gamma} U_N \to M$$

associated to $\varrho \circ \alpha : \Gamma \to U_N$. The G-map Θ induces a U_N-map Θ_ϱ such that the square commutes

(4.22) $$\begin{array}{ccc} P(\varrho \circ \alpha) = V \times_G U_N & \xrightarrow{\Theta_\varrho} & M \times U_N \\ \pi \downarrow & & \downarrow \pi_1 \\ V & \longrightarrow & M \end{array}$$

The product connection θ on $\tilde{M} \times U_N$ descends to a flat U_N-connection on $P(\varrho \circ \alpha)$, denoted by $\theta^{\varrho \circ \alpha}$, which pushes forward under Θ_ϱ to

(4.23) $$\theta^{\overline{\varrho \circ \alpha}} = (\Theta_\varrho)_* (\theta^{\varrho \circ \alpha})$$

on $M \times U_N$.

Define the Cheeger-Chern-Simons character for $P(\varrho \circ \alpha)$ with trivialization Θ_ϱ to be the odd-degree cohomology class

(4.24) $$\mathrm{ch}(\bar{\alpha}, \varrho) = [\mathrm{Tch}(\theta^{\overline{\varrho \circ \alpha}}, \theta)] \in H^{\mathrm{odd}}(M).$$

Theorem 4.7. *Let* V, $\mathscr{F}_\alpha$, Θ, ϱ *and* $d\mu$ *be as above, with* $\pi: V \to M$ *the fibration map. Then in* $H^*(V)$,

(4.25) $$\mathrm{ch}^*[u] \cup [d\mu] = \pi^*(\mathrm{ch}(\bar{\alpha}, \varrho)) \cup [d\mu].$$

Proof. For the adjoint $u^*: V \to U_N$ of u, we have $\mathrm{ch}^*(u) = -\mathrm{ch}(u^*)$ as $[u^*]$ is the K-theory inverse of $[u]$. For the adjoint connection

(4.26) $$\theta^{u^*} = \theta + u^{-1} \cdot du$$

associated to u^* by (4.14), we then have by (4.19) that $\mathrm{ch}^*(u)$ is represented by the differential form $\mathrm{Tch}(\theta, \theta^{u^*})$, so it will suffice to show the equality

(4.27) $$\mathrm{Tch}(\theta, \theta^{u^*}) \wedge d\mu = \pi^* \mathrm{Tch}(\theta^{\overline{\varrho \circ \alpha}}, \theta) \wedge d\mu$$

of differential forms. The second factor $d\mu$ is transverse to the leaves of $\mathscr{F}_\alpha$ in V, so we need only show that the first factors agree when restricted to leaves of $\mathscr{F}_\alpha$. Each leaf is covered by a slice of $\tilde{M} \times \{g\} \in \tilde{M} \times G$, via the quotient map identifying

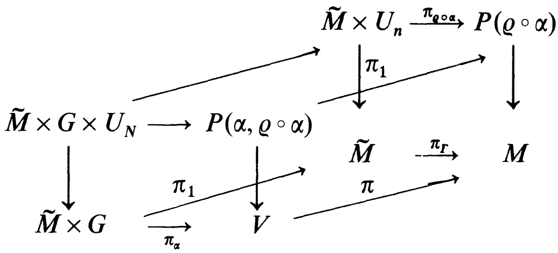
$(\tilde{x}\cdot\gamma, g)\sim(\tilde{x}, \alpha(\gamma)\cdot g)$, so it suffices to show the forms agree when pulled-back to $\tilde{M}\times\{g\}$. This will follow from identifying the connection forms appearing in (4.27).

The connection form $\theta^{\overline{\varrho\circ\alpha}}$ on $M\times U_N\to M$ lifts to the product connection on $\tilde{M}\times U_N\to\tilde{M}$ under the compositions

$$(4.28)\qquad \begin{array}{ccccc} \tilde{M}\times U_N & \xrightarrow{\pi_{\varrho\circ\alpha}} & P(\varrho\circ\alpha) & \xrightarrow{\Theta_\varrho} & M\times U_N \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & M & \longrightarrow & M \end{array}$$

where $\pi_{\varrho\circ\alpha}$ is the quotient map for the Γ action on $\tilde{M}\times U_N$. Also introduce the quotient map $\pi_\alpha:\tilde{M}\times G\to V$ and the covering map $\pi_\Gamma:\tilde{M}\to M$. We then have a commuting cube which exhibits all of the relationships of the various flat bundles under construction:

(4.29)



The flat bundle $P(\alpha,\varrho\circ\alpha)\to M$ is associated to the product representation $(\alpha,\varrho\circ\alpha):\Gamma\to G\times U_N$, and (4.29) exhibits this flat bundle as a pull-back over the map $\pi: V\to M$. We now make several deductions based on (4.29).

The bottom square of (4.29) commutes, so the lift of $\theta^{\overline{\varrho\circ\alpha}}$ to $\tilde{M}\times\{g\}$ is the lift of the product connection on $\tilde{M}$, hence is the product connection, now denoted by $\tilde{\theta}$ on $\tilde{M}\times\{g\}$.

The composition

$$(4.30)\qquad \Theta\circ\pi_\alpha:\tilde{M}\times G\to M\times G$$

is a right G-map, so there exists a smooth map $\varphi:\tilde{M}\to G$, defined by $\varphi(\tilde{x}) =\Theta\circ\pi_2(\tilde{x},e)$ where $e\in G$ is the identity. This satisfies

$$(4.31)\qquad \begin{cases} \Theta\circ\pi_2(\tilde{x},g)=(\pi_\Gamma(\tilde{x},\varphi(\tilde{x}\cdot g))) \\ \varphi(\tilde{x}\cdot\gamma) \qquad =\varphi(\tilde{x})\cdot\alpha(\gamma) \end{cases}$$

The flat bundle $P(\varrho\circ\alpha)$ is associated to $V=P(\alpha)$ by ϱ, so we similarly have that

$$(4.32)\qquad \begin{cases} \Theta_\varrho\circ\pi_{\varrho\circ\alpha}(\tilde{x},A)=(\pi_\Gamma(\tilde{x},\varrho\circ\varphi(\tilde{x})A)) \\ \varrho\circ\varphi(\tilde{x}\cdot\gamma) \qquad =\varrho\circ\varphi(\tilde{x})\cdot\varrho\circ\alpha(\gamma) \end{cases}$$

Introduce a $\mathfrak{u}(N)$-matrix-valued 1-form on $\tilde{M}$

$$(4.33)\qquad \theta_\varphi=(\varrho\circ\varphi)^{-1}\cdot d(\varrho\circ\varphi).$$

Then the product connection $\theta=A^{-1}\cdot dA$ on $M\times U_N$ pulls back to $\tilde{M}\times\{A\}$ as

$$(4.34)\qquad (\Theta_\varrho\circ\pi_{\varrho\circ\alpha})^*\,\theta=A^{-1}\cdot\theta_\varphi\cdot A\equiv\mathrm{Ad}(A)(\theta_\varphi).$$

Finally, under the diagonal map on the left face of (4.29),

$$(\tilde{x},g)\to(\tilde{x},\varrho(g)),$$

this form pulls-back to $\varrho(g)^{-1}\theta_\varphi \cdot \varrho(g)$ on $\tilde{M}\times\{g\}$. Thus, $\mathrm{Tch}(\theta^{\overline{\varrho\circ\alpha}},\theta)$ pulls-back to the form

$$\mathrm{Tch}(\tilde{\theta},\mathrm{Ad}(\varrho(g))\theta_\varphi) \quad \text{on} \quad \tilde{M}\times\{g\}.$$

The second form, $\mathrm{Tch}(\theta,\theta^{u^*})$ on V, is constructed from the product connection θ and $\theta^{u^*}=\theta+u^{-1}du$ on $V\times U_N$. The product connection pulls-back to $\tilde{\theta}$ on $\tilde{M}\times\{g\}$ under π_α.

The composition $u^{-1}\circ\pi_\alpha$ is equal

$$u^{-1}\circ\pi_\alpha(\tilde{x},g)=(\varrho\circ\pi_2\circ\Theta\circ\pi_\alpha(\tilde{x},g))^{-1} \tag{4.35}$$
$$=(\varrho\circ\varphi(\tilde{x})\cdot\varrho(g))^{-1}$$

so that θ^{u^*} pulls-back to the connection

$$\pi_2^*(\theta^{u^*})=\tilde{\theta}+(\varrho\circ\varphi)^{-1}d(\varrho\circ\varphi). \tag{4.36}$$

The restriction of $\pi_\alpha^*(\theta^{u^*})$ to $\tilde{M}\times\{g\}$ is then $\mathrm{Ad}(\underline{\varrho(g)})\theta_\varphi$, and the pull-back of $\mathrm{Tch}(\theta,\theta^{u^*})$ is the same as the pull-back of $\mathrm{Tch}(\theta^{\overline{\varrho\circ\alpha}},\theta)$. □

By combining Theorems 4.3 and 4.7, we obtain the main result of this section:

Theorem 4.8. *Let M be a compact oriented manifold of dimension $m=2l-1$, and V, $\mathcal{F}_\alpha$, Θ, ϱ and $d\mu$ as above. Then for $u=\varrho\circ\pi_2\circ\Theta$,*

$$c_\Phi(u)=(-1)^m\langle\psi^{-1}\mathrm{ch}^*(\xi)\cup\mathrm{Td}(F)\cup\pi^*(\mathrm{ch}(\bar{\alpha},\varrho))\cup[d\mu],[V]\rangle. \quad \square \tag{4.37}$$

5. The sharp transverse cocycle

Fix a representation $\alpha:\Gamma\to G$ and let $\mathcal{F}_\alpha$ and $\mathcal{F}_\pi$ be the foliations on V obtained via the suspension construction. The *sharp transverse cyclic Chern cocycle*, $c^\#$, will be defined on the smooth convolution algebra $C_c^\infty(\pi)$ using either a "sharp transverse parametrix", or a "sharp phase function". The terminology "sharp" is used to indicate that the transverse parametrices are not pseudo-differential on V. The formal procedure for constructing $c^\#$ is exactly parallel to Connes' construction of a transverse cocycle, denoted by $c^\pitchfork$, detailed in (§ 8, [30]). Both $c^\#$ and $c^\pitchfork$ yield the same cohomology class in $H_\lambda^q(C_c^\infty(\pi))$. However, $c^\pitchfork$ is constructed from a transverse parametrix that is pseudo-differential on V. This difference is reflected in the way that $c^\#$ and $c^\pitchfork$ behave as distributions on $C_c^\infty(\pi)$, so that $c^\#$ is renormalizable while $c^\pitchfork$ is not, as discussed further in Sect. 6.

Fix the following data:

(5.1) M is a compact, orientable manifold without boundary; X is a connected Riemannian manifold with bounded geometry; G is a unimodular Lie group with an inclusion $G\subset\mathrm{Isom}(X)$; $\mathcal{F}_\alpha$ is the suspension foliation for a representation $\alpha:\Gamma\to G$ on the quotient $V=\tilde{M}\times_\Gamma X$; $\mathcal{F}_\pi$ is the foliation by fibers of $\pi:V\to M$; $\Theta:P(\alpha)\to M\times G$ is a right G-isomorphism with induced fiber-preserving map $\Theta_X:V\to M\times X$; $\mathcal{F}_{\bar{\alpha}}$ is the push-forward foliation from $\mathcal{F}_\alpha$ to $M\times X$.

(5.2) $(V,\mathcal{F}_\alpha,\mathcal{F}_\pi)$ has a fixed c_1-biregular covering by cylindrical foliation charts

$$\{(U_j,\phi_j)|j\in\mathfrak{J}\}.$$

(5.3) For some $0<\delta<\frac{1}{4}c_1$ (with c_1 to be fixed later) there is an elliptic pseudo-differential operator P_M of order $p\geqq 0$ acting on sections $C^\infty(E_M)$ of $E_M\to M$, with support of P_M contained in a δ-neighborhood of the diagonal in $M\times M$. For $E=\pi^*(E_M)\to V$,

$$P\in\mathscr{D}_\psi^p(E,\mathscr{F}_\alpha,\delta,\infty)$$

is the lift of P_M to a leafwise operator along $\mathscr{F}_\alpha$.

The condition on P is equivalent to stipulating that for $f\in C^\infty(E_M)$ with support in a ball of radius less than $\frac{1}{2}\cdot c_1$, $L\subset V$ a leaf of $\mathscr{F}_\alpha$ and $f_L=f\circ(\pi|_L)$ the lift of f to L, then the restriction of P to L satisfies:

(5.4) $$P_L(f_L)=(P_M f)_L.$$

The operator P acting on $C_c^\infty(E)$ is not elliptic (for X not a point), and is not even pseudo-differential unless $P=D$ is a leafwise differential operator. We use the notation P_V to denote that P is considered to act on $C_c^\infty(E)$. Nonetheless, the integrability of $\mathscr{F}_\alpha$ allows the construction of parametrices and phase functions for P_V.

Definition 5.1.

a) A *sharp parametrix* for P is an $\mathscr{F}_\alpha$-leafwise parametrix

$$Q\in\mathscr{D}_\psi^{-p}(E,\mathscr{F}_\alpha,\delta,\infty)$$

(chosen as in Corollary 3.4) which is the lift of a parametrix Q_M for P_M, considered as an operator on $C_c^\infty(E)$.

b) A *sharp phase operator for* P is an $\mathscr{F}_\alpha$-leafwise phase

$$\Phi\in\mathscr{D}_\psi^0(E,\mathscr{F}_\alpha,\delta,\infty)$$

(chosen as in Proposition 3.5) which is the lift of a phase Φ_M for P_M, considered as an operator on $C_c^\infty(E)$.

Introduce also the notation:

P_V, Q_V and Φ_V for P, Q, Φ considered as operators on $C_c^\infty(E)$.

Π_V for the operator on $C_c^\infty(E^4)$ obtained from the involution Π as in (3.20).

$\tilde{P}_V$ and $\tilde{Q}_V$ for the operators on $C_c^\infty(E^2)$ obtained from $\tilde{P}$ and $\tilde{Q}$ as in (3.43) and (3.44).

The bundle E has a natural parallelization along the fibers of π, so we can define the action of a kernel $k\in C_c^\infty(\pi)$ on a section $f\colon V\to E$ by convolution:

(5.5) $$k*f(z)=\int_{z'\in\pi^{-1}(z)} k(z,z')f(z')\,dv_\pi(z')$$

(cf. Lemma 2, §8, [30]). The action (5.5) is extended to $C^\infty(E^2)$ and $C^\infty(E^4)$ by letting k act trivially on the second summands in $E^2\cong E\oplus E$ and $E^4\cong E\oplus E^3$. Finally, we extend these actions via linearity to the tensor products $C^\infty(E^{2N})\cong C^\infty(E^2)\otimes\mathbf{C}^N$ and $C^\infty(E^{4N})\cong C^\infty(E^4)\otimes\mathbf{C}^N$.

The following result for sharp parametrices corresponds to (Lemma 3, §8, [30]) where it was proven for pseudo-differential (on V) parametrices.

Proposition 5.2. *If $k \in C_c^\infty(\pi)$, then the operators*

a) $[\Pi_V, k]$ *on* $C_c^\infty(E^4)$

b) $\bar{Q}_V[\tilde{P}_V, k]$ *on* $C_c^\infty(E^2)$

are in the Schatten $(m+1)$*-class.*

Proof. We give the details for a) and indicate the modifications necessary for b). It suffices to consider basic kernels k which have support in U_γ for which both the domain $\phi_0 : U_0 \to I^m \times I^n$ and range $\phi_1 : U_1 \to I^m \times I^n$ are cylindrical sets of diameter less that $(4m+4n)^{-1} c_1$, with a common base $\pi(U_0) = \pi(U_1)$ in M. There is then a coordinate chart (U, ϕ) on M containing the δ-neighborhood of $\pi(U_0)$. Use (U, ϕ) to define a cylindrical chart, U_γ, on V that contains the supports of $\Pi_V \circ k$ and $k \circ \Pi_V$. By (5.4), Π_V in local coordinates (U_i, ϕ_i), $i=0,1$, is given by operators on plaque sections $C^\infty(I^m \times \{y\} \times \mathbf{C}^{4k})$, denoted by Π_y. (However, note that Π_y is independent of y and i.) Let $k : I^m \times I^n \times I^n \to \mathbf{C}$ also denote the coordinate expression for k.

The m-torus T^m is obtained from I^m by identifying opposite sides, and similarly T^n from I^n. As k is compactly supported, it admits a smooth extension to the product $T^m \times T^m \times T^n$. Similarly, the operators $\Pi_V \circ k$ and $k \circ \Pi_V$ extend to operators on $C^\infty(T^m \times T^n \times \mathbf{C}^{4k})$, and it will suffice to show that $[\Pi_V, k]$ is in the Schatten $(m+1)$-class on the Hilbert space closure

$$\mathscr{H}_T = L^2(T^{m+n} \times \mathbf{C}^{4k}).$$

We follow the usual approach of estimating the trace of a power of the self-adjoint operator $[\Pi_V, k]^*[\Pi_V, k]$ by introducing an orthonormal basis of $\mathscr{H}_T$, then calculating the appropriate sum of powers of the norms of $[\Pi_V, k]$ applied to the vectors in the fixed basis.

For $I = (i_1, \dots, i_m) \in \mathbf{Z}^m$, let

$$\phi_I(x) = \exp(\pi i (I \cdot x)) \tag{5.6}$$

be the character on T^m corresponding to I via the identification $\hat{T}^m \cong \mathbf{Z}^m$. Similarly, for $J = (j_1, \dots, j_n) \in \mathbf{Z}^n$, let

$$\psi_J(y) = \exp(\pi i (J \cdot y)) \tag{5.7}$$

be the character on T^n for $J \in \mathbf{Z}^n \cong \hat{T}^n$.

The extension of k to $T^m \times T^n \times T^n$ is smooth so admits a Fourier expansion

$$k(x, y, y') = \sum a_{IJJ'} \phi_I(x) \psi_J(y) \overline{\psi_{J'}(y')} \tag{5.8}$$

where the norms $|a_{IJJ'}|$ are rapidly decreasing with respect to the index

$$\|I, J, J'\|^2 = \sum_{l=1}^{m} (i_l)^2 + \sum_{l=1}^{n} (j_l)^2 + \sum_{l=1}^{n} (j_l')^2 . \tag{5.9}$$

A function $\phi \in C^\infty(T^m)$ defines a multiplication operator on $\mathscr{H}_T$ by first lifting ϕ to $T^m \times T^n$ by declaring it constant along T^n, and then letting ϕ act on sections via the module structure (3.18) extended to $\mathbf{C}^{4k}$.

Let $\lambda : I^m \to [0, 1]$ be a smooth function with compact support that is identically 1 on a δ-neighborhood of the I^m-support of k, so that

$$k(x, y, y') \lambda(x) \equiv k(x, y, y') \forall x \in I^m .$$

Let $\tilde{\Pi}_y = \lambda \cdot \Pi_y \cdot \lambda$ be the compression of the plaquewise-operator Π_y where now we let λ act diagonally on $\mathbf{C}^{4k}$. Then $\tilde{\Pi}_y$ extends to an order zero pseudo-differential operator $\tilde{\Pi}_V$ on $C^\infty(T^m \times T^n \times \mathbf{C}^{4k})$ so that

$$k \circ \tilde{\Pi}_V = k \circ \tilde{\Pi}_V. \tag{5.10}$$

For each J and $J' \in \mathbf{Z}^n$, introduce the rank-one operator on $C^\infty(T^n)$ with kernel

$$k_{JJ'}(y, y') = \psi_J(y)\overline{\psi_{J'}(y')}. \tag{5.11}$$

These operators extend to $\mathscr{H}_T$ via the product structure on T^{m+n} and the module action (3.18) on $\mathbf{C}^{4k}$.

Lemma 5.3. *On* $\mathscr{H}_T$

$$[\tilde{\Pi}_V, k] = \sum a_{IJJ'}[\tilde{\Pi}_y, \phi_I] \circ k_{JJ'}. \tag{5.12}$$

Proof. The restrictions Π_y commute with the operators $k_{JJ'}$, so (5.12) follows from (5.8) and (5.10). □

From (5.12) we estimate the Schatten $(m+1)$-norm of $[\Pi_V, k]$ by

$$\| [\tilde{\Pi}_V, k] \|_{m+1}^2 \leqq \sum |a_{IJJ'}|^2 \quad \| [\tilde{\Pi}_y, \phi_I] \|_{m+1}^2. \tag{5.13}$$

Since the $|a_{IJJ'}|$ are rapidly decreasing, the proof of a) follows from:

Lemma 5.4. *There is a constant $c_2 > 0$ depending only on Π, the chart U_γ and λ such that*

$$\| [\tilde{\Pi}_y, \phi_I] \|_{m+1} \leqq c_2(1 + \|I\|), \quad \textit{where} \quad \|I\|^2 = \sum_{l=1}^{m} (i_l)^2. \tag{5.14}$$

Proof. It will suffice to prove (5.14) in a coordinate chart on T^m for the compression of $\tilde{\Pi}_y$ and ϕ_I by a smooth function compactly supported in the coordinate neighborhood. Let $p(x, \xi) \in S^0_{1,0}(\mathbf{R}^m, \mathbf{C}^{4k})$ be a symbol of order 0 which is compactly supported in x such that the ψDO, $p(x, D)$, on $C^\infty(\mathbf{R}^m, \mathbf{C}^{4k})$ represents the compression of $\tilde{\Pi}_y$. (We use the notation and results of basic ψDO theory, as given in Chap. 2, [87].) Let $\phi \in C_c^\infty(\mathbf{R}^m)$ be a test function for which we must estimate the $(m+1)$-Schatten norm of $[p(x, D), \phi]$. This commutator is again a ψDO of order -1, with symbol denoted $q(x, \xi) \in S^{-1}_{1,0}(\mathbf{R}^m)$. If $c > 0$ is a constant such that

$$|q(x, \xi)| \leqq c \cdot (1 + \|\xi\|)^{-1} \ \forall (x, \xi) \in T^*\mathbf{R}^m, \tag{5.15}$$

then a well-known consequence of the Sobolev Lemma is that the Schatten $(m+1)$-norm of $q(x, D)$ is estimated by a fixed multiple of c.

To obtain an estimate (5.15), observe from direct calculation that

$$\begin{aligned} q(x, \xi) &= \int_{\mathbf{R}^m} p(x, z+\xi) e^{ix \cdot z} \hat{\phi}(z) dz - \phi(x) p(x, \xi) \\ &= \int_{\mathbf{R}^m} \{p(x, z+\xi) - p(x, \xi)\} e^{ix \cdot z} \hat{\phi}(z) dz \end{aligned} \tag{5.16}$$

where we use that $p(x, \xi)$ and $\phi(x)$ commute, and

$$\hat{\phi}(z) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbf{R}^m} e^{-iz \cdot w} \phi(w) dw \tag{5.17}$$

is the Fourier transform of ϕ. The symbol $p(x, \xi)$ has a partial asymptotic expansion

$$p(x, \xi) = p_0(x, \xi) + p_-(x, \xi) \tag{5.18}$$

where $p_0(x, \xi)$ is homogeneous of order 0, and $p_-(x, \xi)$ has order -1. Using the decomposition (5.18) and the differentiability of $p_0(x, \xi)$, there is a constant c' so that

$$|p(x, z+\xi) - p(x, \xi)| \leqq c' \cdot \frac{1+\|z\|}{1+\|\xi\|}. \tag{5.19}$$

Combining (5.16) and (5.19) we obtain

$$|q(x, \xi)| \leqq \frac{c'}{1+\|\xi\|} \int (1+\|z\|) \cdot |\hat{\phi}(z)| dz. \tag{5.20}$$

For ϕ the compression of ϕ_I to a coordinate chart, elementary estimates yield a constant c'' such that

$$\int (1+\|z\|) |\hat{\phi}(z)| dz \leqq c''(1+\|I\|)$$

so that we can take $c = c' \cdot c''(1+\|I\|)$ in (5.15). □

The proof of b) reduces as above to estimating the $(m+1)$-Schatten norm of $\tilde{Q}_V[\tilde{P}_V, k]$ supported in a cylindrical foliation chart. This, in turn, follows from a uniform estimate along plaques of the type

$$\|\tilde{Q}_y[\tilde{P}_y, \phi_I]\|_{m+1} \leqq c_3 \cdot (1+\|I\|), \tag{5.21}$$

which is proved by the same methods as used for Lemma 5.4. □

The reader familiar with the proof by Connes of (Lemma 3, § 8, [30]) will recognize that the above proof follows the same outline, but the technical details differ as $\tilde{\Pi}_V$ and $\tilde{Q}_V$ are not ψDO's necessitating the use of Lemma 5.4.

For odd $q \geqq m$, by Lemma 5.2 we can define for any $k_0, \ldots, k_q \in C_c^\infty(\pi)$:

$$\begin{cases} c^\#(k_0, \ldots, k_q) = \\ -(-4)^{-\left(\frac{q+1}{2}\right)} \mathrm{Tr}_{\mathscr{H}}\{\Pi_V[\Pi_V, k_0] \ldots [\Pi_V, k_q]\} \end{cases} \tag{5.22}$$

Proposition 5.5. *The $(q+1)$-multilinear functional $c^\#$ is a cyclic q-cocycle over the convolution algebra $C_c^\infty(\pi)$.*

Proof. $\Pi_V^2 = \mathrm{Id}$ on $C^\infty(E^4)$, and by Lemma 5.2 the trace law holds,

$$\mathrm{Tr}_{\mathscr{H}}(S \circ T) = \mathrm{Tr}_{\mathscr{H}}(T \circ S),$$

where S is a product of l-commutators, and T is a product of $(q+1-l)$-commutators. Therefore, the algebraic methods which prove Proposition 3.9 also establish that $c^\#$ is cyclic and closed. □

For $q = 2l \geqq m$, given a graded operator (P, ε) we define a cocycle for $k_0, \ldots, k_q \in C_c^\infty(\pi)$:

$$c^\#(k_0, \ldots, k_q) = -(2\pi i)^l l! \cdot \mathrm{Tr}_{\mathscr{H}}\{\tilde{\varepsilon} \cdot \tilde{Q}_V[\tilde{P}_V, k_0] \ldots \tilde{Q}_V[\tilde{P}_V, k_q]\}. \tag{5.23}$$

We next consider the relation between the cocycle $c^\#$ and the cocycle $c^\pitchfork$ of Connes. Recall that $c^\pitchfork$ is defined using the same formulas as $c^\#$, but Q_V and Π_V are required to be ψDO's on V whose symbols are specified for $\xi \in T^* V$ vanishing on $T\pi \subset TM$ as discussed below. There is a natural identification $\pi^* : T^*M \to T^*(\pi)$ of the cotangent bundle to M with the sub-bundle of cotangent vectors on V which vanish along the fibers of π. For the principal symbol σ_p on $T^* V$

$$\sigma_P(\pi^*(\xi)) = \sigma_{P_M}(\xi), \quad \xi \in T^*M. \tag{5.24}$$

A ψDO on V is transversally elliptic for $\mathcal{F}_\pi$ if its symbol is invertible for all $0 \neq \xi \in T^*(\pi)$ (cf. [1], [81], [30]). If P_M is a diferential operator on M, then P will be a ψDO on V, and (5.24) implies that P is transversally elliptic to $\mathcal{F}_\pi$.

For (P, ε) a graded operator, Connes observes in (§ 8, [30]) that there is a graded parametrix (Q, ε) which is ψDO of negative order on V, and

$$\sigma_Q(\xi) \cdot \sigma_P(\xi) = \mathrm{Id}, \quad \xi \in T^*(\pi). \tag{5.25}$$

Similarly, for ungraded P there is a phase ψDO, Φ, on V of order 0. The even cocycle c^{Φ} of Connes is obtained by using this Φ in place of a sharp parametrix in formula (5.23), and one can similarly define an odd cocycle c^{Φ} by using the ψDO, Φ, in the construction of $c^{\#}$. The proof of the following was communicated to us by J. Roe [77].

Proposition 5.7. *For $q > \dim V$, the cyclic cohomology classes of $c^{\#}$ and c^{Φ} agree in $H^q_\lambda(C^\infty_c(\pi))$.*

Proof. We prove only the odd case, and leave the even case to the reader. Introduce a family $\{S_t | 0 < t \leqq 1\}$ of smoothing operators on $L^2(X)$ which satisfy

(5.26) S_t is represented by a smooth kernel k_t on $X \times X$ with support contained in a t-neighborhood of the diagonal.

(5.27) S_t converges weakly to the identity as $t \to 0$.

(5.28) k_t is invariant under the left G-action on $X \times X$.

Each operator S_t extends to an operator S^*_t on the product space $L^2(U_0 \times X)$, for $U_0 \subset M$, via the product structure. As V is obtained by identifying product sets $U_0 \times X$, via the action of G on X associated to the structure cocycle of $P(\alpha) \to M$, the operators S^*_t agree on overlaps using (5.28), so define a global operator S^*_t on $L^2(V)$. We extend S^*_t to also act on $L^2(E^{4n})$ via the diagonal action on fibers.

The operators Π and S^*_t commute, for in a cylindrical coordinate system Π restricts to plaques of $\mathcal{F}_\alpha$ and is constant in the transverse parameter; similarly S^*_t restricts to plaques of $\mathcal{F}_\pi$ and is constant in the transverse parameter. Thus, they totally decouple in coordinates. A second consequence of decoupling is that the product $\Pi_t = \Pi \circ S_t$ is a ψDO on V of order 0. Moreover,

$$\sigma_{\Pi_t}(\xi) = \sigma_\Pi(\xi), \quad \xi \in T^*(\pi) \tag{5.29}$$

so that we can construct a cocycle c^{Φ}_t using $\tilde{\Pi}_t$, and this represents the odd degree analogue of the Connes transverse cocycle.

The assumption $q > \dim V$ implies that $[c^{\#}]$ and $[c^{\Phi}_t]$ are stabilized classes, so are determined by their pairings with cyclic homology. Consequently it suffices to show that $c^{\#}(u) = c^{\Phi}(u)$ for a unitary $u \in M(N, C^\infty_c(\pi))$. The algebra $C^\infty_c(\pi)$ consists of leafwise smoothing operators with compact support. Therefore, we can apply (5.27) and the dominated convergence theorem to deduce that

(5.30) $$c^{\#}(u) = \lim_{t\to 0} c_t^{\dagger}(u).$$

Since $c_t^{\dagger}(u)$ is independent of t for u a unitary, this proves the proposition. □

The algebra $C_c^\infty(\pi)$ is Morita equivalent to the commutative algebra $C^\infty(M)$. An operator (P_M, ε) on $C^\infty(E_M)$ determines an even degree Chern character $\operatorname{ch}(P_M, \varepsilon) \in H_\lambda^q(C^\infty(M))$, and one can show using (5.4) that this class corresponds to $[c^{\#}]$ under the isomorphism

(5.31) $$H_\lambda^q(C_c^\infty(\pi)) \cong H_\lambda^q(C^\infty(M))$$

induced by Morita equivalence. Since $\operatorname{ch}(P_M, \varepsilon)$ is independent of $\mathscr{F}_\alpha$, it follows that $[c^{\#}]$ yields no information about $\mathscr{F}_\alpha$.

For an ungraded operator P_M on $C^\infty(E_M)$, similar remarks apply to the odd degree cyclic cohomology classes $[c^{\#}]$ and $\operatorname{ch}(P_M)$, so that the *cohomology class* of $c^{\#}$ in $C_c^\infty(\pi)$ is independent of α.

6. Renormalization and transfer

In this section we introduce a renormalization procedure for the transverse cocycle $c^{\#}$ constructed in Sect. 5, which produces a cyclic cocycle of the same degree over the commutative algebra $C_c^\infty(V)$. Renormalization is a type of transfer process, corresponding to lifting a de Rham current from the quotient $M = V/\mathscr{F}_\pi$ up to V. The basic idea is to introduce a family of approximate units $\{k_l \mid l = 1, 2, \ldots\} \subset C_c^\infty(\pi)$ which satisfy the Fölner condition (6.5) of Definition 6.1. We use the left-module action of $C_c^\infty(V)$ on $C_c^\infty(\pi)$ to define a sequence of embeddings $\tau_l : C_c^\infty(V) \to C_c^\infty(\pi)$ via the $\{k_l\}$. The maps τ_l are not algebra morphisms, but the Fölner condition guarantees that they are asymptotically so with respect to the relevant Schatten m-norms. Thus, the cocycle $c^{\#}$ pulls-back via τ_l to cyclic cochains $\hat{c}_l$, which after renormalization, converge in the limit $l \to \infty$ to the *renormalized cocycle* $\hat{c}$. The final result of this section is that by suitably choosing the $\{k_l\}$, the limit $\hat{c}$ is precisely the longitudinal cocycle of Sect. 3.

We again fix $(V, \mathscr{F}_\alpha, \Theta, P)$ satisfying (5.1) to (5.3). On the fiber X, we specify a class of approximate identities that are suitable for renormalization.

Definition 6.1. A sequence of smooth symmetric real-valued kernels $\{k_l \mid l = 1, 2, \ldots\}$ on X is said to be an FAI (*fancy approximate identity*) if it satisfies:

(6.1) $$k_l(x, y) = 0 \quad \text{if} \quad \operatorname{dist}_X(x, y) > \varepsilon_l \quad \text{where} \quad \lim_{l\to\infty} \varepsilon_l = 0.$$

(6.2) $k_l * f$ converges C^0-uniformly to f for each $f \in C_c^\infty(X)$, where a convolution operator is defined on $C_c^\infty(X)$ by

$$k_l * f(x) = \int_X k_l(x, y) f(y) dv_X(y)).$$

(6.3) $$\|k_l\| \leq c\{k_l\}, \quad l > 0, \quad \text{for some fixed constant} \quad c\{k_l\}.$$

Let $\|A\|_p$ denote the Schatten p-norm of a linear operator A on $L^2(X)$.

(6.4) For $\lambda \in C_c^0(X)$ and integers $l, l_1, \ldots, l_r, s_1, \ldots, s_r > 0$ such that

$$(s_1 l_1)^{-1} + \ldots + (s_r l_r)^{-1} = l^{-1},$$

then as l tends to infinity,

$$\|\lambda(k_{s_1 l_1} * \ldots * k_{s_r l_r} - k_l)\|_1 = o(\|\lambda \cdot k_l\|_1).$$

(6.5) For $f \in C_c^1(X)$ and any non-negative function $\lambda \in C_c^0(X)$ which is identically 1 on the support of f,

$$\|[f, k_l]\|_1 = o(\|\lambda \cdot k_l\|_1) \quad \text{as} \quad l \to \infty.$$

(6.6) For all $g \in G$ and $x, y \in X$,

$$k_l(gx, gy) = k_l(x, y).$$

(6.7) There exists a sequence of positive constants $\{d_l\}$ such that

$$|k_l(x, x) - d_l| = o(d_l) \quad \text{as} \quad l \to \infty$$

uniformly in $x \in X$. □

We remark on the interpretations and applications of these conditions. (6.1) and (6.2) specify that the convolution operators $\{k_l\}$ form an approximate identity on both $C_c^0(X)$ and $L^2(X)$.

(6.3), (6.4) and (6.5) regulate the convergence of the $\{k_l\}$ to the identity in the Schatten 1-norm. The condition (6.4) is a weak type of semi-group property for the sequence, while (6.5) is called the Fölner condition, a notation which will be further justified in Sect. 8 below.

(6.6) is the condition that the kernels are G-invariant, and hence define elements of $C_c^\infty(\pi)$ via suspension.

(6.7) is a uniformity condition to ensure that the diagonal asymptotics of the k_l are uniform in X. If there exists a dense (or better, transitive) orbit of G on X, then (6.7) is a consequence of (6.6).

Proposition 6.2. *Let X be a connected Riemannian manifold with bounded geometry (and injectivity radius greater than $2c_0 > 0$.) If $G \subset \mathrm{Isom}(X)$, then there exists an* FAI *for X.*

Proof. Let Δ be the Laplacian acting on smooth functions on X, and $\exp(-t\Delta)$ the heat operator for Δ. We use the following properties of this family:

(6.8) $\exp(-t\Delta)$ has norm one on $L^2(X)$, and converges weakly to the identity.

(6.9) There is a continuous family $\{h_t \mid t > 0\}$ of smooth, symmetric real-valued kernels on X which represent the heat operator.

(6.10) Isometry invariance of Δ implies $h_t(gx, gy) = h_t(x, y)$ for all $g \in G$, $x, y \in X$.

The bounded geometry and positive injectivity radius of X imply that there exists constants c_4, $c_5 > 0$ so that for all $t > 0$, $x, y \in X$

$$0 < h_t(x, y) \leqq c_4 \cdot t^{-n/2} \cdot \exp\left\{-\frac{c_5 \cdot r(x, y)^2}{t}\right\} \tag{6.11}$$

where $r: X \times X \to [0, \infty)$ is the Riemannian distance function. A proof of (6.11) can be found in [21]. A stronger version of (6.11) can be proven, estimating the covariant derivatives of the kernels $\{h_t\}$. Corollary 8 of [21] specializes in our context to yield: Let ∇ denote the covariant differentiation operator on functions. Then for all $l>0$, there exists a constant $c(l)>0$ so that for all l-tuples of unit vectors $v_1,\dots,v_l\in T_xT$,

$$|\nabla_{v_1}\dots\nabla_{v_l}h_t(x,y)|\leqq c(l)t^{-(n+l)/2}\exp\left\{-\frac{c_5\cdot r(x,y)^2}{t}\right\}. \tag{6.12}$$

From (6.11) we see that the kernels $\{h_t\}$ are not compactly supported. We propose to take for an FAI a sequence of appropriately cut-off heat kernels. Choose a smooth function $\lambda:[0,\infty)\to[0,1]$ such that

$$\lambda(s)=\begin{cases}1 & \text{if } s\leqq 1\\ 0 & \text{if } s\geqq 2.\end{cases} \tag{6.13}$$

We can assume without loss that $1>c_0>0$. Then set

$$k_l(x,y)=\lambda(l^{1/4}\cdot r(x,y))\cdot h_{1/l}(x,y). \tag{6.14}$$

We will prove that the collection $\{k_l|l=1,2,\dots\}$ is an FAI for X. Note that property (6.1) for $\varepsilon_l=2(l^{-1/4})$ and (6.6) are immediate.

The operators $\exp(-t\Delta)$ for $t>0$ all have norm 1, so from the inequality $0\leqq k_l(x,y)\leqq h_{1/l}(x,y)$ we obtain (6.3) with $c(\{k_l\})=1$.

The estimates (6.2), (6.4) and (6.5) follow from the basic

Lemma 6.3. *For all* $x\in X$,

$$\lim_{t\to 0}\int_X h_t(x,y)dv_X(y)=1. \tag{6.15}$$

Proof. For each test function $f\in C_c(X)$, $\lim_{t\to 0}h_t*f=f$ uniformly on X. The manifold X has bounded geometry, so there exists a constant, c_X, so that for all $x\in X$

$$\mathrm{vol}_X B(x,r)\leqq\exp\{c_X\cdot r\}. \tag{6.16}$$

Fix x, and introduce functions

$$f_s(y)=\lambda\left(\frac{1}{s}\cdot r(x,y)\right),\qquad s>0.$$

For fixed s, we obtain

$$\begin{aligned}1&=\lim_{t\to 0}\int_X h_t(x,y)f_s(y)d_X(y)\\ &=\lim_{t\to 0}\int_X h_t(x,y)d_X(y)\\ &\quad+\lim_{t\to 0}\int_{X-B(x,s)}\{f_s(y)-1\}\cdot h_t(x,y)d_X(y).\end{aligned}$$

The absolute value of the integrand of the last term has an upper estimate, by combining (6.11) and (6.16), of

$$c_y\cdot\int_s^\infty t^{-n/2}\exp\left\{c_X\cdot r-\frac{c_5\cdot r^2}{t}\right\}dr$$

which tends to zero with t. □

Condition (6.2) now follows from (6.15) and (6.11), using the uniform continuity of $f \in C_c(X)$. Details are left to the reader.

Condition (6.5) is based on a simple local estimate. For $f \in C_c(X)$ and $a > 0$, we say that f is a-Hölder if there exists a constant $c_a(f) > 0$ with

$$(6.17) \qquad |f(x) - f(y)| \leqq c_a(f) \cdot r(x, y)^a; \qquad x, y \in X.$$

The absolute value of the skew-symmetric operator $[f, k_l]$ is bounded above by the positive operator with kernel $|f(x) - f(y)| k_l(x, y)$. If f is a-Hölder, then this in turn is dominated by the symmetric operator

$$c_a(f)\left(\frac{2}{l} \cdot c_0\right)^a k_l(x, y).$$

From this it follows that

$$(6.18) \qquad \begin{aligned} \|[f, k_l]\|_1 &\leqq c_a(f)\left(\frac{2}{l^{1/4}}\right)^a \cdot \int_{\operatorname{spt}(f)} k_l(y, y)\, dv_X(y) \\ &\leqq c_a(f)\left(\frac{2}{l^{1/4}}\right)^a \cdot \|\lambda \cdot k_l\|_1, \end{aligned}$$

where λ_1 is a non-negative test function as in (6.5), which yields (6.5).

The Weyl asymptotic theorem for compact manifolds has a local version applicable to open manifolds (cf. Chap. 5, [78]) which states

$$(6.19) \qquad h_t(x, x) \sim (4\pi t)^{-n/2}.$$

Setting $d_l = \left(\frac{4\pi}{l}\right)^{-n/2}$ we obtain (6.7).

The last property to establish, (6.4), follows from (6.7), (6.11) and the semi-group property $h_t * h_s = h_{t+s}$, which implies that for $l, l_1, \ldots, l_r, s_1, \ldots, s_r$ as given in (6.4), then

$$\|f \cdot h_{\frac{1}{s_1 l_1}} * \ldots * h_{\frac{1}{s_r l_r}} - f \cdot h_{1/l}\|_1 = 0.$$

To establish (6.4), it will thus suffice to show that for integers $l > 0$,

$$(6.20) \qquad \|f \cdot (h_{1/l} - k_l)\|_1 = o(1) \quad \text{as} \quad l \to \infty,$$

since $\|h_t\| = 1$ and $\|k_l\| \leqq 1$. But $(h_{1/l} - k_l)$ is a kernel which vanishes for $r(x, y) \leqq c_0 \cdot l^{-1/4}$, and otherwise satisfies

$$|h_{1/l}(x, y) - k_l(x, y)| \leqq c_r \cdot l^{n/2} \cdot \exp\{-c_5 \cdot l \cdot d(x, y)^2\}$$

so that (6.20) follows from (6.16). This completes the proof of Proposition 6.2. □

Assume that an FAI, $\{k_l\}$, for X is given, and let $\mathfrak{k}_l$ denote the suspension of k_l to an operator along the leaves of $\mathscr{F}_\pi$. Introduce the algebra $\mathscr{P}\{\mathfrak{k}_l\}$ consisting of polynomials in the operators $\{\mathfrak{k}_l\}$ with constant real-valued coefficients. Given a function $f \in C_c^\infty(V)$ and $\mathfrak{p} \in \mathscr{P}\{\mathfrak{k}_l\}$, the product $f \cdot \mathfrak{p} \in C_c^\infty(\pi)$ since f has compact support in V and the $\{k_l\}$ are supported in a uniform neighborhood of the diagonal. Also note that the elements of $\mathscr{P}\{\mathfrak{k}_l\}$ acts as multipliers on the algebra $C_c^\infty(\pi)$.

Each $\mathfrak{p} \in \mathscr{P}\{\mathfrak{k}_l\}$ is a finite sum of products of the $\mathfrak{k}_l$, so $\mathfrak{p}$ is represented by a kernel on V with support contained in a uniform neighborhood of the diagonal. Let $r(\mathfrak{p})$

denote the least $r>0$ so that the support of the kernel $\mathfrak{p}(x,y)$ of $\mathfrak{p}$ is contained in the r-uniform tube around the diagonal.

Introduce the operator $|\mathfrak{p}|$ in $C_c^0(\pi)$ represented by the kernel $|\mathfrak{p}(x,y)|$ which is pointwise the absolute value of the kernel for $\mathfrak{p}$.

For each integer $p>0$, introduce the uniform norm for order p-derivatives on $C_c^\infty(V)$, given by

$$\|f\|_{p,\infty}=\sup_{x\in V}|f(x)|+\sup_{\substack{v\in TV\\ \|v\|=1}}|\nabla_v(f)|+\dots + \sup_{\substack{v_1,\dots,v_p\in T_xV\\ \|v_i\|=1}}|\nabla_{v_1}\circ\nabla_{v_2}\circ\dots\circ\nabla_{v_p}(f)| \tag{6.21}$$

Let $\xi\in C_\lambda^q(C_c^\infty(\pi))$ be a cyclic q-cochain. Recall this assumes that ξ is a continuous functional for the Frechet topology on $C_c^\infty(\pi)$. We will assume that the Lebesgue number c_1 of (5.2) satisfies $0<c_1<1$.

Definition 6.4. A q-cochain ξ is *normal* with respect to an F.A.I. $\{k_l\}$ if it satisfies:

(6.22) There exist constants $c(\xi)$ and $c_6>0$ so that for all $f_0,\dots,f_q\in C_c^\infty(V)$ and

$$\mathfrak{p}_0,\dots,\mathfrak{p}_q\in\mathscr{P}\{\mathfrak{k}_l\}\quad\text{with}\quad r(\mathfrak{p}_i)<c_6,$$

then for any non-negative function $\lambda\in C_c^\infty(V)$ which is identically 1 on a $\frac{1}{4}c_1$-neighborhood of a compact set $K\subset V$ containing the support of all of the f_i, it follows that

$$|\xi(f_0\mathfrak{p}_0,\dots,f_q\mathfrak{p}_q)|\leqq c(\xi)\cdot\|f\|_{2,\infty}\cdot\|f_q\|_{2,\infty}\cdot\mathrm{Tr}\{\lambda\cdot|\mathfrak{p}_0|*\dots*|\mathfrak{p}_q|\}.$$

For all $g_0,\dots,g_p\in C_c^\infty(\pi)$

$$|\xi(g_0\mathfrak{k}_l,g_1,\dots,g_q)-\xi(g_0,\mathfrak{k}_lg_1,\dots,g_q)|=o(d_l) \tag{6.23}$$

and for all $f\in C_c^\infty(V)$,

$$|\xi([f,\mathfrak{k}_l]g_0,g_1,\dots,g_q)|=\|f\|_{2,\infty}\cdot o(d_l) \tag{6.24}$$

uniformly in f. □

The left action of $C_c^\infty(V)$ on $C^\infty(\pi)$ yields a continuous linear map:

$$\begin{aligned}\tau_l: C_c^\infty(V)&\to C_c^\infty(\pi)\\ f\quad&\to f\cdot\mathfrak{k}_l\end{aligned} \tag{6.25}$$

A cyclic cochain ξ on $C_c^\infty(\pi)$ lifts via τ_l to a cyclic cochain denoted ξ_l. This lifting does not, in general, preserve cocycles, but asymptotically this will be the case if ξ is normal.

Theorem 6.5. *Let ξ be a normal cyclic q-cocycle over $C_c^\infty(\pi)$. Then the cochains $\{\xi_l|l=1,2,\dots\}$ have a rescaling which admit weak-$*$ limits which are cyclic q-cocycles over $C_c^\infty(V)$.*

Remark 6.6. We call $\hat{\xi}$ the renormalization (with respect to $\{k_l\}$) of ξ. The ambiguity of the rescaling process can be removed for X compact and ξ Morita equivalent to a cocycle ξ_M over $C_c^\infty(M)$: we require that the push-down $\pi_*(\xi)$ be cohomologous to ξ_M, where $\pi^*: C_c^\infty(M) \to C^\infty(V)$ induces the push-down map π_* on cyclic cocycles. This property is reminiscent of the *transfer* in algebaric topology.

Proof. Define a sequence of cyclic cochains

$$\psi_s = d_s^{-1} \cdot \xi_{(q+1)\cdot s}, \qquad s = 1, 2, \dots$$

For $K \subset V$ a compact smooth submanifold, let $C_K^\infty(V)$ denote the algebra of smooth functions with support in K. By (6.22) and (6.7), the family $\{\psi_s\}$ is uniformly bounded with respect to the uniform C^2-norm on $C_K^\infty(V)$. The key point of the proof of the theorem is to obtain a similar conclusion for the coboundaries.

Lemma 6.7. *The family of coboundaries $\{b\psi_s\}$ is uniformly bounded with respect to the uniform C^2-norm on $C_K^\infty(V)$, for each compact K.*

Proof. Let $f_0, \dots, f_{q+1} \in C_c^\infty(V)$ have support in K. Choose a non-negative $\lambda \in C_c^\infty(V)$ which is identically 1 on K, and set

$$l = (q+1)s; \quad p = (q+2)s. \tag{6.26}$$

We will show that

$$|b\xi_l(f_0, \dots, f_{q+1})| = \{\|f_0\|_{2,\infty} \cdot \dots \cdot \|f_{q+1}\|_{2,\infty}\} \cdot o(d_l) \tag{6.27}$$

uniformly in $f_0, \dots, f_{q+1}$, so that the lemma follows by rescaling (6.27) with the factor d_s^{-1}.

The idea of the proof of (6.27) is to observe that $b\xi = 0$ implies $\widehat{(b\xi)}_p = 0$, so it suffices to estimate the difference $b(\xi_l) - \widehat{(b\xi)}_p$. From (3.28) and using the cyclic property of ξ, each term of this difference is typically given by

$$\begin{aligned}
&\xi(f_0 f_1 \mathfrak{k}_l, f_2 \mathfrak{k}_l, \dots, f_{q+1}\mathfrak{k}_l) \\
&\quad - \xi(f_0 \mathfrak{k}_p * f_1 \mathfrak{k}_p, f_2 \mathfrak{k}_p, \dots, f_{q+1}\mathfrak{k}_p) \\
&\quad = \xi(f_0 f_1 \mathfrak{k}_l, f_2 \mathfrak{k}_l, \dots, f_{q+1}\mathfrak{k}_l) \\
&\quad - \xi(\mathfrak{k}_p * f_0 f_1 \mathfrak{k}_p, f_2 \mathfrak{k}_p, \dots, f_{q+1}\mathfrak{k}_p) \\
&\quad + \|f_0\|_{2,\infty} \cdot o(d_l).
\end{aligned} \tag{6.28}$$

The difference $\mathfrak{k}_p - (\mathfrak{k}_{(q+1)p})^{q+1} \in \mathscr{P}\{\mathfrak{k}_l\}$ and by (6.4) there is the estimate

$$\mathrm{Tr}_\pi\{\lambda |\mathfrak{k}_p - (\mathfrak{k}_{(q+1)p})^{q+1}|\} = o(d_l).$$

By (6.22) we can therefore replace the leading term $\mathfrak{k}_p$ in (6.28) with $(\mathfrak{k}_{(q+1)p})^{q+1}$ and introduce an error dominated by

$$\begin{aligned}
&\|f_0 f_1\|_{2,\infty} \cdots \|f_{q+1}\|_{2,\infty} \cdot o(d_l) \\
&\quad \leqq \{\|f_0\|_{2,\infty} \cdot \dots \cdot \|f_{q+1}\|_{2,\infty}\} \cdot o(d_l).
\end{aligned}$$

Then using the cyclic property of ξ and repeated applications of (6.23) and (6.24) we arrive at the difference

$$\begin{aligned}
&\xi(f_0 f_1 \mathfrak{k}_l, f_2 \mathfrak{k}_l, \dots, f_{q+1}\mathfrak{k}_l) \\
&\quad - \xi(f_0 f_1 \mathfrak{k}_p * \mathfrak{k}_{(q+1)p}, f_2 \mathfrak{k}_p * \mathfrak{k}_{(q+1)p}, \dots, f_{q+1}\mathfrak{k}_p * \mathfrak{k}_{(q+1)p}).
\end{aligned} \tag{6.29}$$

Finally, observe that from (6.26),

$$\frac{1}{p}+\frac{1}{(q+1)p}=\frac{1}{p}\cdot\left(\frac{q+2}{q+1}\right)=\frac{1}{l}$$

so that $\mathrm{Tr}_\pi\{\lambda|\mathfrak{k}_l-\mathfrak{k}_p*\mathfrak{k}_{(q+1)p}|\}=o(d_l)$ and repeated application of (6.22) to (6.29) yields a zero difference, with error estimated by the right-side of (6.27), which establishes this estimate. □

The multilinear functions ξ and $b\xi$ are thus bounded with respect to the uniform C^2-norm on $C^\infty(K\times\ldots\times K)$ for each compact $K\subset V$. From the weak-* compactness of the unit ball in the dual spaces, there exists a weak-* limit of the pairs $(\psi_s, b\psi_s\}$. Furthermore, V is paracompact so can be exhausted by a nested sequence of compact submanifolds. Using a diagonal process we obtain a weak-* limit

$$\xi=\overset{*}{\lim}\,\psi_s \tag{6.30}$$

defined on all of $C_c^\infty(V\times\ldots\times V)$, for which the sequence $\{b\psi_s\}$ has a weak-* limit denoted by $b\xi$. Each ψ_s is a cyclic co-chain, so that ξ will be cyclic. Observe that the uniform C^2-norm estimate (6.27) on $b\psi_s$ converges to zero uniformly on compact sets in V, so we obtain $b\xi=0$. The boundary operator b is continuous in the C^2-topology when applied to ψ_l by (6.22), so that as the notation suggests we have $b(\xi)=\widehat{(b\xi)}=0$. This completes the proof of the theorem. □

Remark 6.8. Theorem 6.5 gives one method of constructing a "transfer" between $C_c^\infty(\pi)$ and $C_c^\infty(V)$. We will next show that the sharp cocycle, $c^\#$ of Sect. 5, satisfies the conditions (6.21), (6.22) and (6.23) so that this renormalization method applies in the cases of interest for this work. It would be very interesting to have a general theory of the transfer. The above renormalization of $c^\#$ yields a cocycle $\hat{c}$ over $C_c^\infty(V)$ that incorporates secondary data about the foliation $\mathscr{F}_\alpha$, which suggests that the general analytic transfer is a process related to the construction of Cheeger-Chern-Simons invariants, wherein, a nonclosed universal transgressed Chern form on a Stieffel bundle yields a closed odd-degree form by restricting to a precise class of cycles in the Stieffel bundle. In our context, the existence of an F.A.I. is the equivalent of being able to restrict the Chern form of $(\hat{c}^\#)_l$ to such cycles. In this regards, the remarks on pages 55–56 of [20] are particularly relevant.

Theorem 6.9. *For $q>m$, the cyclic q-cocycle $c^\#$ of Sect. 5 is normal for the* F.A.I. *defined by* (6.14).

Proof. We give the proof for the case q is odd and $c^\#$ is defined by (5.22), and leave the case q is even and $c^\#$ is defined by (5.23) to the reader.

The first reduction of the problem of establishing (6.21), (6.22) and (6.23) is that we can assume $f_0,\ldots,f_q\in C_c^\infty(V)$ all have support in a common compact subset $K\subset V$ with diameter $(K)<\frac{1}{2}c_1$. To reduce to this case, first observe that the bounded geometry of V implies there is a partition-of-unity $\{\tilde{\lambda}_\alpha|\alpha\in\mathscr{A}'\}$ for V which is uniformly locally finite with each $\tilde{\lambda}_\alpha$ having support of diameter $<\frac{1}{10}c_1$. Moreover, by requiring that each $\tilde{\lambda}_\alpha$ have support containing a ball of radius at least $\frac{1}{100}c_1$, we obtain for each compact set $K\subset V$ the number of $\alpha\in\mathscr{A}'$ such that support $(\tilde{\lambda}_\alpha)\cap K\neq\emptyset$ is bounded above by a fixed constant times $\mathrm{vol}(K)$. Finally, we assume that there is a uniform upper bound, $\tilde{c}$, on the family of norms, $\{\|\tilde{\lambda}_\alpha\|_{2,\infty}|\alpha\in\mathscr{A}'\}$.

Given functions $\{f_0,\dots,f_q\}$ with support in a compact set K, write each $f_i=\sum_\alpha f_{i,\alpha}$. Note that each $f_{i,\alpha}$ has support of diameter $<\frac{1}{10}c_1$, and the norm $\|f_{i,\alpha}\|_{2,\infty}$ is uniformly estimated by a constant multiple of $\tilde{c}\|f_i\|_{2,\infty}$.

To begin the estimate (6.22), observe

$$(6.31)\qquad \begin{aligned}&|c^{\#}(f_0\mathfrak{p}_0,\dots,f_q\mathfrak{p}_q)|\\ &\leqq \sum_{\alpha_0,\dots,\alpha_q} |\mathrm{Tr}_{\mathscr{H}}[\Pi_V,f_{0,\alpha_0}\mathfrak{p}_0]\dots[\Pi_V,f_{q,\alpha_q}\mathfrak{p}_q]|.\end{aligned}$$

The distributional kernel of Π_V is supported in a $3\delta=\{10(q+2)\}^{-1}\cdot c_1$-uniform neighborhood of the diagonal. We also assume the each $\mathfrak{p}_i$ satisfies $r(\mathfrak{p}_i)<c_6=\{10(q+1)\}^{-1}\cdot c_1$. With these two assumptions, once a choice $\alpha_0\in\mathscr{A}'$ is made, then there is a uniformly finite (independent of α_0) number of choices of $(\alpha_1,\dots,\alpha_q)$ for which the summand on the right-side of (6.31) is non-zero. Hence, it will suffice to prove (6.22) for all f_i with support of diameter $<\frac{1}{10}c_1$. In fact, observe that $c^{\#}(f_0\mathfrak{p}_0,\dots,f_q\mathfrak{p}_q)=0$ if any pair of f_i,f_j have support separated by more than $\frac{1}{10}c_1$, so that we can also assume all of the functions have support in a compact set K of diameter at most $\frac{1}{2}c_1$. We assume then that λ is identically one on the $\frac{3}{4}c_1$-neighborhood of K with support (λ) of diameter $<c_1$.

Let (U,ϕ) be a cylindrical foliation chart containing the support of λ. Then each commutator $[\Pi_V,f_ip_i]$ and their products have support in U, so that we can reduce to coordinates to make the estimate (6.22). We follow the notation of the proof of Proposition 6.2. The operator Π_V restricts to a family of leafwise operators Π_y, acting on $C^\infty(I^m,\mathbf{C}^{4k})$ and independent of $y\in I^n$. Each $\mathfrak{p}_i$ is represented by a kernel $\mathfrak{p}_i(y,y')$ on I^n which is independent of $x\in I^m$. For each i and $x\in I^m$, we set $f_{i,y}(x)=f_i(x,y)$ in these coordinates. Then introduce operators

$$(6.32)\qquad A_i(y)=[\Pi_y,f_{i,y}]$$

on $L^2(I^m,\mathbf{C}^{4k})$. The operators Φ and $\mathfrak{p}_i$ commute, so the perturbation Π_V satisfies

$$(6.33)\qquad [\Pi_V,f_i\mathfrak{p}_i]=[\Pi_V,f_i]*\mathfrak{p}_i.$$

We can then use the Fubini theorem and formula (2.10) to estimate

$$(6.34)\qquad \begin{aligned}&|c^{\#}(f_0\mathfrak{p}_0,\dots,f_q\mathfrak{p}_q)|\\ &=\left|\int_{I^m\times I^n}\mathrm{Tr}\{A_0(y_0)*\mathfrak{p}_0*A_1*\mathfrak{p}_1*\dots\right.\\ &\qquad\left. *A_q*\mathfrak{p}_q(y_q,y_0)\cdot dv(g_{y_0})\cdot dv_X(y_0)\right|\\ &\leqq \mathrm{Tr}_\pi\{\|A_0(y_0)\|_{q+1}\dots\|A_q(y_q)\|_{q+1}\cdot\lambda\cdot|\mathfrak{p}_0|*\dots*|\mathfrak{p}_q|\},\end{aligned}$$

where $\|\ \|_{q+1}$ is the Schatten $(q+1)$-norm on $L^2(I^m,C^{4k})$. The proof of Lemma 5.4 established a slightly stronger statement that we give as

Lemma 6.10. *There exists a constant c_7 depending only on Π_V and the data* (5.1) *to* (5.3) *such that for all $f\in C_c^\infty(U)$,*

$$(6.35)\qquad \|[\Pi_y,f]\|_{q+1}\leqq c_7\cdot\|f_y\|_{1,\infty}.\quad\square$$

Combining (6.34) and (6.35) yields (6.22) for the local case, and by the previous remarks we obtain (6.22) in general.

Condition (6.23) is evident for $c^{\#}$, as Φ and each $\mathfrak{k}_l$ commute, so computation yields

$$[\Pi_V, g_0\mathfrak{k}_l][\Pi_V, g_1] = [\Pi_V, g_0]\cdot[\Pi_V, \mathfrak{k}_l g_1],$$

and this implies (6.23).

It remains to prove (6.24), which follows from a more precise form of (6.34) and the calculation

$$\lambda\cdot[\Pi_V, [f, \mathfrak{k}_l]g_0] = \lambda[[\Pi_V, f], \mathfrak{k}_l]g_0. \tag{6.36}$$

First we remark that the reduction to a foliation chart (U, ϕ) used to prove (6.22) can also be applied here, as $r(k_l)\to 0$ with l. Thus, we can assume that all λ, f, $f_0,\dots,f_q\in C_c^\infty(U)$ as before.

There exists a constant c' depending only on (5.1) and (5.2) so that for $f\in C_c^\infty(U)$ and $y, y_0\in I^n$,

$$\|f_y - f_{y_0}\|_{1,\infty} \leqq c'\cdot\|f\|_{2,\infty}\cdot\|y-y_0\|. \tag{6.37}$$

Integrate the left-side of (6.34) with respect to I^m to obtain for $A_y = [\Pi_y, f_y]$,

$$|c^{\#}([f, \mathfrak{k}_l]g_0, g_1,\dots,g_q)| \leqq c_8\cdot\mathrm{Tr}_\pi\{\lambda\cdot|[A_y, \mathfrak{k}_l]| * |g_0| * \dots * |g_q|\} \tag{6.38}$$

where c_8 depends upon Π_V and appropriate uniform C^1-norms of the $\{g_0,\dots,g_q\}$. Combining the estimates (6.35) and (6.37), we conclude by (6.5) that

$$\mathrm{Tr}_\pi\{\lambda|[A_y, \mathfrak{k}_l]|\} = \|f\|_{2,\infty}\cdot o(d_l). \tag{6.39}$$

Thus, we obtain the more precise form of (6.23), that the limit is $o(d_l)$, uniform in $\|f\|_{2,\infty}$ and appropriate C^1-norms on the $g_0,\dots,g_q$. □

Combining Theorems 6.5 and 6.9, we conclude that the transverse cocycle $c^{\#}$ of Sect. 5 over $C_c^\infty(\pi)$ can be renormalized to yield a cocycle $\hat{c}$ over $C_c^\infty(V)$. The proof of Theorem 6.9 contains the techniques needed to identify $\hat{c}$ more explicity.

Theorem 6.11. *Let $\{k_l\}$ be the F.A.I. given by* (6.14). *Then the renormalized cocycle $\hat{c}$ is identically equal to the odd degree longitudinal cocycle, c_Φ, defined by* (3.24). *In particular, the weak-$*$ limit* (6.30) *is unique.*

Proof. We must prove that

$$c_\Phi(f_0,\dots,f_q) = \lim_{s\to\infty} d_s^{-1}\cdot c^{\#}(f_0 k_{(q+1)s},\dots,f_q k_{(q+1)s}). \tag{6.40}$$

We can assume that each f_i has support of diameter $<\frac{1}{10}c_1$, and as $r(k_l)\to 0$, that all of the supports are contained in a compact set K of diameter $<\frac{1}{2}c_1$. We then introduce notation as in the proof of Theorem 6.9, and also define

$$A(y_0,\dots,y_q) = -(-4)^{-\frac{q+1}{2}}\Pi_{y_0}[\Pi_{y_0}, f_{0,y_0}]\dots[\Pi_{y_q}, f_{q,y_q}]. \tag{6.41}$$

$$\begin{aligned} a(y_0,\dots,y_q) &= \mathrm{Tr}_{L^2(I^m)}\{A(y_0,\dots,y_q)\}\\ &= \int_{I^m}\mathrm{Tr}\{A(y_0,\dots,y_q)\}(x,x)\,dv(x). \end{aligned} \tag{6.42}$$

The key point is to observe that by (6.35) and (6.37), if

$$\|y_i - y_0\| < \varepsilon < 1 \quad \text{for all} \quad 0<i<q,$$

then there exists a constant c_q so that

(6.43) $$|a(y_0, y_1, \ldots, y_q) - a(y_0, y_0, \ldots, y_0)| < c_q \cdot \varepsilon \cdot \|f_0\|_{2,\infty} \cdots \|f_q\|_{2,\infty}.$$

Fix $l=(q+1)s$ such that $r(k_l) < \dfrac{\varepsilon}{(q+1)}$. Then we estimate

(6.44) $$c^{\#}(f_0 \mathfrak{k}_l, \ldots, f_q \mathfrak{k}_l)$$
$$= \iiint\limits_{I^n \times \ldots \times I^n} a(y_0, \ldots, y_q) \cdot \mathfrak{k}_l(y_0, y_1) \ldots \mathfrak{k}_l(y_q, y_0)\, dv_\pi(y_0, \ldots, y_q)$$

(6.45) $$= \int\limits_{I^n} a(y_0, \ldots, y_0) \cdot \mathfrak{k}_l * \ldots * \mathfrak{k}_l(y_0, \ldots, y_0)\, dv_\pi(y_0)$$
$$\pm c_q \cdot \varepsilon \cdot \|f_0\|_{2,\infty} \cdots \|f_q\|_{2,\infty} \cdot \mathrm{Tr}_\pi(\lambda \cdot \mathfrak{k}_l * \ldots * \mathfrak{k}_l).$$

By (6.4) the error term in (6.45) tends to zero with ε, which tends to zero with $l \to \infty$. By (6.7) and (6.4), the first integral in (6.45) scaled by d_s^{-1} tends to

(6.46) $$\int\limits_{I^n} a(y_0, \ldots, y_0) \cdot dv_\pi(y_0).$$

Examining the definition of $c_\Phi(f_0, \ldots, f_q)$ in a cylindrical flow chart, we obtain that (6.46) is precisely $c_\Phi(f_0, \ldots, f_q)$. Therefore,

(6.47) $$d_s^{-1} c^{\#}(f_0 \mathfrak{k}_l, \ldots, f_q \mathfrak{k}_l) \to c_\Phi(f_0, \ldots, f_q)$$

as was to be shown. □

There is a corollary of the proof above, that will be important for the proof of Theorem 8.1 below.

Corollary 6.12.

(6.48) $$\hat{c}(f_0, \ldots, f_q)$$
$$= -(-4)^{-\frac{q+1}{2}} \lim_{s \to \infty} d_s^{-1} \cdot \mathrm{Tr}_{\mathscr{H}^{4N}}(\Pi_V \cdot [\Pi_V, f_0] \ldots [\Pi_V, f_q] \circ \mathfrak{k}_s). \quad \square$$

Remark 6.13. The restriction on the support of the sharp phase function Φ_V used in the construction of $c^{\#}$ is primarily for notational convenience. It guarantees that the supports of all relevant products will lie in an appropriate plaque chart I^m. With some additional restrictions on the geometry of $\mathscr{F}$, this requirement on supports can be relaxed to allow Φ_V with arbitrarily large, but uniform, support. For example, if $\mathscr{F} = \mathscr{F}_\alpha$ is a suspension foliation on $V = \tilde{M} \times_\Gamma G$ for G-compact, then every leaf of $\mathscr{F}_\alpha$ is diffeomorphic to the covering $M_\alpha \to M$ associated to $\alpha: \Gamma \to G$. We can then replace the charts I^m appearing in (6.34) and (6.42) with integration over the images of diagonal sets in $M_\alpha \times M_\alpha$ which are compact, but contain the supports of all appropriate products. The estimates of the above proofs depend only on compactness, so extend in this context. We can thus apply, for example, Theorem 6.11 for these more general $c^{\#}$ obtained from Φ_V with large support. This will play a key role in Sect. 8.

We conclude this section with a remark on renormalization. Our choice of using an F.A.I. to make the transfer is based on the properties of heat kernels, which can be applied in the general context of this section. The authors' original approach to the transfer applied only to the context where $X = G$ is a compact connected Lie group. The Peter-Weyl theorem yields a sequence of finite-rank *smooth* projections in $L^2(G)$ which converge weakly to the identity. Renormalization can be effected via these kernels, with slight modifications of the above arguments. The main differences arise first from replacing (6.4) with the projection law, and second the Fölner condition (6.5) must be proven using fairly tedious representation theory of a compact group. For the circle case, $G=S^1$, this method of renormalization was carried out in [43], and can be viewed as approximating the longitudinal operator $\tilde{c}_\Pi(f_0,\dots,f_q)$ of (3.23) by its finite Fourier sums in the S^1-parameter. The group methods thus correspond to lattice renormalization, while the approach of this section via an F.A.I. corresponds to heat kernel regularization. These ideas appear also in the recent works [33] and [57, 58].

7. Eta distributions

In this section we introduce the eta-distribution

(7.1) $$\eta(D_M,\alpha): R_\infty(G)\to\mathbf{R}$$

associated to a geometric operator D_M on the odd-dimensional manifold M and a homomorphism $\alpha:\Gamma\to G$. We will give two definitions of (7.1); the first is a direct extension of the usual definition of the eta-invariant, and only mildly uses the structure theory of the representation ring of a compact Lie group. The second definition uses the lifted operator D_V acting on sections over the principal bundle V, where D_V is transversally elliptic to the G-action.

The main new result of this section is the observation that the eta-distribution is *tempered.* That is, there is a constant $c(D_M)$ such that for a smooth class function $\psi\in R_\infty(G)$,

(7.2) $$|\eta(D_M,\alpha,\psi)|\leqq c(D_M)\cdot\sum_{\chi\in\hat{G}}|a(\psi,\chi)|\cdot N(x)^2$$

with the notation explained below. This will follow from results of Cheeger and Gromov.

Distributional eta and zeta-invariants have appeared in the literature in several contexts. For a finite group G, the distributional eta invariant was defined by Atiyah, Patodi and Singer in (Formula (2.13), page 412, [5]) as part of their study of index formula for G-manifolds with boundary. Donnelly proved in [36] the relevant index theorem in this context. Note that for G finite, the manifold V is a finite covering of M so that the lift D_V is also elliptic, hence the analytic subtleties encountered for non-elliptic operators do not arise in their work.

For G a compact connected Lie group, Šubin [86] and Smagin and Šubin [84] introduced a distributional zeta function for D_V, which is now only G-transversally elliptic. Via the methods of Wodzicki [89], this can be used to define the distributional eta-invariant. Finally, Donnelly in [37] studied the multiplicative property of eta-invariants for the fibered manifolds $V_\alpha=\tilde{M}\times_\Gamma X$. The relation between the eta-invariants of V_α and the eta-distributions we define is a topic for further exploration.

For the rest of this section we fix a geometric operator D_M acting on sections $C^\infty(E_M)$, in the sense of [49]. We assume that M has odd dimension equal m. The Riemannian metric on TM is that obtained from the principal symbol of D_M, so that D_M has the customary nice properties (cf. § 1, [49]). In particular, D_M is symmetric and has a unique self-adjoint closure in $L^2(E_M)$. The compact group G and homomorphism α are also fixed.

There is a unique lift of D_M to a leafwise operator D along the leaves of the foliation $\mathscr{F}_\alpha$ on the principal bundle $\pi: V \to M$. As before, we let D_V denote the first-order differential operator acting on sections of

$$E = \pi^*(E_M) \to V.$$

Using the bi-invariant Haar measure on G, we obtain a Riemannian volume form on V as in Sect. 2, and use this to give $C^\infty(E)$ an inner product. The lifted operator D_V is symmetric on $C^\infty(E)$, and it is shown in (Chap. 5, [56], cf. [22]) that D_V has a unique self-adjoint closure in $L^2(E)$. In fact, there is a more precise result.

Proposition 7.1. *The operator D_V has real pure-point spectrum* (p.p.s.), *and for each eigenvalue $\lambda \in \mathbf{R}$, the corresponding eigenspace $H_\lambda \subset L^2(V)$ is G-invariant with each irreducible character $\chi \in \hat{G}$ having finite multiplicity in H_λ.*

Proof. Let Δ_G denote the G-Laplacian acting along fibers of $\pi: V \to M$ constructed in Sect. 6 for $X = G$. There is a natural extension of Δ_G to a second-order operator on $C^\infty(E)$. The operators D_V^2 and Δ_G commute, and their sum $\Delta_G + D_V^2$ is a symmetric, elliptic second order operator on $C^\infty(E)$ which commutes with D_V. As $\Delta_G + D_V^2$ has real p.p.s. and D_V has a unique self-adjoint extension, this implies the same for D_V. The standard estimates on the growth rate of the eigenvalues (counted with multiplicity) of $\Delta_G + D_V^2$ implies that the multiplicity of $\chi \in \hat{G}$ in H_λ is finite. (Estimates on the growth rate of $\dim(H_\chi)$ can also be obtained, as in Sect. 2 of [1].) □

Let $\{\lambda_i\}$ be the set of eigenvalues of D_M, listed with multiplicity. The eta-function of D_M is the sum

$$\eta(D_M, s) = \sum_{\lambda_i > 0} \lambda_i^{-s} - \sum_{\lambda_i < 0} (-\lambda_i)^{-s} + \dim(\ker D_M). \tag{7.3}$$

By general facts from [4], this function is holomorphic in $s \in \mathbf{C}$ for $\mathrm{Re}(s) > m$, and admits a meromorphic extension to all of $\mathbf{C}$ with isolated simple poles at $s = m - l$, for natural numbers $l \geqq 0$. Moreover, for a geometric operator D_M, $\eta(D_M, s)$ is holomorphic for $\mathrm{Re}(s) > -2$ due to supersymmetric cancellations (cf. [13]). We define

$$\eta(D_M) = \eta(D_M, 0). \tag{7.4}$$

Given a unitary representation $\varrho: G \to U_N$, let $E(\varrho \circ \alpha) \to M$ be the Hermitian flat $\mathbf{C}^N$-bundle associated to the flat principal bundle $P(\varrho \circ \alpha)$ of (4.17). The differential operator D_M has a canonical extension to $C^\infty(E \otimes E(\varrho \circ \alpha))$ using the flat structure, which will be denoted by $D_M \otimes \nabla^{\varrho \circ \alpha}$. This extension is again symmetric and of geometric type, so it has a well-defined eta-invariant. We set

$$\eta(D_M, \alpha, \varrho) = \eta(D_M \otimes \nabla^{\varrho \circ \alpha}). \tag{7.5}$$

Introduce the vector subspace $Cl_f(G) \subset C^\infty(G)$ whose elements are finite linear combinations of characters of finite dimensional representations ϱ. Let $\hat{G}$ denote the set of irreducible representations of G, indexed by their characters χ. The dimension of the representation space $V(\chi)$ for χ will be denoted by $N(\chi)$. A typical element

$\psi \in Cl_f(G)$ can then be written

$$\psi = \sum_{\chi \in G} N(\chi) \cdot a(\psi, \chi) \cdot \chi \tag{7.6}$$

where at most finitely many of the "Fourier coefficients" $a(\psi, \chi)$ are non-zero. Note that we scale χ by $N(\chi)$ in (7.6), as the basic characters for this work will be those for the isotypical summands of the right regular representation of G on $L^2(G)$.

Recall that the left and right regular representations of G on $L^2(G)$ denoted by λ and ϱ, respectively, are given by

$$\begin{aligned} \lambda(g)(f)(h) &= f(g^{-1}h) \\ \varrho(g)(f)(h) &= f(hg) \end{aligned} \qquad \text{for } f \in L^2(G) \text{ and } g \in G. \tag{7.7}$$

If we consider $V(\chi)$ as a right G-space, then the dual $V(\chi)^* = \mathrm{Hom}(V(\chi), \mathbf{C})$ is a left G-space and $M(\chi) \equiv V(\chi)^* \otimes V(\chi)$ is a $G \times G$-module. By the Peter-Weyl Theorem, there is a $G \times G$-invariant subspace $L^2(G, \chi) \subset L^2(G)$ and an isomorphism of $G \times G$-modules

$$M(\chi) \cong L^2(G, \chi). \tag{7.8}$$

Let λ_χ and ϱ_χ denote the restrictions of λ and ϱ to $L^2(G, \chi)$, which intertwine via (7.8) with the fixed left and right G-actions on $M(\chi)$. Observe then that

$$\begin{aligned} \mathrm{Tr}(\varrho_\chi) &= N(\chi) \cdot \chi \\ \mathrm{Tr}(\lambda_\chi) &= N(\chi) \cdot \bar{\chi}. \end{aligned} \tag{7.9}$$

The $G \times G$-module $L^2(G)$ admits a Hilbert space decomposition into irreducible summands

$$L^2(G) \cong \hat{\bigotimes_\chi} L^2(G, \chi). \tag{7.10}$$

The orthogonal projection P_χ onto the summand $L^2(G, \chi)$ is given by convolution with $N(\chi) \cdot \chi$; that is, for $f \in L^2(G)$

$$P_\chi(f)(h) = f_\chi(h) = \int_G N(\chi) \cdot \chi(hg^{-1}) f(g) dg. \tag{7.11}$$

Note that χ is a smooth function, so that each $f \in L^2(G, \chi)$ is smooth on G.

We introduce the ring of central smooth functions on G,

$$R_\infty(G) = \{\psi \in C^\infty(G) | \psi(gh) = \psi(hg) \text{ all } h, g \in G\}. \tag{7.12}$$

Clearly, $Cl_f(G) \subset R_\infty(G)$ and the Peter-Weyl Theorem implies that each $\psi \in R_\infty(G)$ can be written as an infinite sum

$$\psi = \sum_\chi a(\psi, \chi) N(\chi) \cdot \chi \tag{7.13}$$

where the $|a(\psi, \chi)|$ are rapidly decreasing as a function of the weight of the representation χ. We let $w(\chi)$ denote the weight of χ.

For each $\psi \in R_\infty(G)$, define a kernel

$$k_\psi(h, g) = \psi(hg^{-1}), \tag{7.14}$$

with corresponding convolution operator on $L^2(G)$:

(7.15) $$f_\psi(h)=k_\psi * f(h)=\int_G \psi(hg^{-1})f(g)dg.$$

Combining (7.11), (7.13) and (7.15) we have:

Lemma 7.2. *For $\psi\in R_\infty(G)$, $f\in L^2(G)$*

(7.16) $$k_\psi * f=\sum_\chi a(\psi,\chi)\cdot P_\chi(f). \quad \square$$

After these preliminaries, we can now introduce the eta-distribution. For $\psi\in Cl_f(G)$, define via (7.5):

(7.17) $$\eta(D_M,\alpha,\psi)=\sum_\chi a(\psi,\chi)\cdot\eta(D_M,\alpha,\lambda_\chi).$$

The following important property of the functional $\eta(D_M,\alpha)$ on $Cl_f(G)$ defined by (7.17) is a direct consequence of estimates of Cheeger and Gromov:

Proposition 7.3. *There exists a constant $c(D_M)$ such that (7.2) holds for all $\psi\in Cl_f(G)$. In particular, if each $a(\chi,\psi)$ is non-negative real, then*

(7.18) $$|\eta(D_M,\alpha,\psi)|\leqq c(D_M)\psi(e).$$

Proof. For the case when D_M is the signature operator $d* - *d$, Theorem 1.2 of [17] shows that there exists a constant $c(D_M)$ such that for each representation $\beta:\Gamma\to U_N$,

(7.19) $$|\eta(D_M\otimes V^\beta)|\leqq c(D_M)\cdot N.$$

Moreover, Remark 4.1 of [18] asserts that (7.19) holds for D_M an arbitrary geometric operator. A proof of this estimate for D_M a geometric operator is given more recently by Ramachandran [71]. Clearly (7.2) is a consequence of (7.19), and then (7.18) follows from noting that $\mathrm{Tr}(\lambda_\chi(e))=N(\chi)\cdot\chi(e)=N(\chi)^2$. $\square$

Corollary 7.4. *The functional $\eta(D_M,\alpha)$ extends to $R_\infty(G)$.*

Proof. There exists constants c_{10}, $r>0$ so that the function

$$\mu(N)=\#\{\chi\in\tilde{G}\mid w(\chi)\leqq N\}\leqq c_{10}\cdot N^r.$$

It then follows that the sum (7.17) is uniformly convergent for $|a(\psi,\chi)|$ rapidly decreasing in $w(\chi)$ by the estimate (7.2). $\square$

The functional $\eta(D_M,\alpha)$ on $R_\infty(G)$ can be reformulated in terms of the spectral theory of D_V acting on $C^\infty(E)$. It is this alternate formulation of $\eta(D_M,\alpha)$ that is used to identify the renormalized cyclic Chern character of Sect. 6 with a spectral invariant of D_M.

The Hilbert space $L^2(E)$ is a right G-module, and there is an orthogonal decomposition into G-spaces

(7.20) $$L^2(E)\cong\hat{\bigoplus_\chi} L^2(E,\chi)$$

where $L^2(E,\chi)$ is the closed subspace which transforms by the right representation ϱ_χ. Let

(7.21) $$e(\chi):L^2(E)\to L^2(E,\chi)$$

be the orthogonal projection, which is defined by convolution (5.5) with the smooth kernel $N(\chi)\cdot k_\chi^* \in C^\infty(\pi)$,

$$(7.22) \qquad k_\chi^*(y, y\cdot g)=\chi(g^{-1})\,; \quad y\in V, \quad g\in G\,.$$

The operator D_V is G-invariant, thus its closure commutes with each projection $e(\chi)$, and the compressions $D(\chi)=D_V\circ e(\chi)$ are densely defined on $L^2(E,\chi)$.

Proposition 7.5. *For each $\chi\in\hat{G}$, there is a natural isomorphism*

$$(7.23) \qquad L^2(E,\chi)\cong L^2(E_M\otimes E(\lambda_\chi\circ\alpha))$$

which identifies $D(\chi)$ with $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}$.

Proof. The quotient map $\tilde{M}\times G\to V$ identifies pairs $(\tilde{x},g)\sim(\tilde{x}\circ\gamma^{-1},\alpha(\gamma)\cdot g)$ for $\gamma\in\Gamma$, and commutes with the right G-action. Let $\tilde{E}_M\to\tilde{M}$ be the lift of E_M, and $\tilde{E}\to\tilde{M}\times G$ the lift of E. There is an isomorphism of right G-modules

$$(7.24) \qquad \begin{aligned} C^\infty(\tilde{E},\chi)&\cong C^\infty(\tilde{E}_M)\otimes L^2(G,\chi)\\ &\cong C^\infty(\tilde{E}_M)\otimes M(\chi)\,. \end{aligned}$$

For a smooth section $s\in C^\infty(E,\chi)$, the lift $\tilde{s}\in C^\infty(\tilde{E},\chi)$ satisfies

$$(7.25) \qquad \begin{aligned} \tilde{s}(\tilde{x}\cdot\gamma,g)&=\tilde{s}(\tilde{x},\alpha(\gamma)g)\\ &=\lambda_\chi\circ\alpha(\gamma^{-1})\cdot\tilde{s}(\tilde{x},g) \end{aligned}$$

via the first isomorphism of (7.24). A section of the flat bundle $E(\lambda_\chi\circ\alpha)\to M$ is equivalent to a map $\psi:\tilde{M}\to\mathbf{C}^{N(\chi)}$ such that

$$\psi(\tilde{x}\cdot\gamma)=\lambda_\chi\circ\alpha(\gamma^{-1})\cdot\psi(\tilde{x}),$$

since we require that

$$(\tilde{x}\cdot\gamma,\psi(\tilde{x}\cdot\gamma))\sim(\tilde{x},\lambda_\chi(\gamma)\cdot\psi(\tilde{x}\cdot\gamma))=(\tilde{x},\psi(\tilde{x}))\,.$$

By the second isomorphism of (7.24), we can thus identify $\tilde{s}$ with the lift of a section $\hat{s}\in C^\infty(E_M\otimes E(\lambda_\chi\circ\alpha))$. The linear map $s\to\hat{s}$ is an isomorphism of smooth section spaces

$$(7.26) \qquad C^\infty(E,\chi)\cong C^\infty(E_M\otimes E(\lambda_\chi\circ\alpha)),$$

which extends to the isometry (7.23) as (7.8) and (7.10) are isometries.

The isomorphism (7.26) becomes an identification when lifted to $\tilde{M}\times G$, so to show $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}$ and $D(\chi)$ agree under (7.26), it will suffices to show they agree on $\tilde{M}\times G$. The first operator, $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}$, which is defined via the flat Hermitian connection on $E(\lambda_\chi\circ\alpha)$, lifts to the operator $\tilde{D}_M\otimes I_{M(\chi)}$. Here, $\tilde{D}_M$ is the lift of D_M to $C^\infty(\tilde{E}_M)$, then we extend via the identity map on $M(\chi)$.

The operator D_V on $C^\infty(E)$ lifts to $\tilde{D}_V=\tilde{D}_M\otimes I_{C^\infty(G)}$ on $C^\infty(\tilde{E})\cong C^\infty(\tilde{E}_M)\hat{\otimes}C^\infty(G)$. The restriction $D(\chi)$ thus lifts to $\tilde{D}(\chi)=\tilde{D}_M\otimes I_{M(\chi)}$, which concludes the proof. □

The operator D_V has p.p.s. by Proposition 7.1, so for $s\in\mathbf{C}$ define

$$(7.27) \qquad D_V^{-s}=\begin{cases}\lambda^s;\ \lambda>0, & \text{on } \lambda\text{-eigenspace of } D_V\\ -(\lambda)^s;\ \lambda<0, & \text{on } \lambda\text{-eigenspace of } D_V\\ 1\,; & \text{on nullspace of } D_V\,.\end{cases}$$

Let Tr_E denote the usual Hilbert space trace on $L^2(E)$. An immediate application of Proposition 7.3 yields

Corollary 7.6. *For each* $\chi \in \hat{G}$,

$$\mathrm{Tr}_E(D_V^{-s} \circ e(\chi)) = \eta(D_M \otimes \nabla^{\lambda_\chi \circ \alpha}, s). \tag{7.28}$$

The second formulation of the distributional trace is suggested by formula (7.28). For each $\psi \in R_\infty(G)$, let $k_\psi^* \in C^\infty(\pi)$ be defined by $k_\psi^*(y \cdot g, y) = \psi(g)$. Then set

$$\tilde{\eta}(D_M, \alpha, \psi) = \mathrm{Tr}_E(D_V^{-s} \circ k_\psi^*)|_{s=0}. \tag{7.29}$$

This is well-defined by (7.28) and Proposition 7.3, and by (7.28) we have:

Corollary 7.7. *For* $\psi \in R_\infty(G)$

$$\tilde{\eta}(D_M, \alpha, \psi) = \eta(D_M, \alpha, \psi). \quad \square \tag{7.30}$$

For our applications in Sect. 8, we need to extend the definition of $\eta(D_M, \alpha)$ to include an auxillary flat bundle. Let $\varrho : G \to U_N$ be a representation; then $D_M \otimes \nabla^{\varrho \circ \alpha}$ acting on $C^\infty(E_M \otimes E(\varrho \circ \alpha))$ is again a geometric operator. We can thus define

$$\eta(D_M, \alpha, \varrho, \psi) = \eta(D_M \otimes \nabla^{\varrho \circ \alpha}, \alpha, \psi) \tag{7.31}$$

where the right-hand-side of (7.31) is defined as in (7.17). The estimate (7.18) is correspondingly scaled by the dimension N of $E(\varrho \circ \alpha)$:

Corollary 7.8. *Let* $\psi \in R_\infty(G)$ *have all* $a(\psi, \chi)$ *real and non-negative. Then*

$$|\eta(D_M, \alpha, \varrho, \psi)| \leqq c(D_M) \cdot N \cdot \psi(e). \quad \square \tag{7.32}$$

When the representation α defines a topologically trivial bundle $P(\alpha) = V$, then the choice of a trivializing G-map $\Theta : V \to M \times G$ yields $\bar{\alpha} = (\alpha, \Theta)$. We use the definition (7.3) to define a linear functional $\eta(D_M, \bar{\alpha}, \varrho)$ on $R_\infty(G)$.

The trivialization Θ induces a product structure,

$$\Theta^* : E(\varrho \circ \alpha) \cong M \times \mathbf{C}^N \equiv \varepsilon^N, \tag{7.33}$$

which carries the Hermitian flat connection on $E(\varrho \circ \alpha)$ over to a connection $\nabla^{\overline{\varrho \circ \alpha}}$ on ε^N. The product flat connection on ε^N will be denoted by ∇^0.

The differential operator D_M has two essentially self-adjoint extensions to $C^\infty(E_M \otimes \varepsilon^N)$ defined in conjunction with $\bar{\alpha}$. The first is $D_0 = D_M \otimes \nabla^0$, obtained by using the product structure to extend the coefficients. The second is $D_1 = D_M \otimes \nabla^{\overline{\varrho \circ \alpha}}$, obtained by letting D_M act on the coefficients of parallel sections for $\nabla^{\overline{\varrho \circ \alpha}}$ of $C^\infty(E_M \otimes \varepsilon^N)$. Define a smooth 1-parameter family

$$D_t = t \cdot D_1 + (1-t) D_0 \tag{7.34}$$

of essentially self-adjoint operators on $C^\infty(E_M \otimes \varepsilon^N)$. The 1-parameter family of eta-invariants $\eta(D_t)$ varies smoothly in t, except for a finite number of integer jumps [6]. Thus, there is a well-defined derivative $\dot{\eta}(D_t)$ which depends continuously on t, and moreover $\dot{\eta}(D_t)$ is a local invariant of the family $\{D_t\}$. Define

$$\eta(D_M, \bar{\alpha}, \varrho) = \int_0^1 \dot{\eta}(D_t) dt. \tag{7.35}$$

As discussed in (§6, page 90, [6]) the correspondence

(7.36) $$(\bar{\alpha}, \varrho) \rightarrow \eta(D_M, \bar{\alpha}, \varrho)$$

defines a linear functional on the odd-degree, real-valued K-theory of M. It is this topological invariant that is naturally related to the foliation index invariants of Sect. 3.

In the definition (7.35), we can replace D_M with $D_M \otimes \nabla^{\lambda_\chi \circ \alpha}$ acting on $C^\infty(E_M \otimes E(\lambda_\chi \circ \alpha))$, for any $\chi \in \hat{G}$. The resulting family will be denoted D_t^χ, and we set

(7.37) $$\eta(D_M, \bar{\alpha}, \varrho, \chi) = \int_0^1 \dot{\eta}(D_t^\chi)\, dt .$$

This is a purely formal device by the next result.

Lemma 7.9.

(7.38) $$\eta(D_M, \bar{\alpha}, \varrho, \chi) = N(\chi)^2 \cdot \eta(D_M, \bar{\alpha}, \varrho) .$$

Proof. The derivative $\dot{\eta}(D_t)$ is local, so extending the operators D_t to $D_t^\chi = D_t \otimes \nabla^{\lambda_\chi \circ \alpha}$ multiplies this derivative by $\dim E(\lambda_\chi \circ \alpha) = N(\chi)^2$, as $\dot{\eta}(D_t^\chi)$ depends only on the local coefficients of D_t^χ. Integrating then gives (7.38). □

Now extend (7.37) to all of $R_\infty(G)$ by setting

(7.39) $$\begin{aligned} \eta(D_M, \bar{\alpha}, \varrho, \psi) &= \sum_\chi a(\psi, \chi) \cdot \eta(D_M, \bar{\alpha}, \varrho, \chi) \\ &= \left\{ \sum_\chi N(\chi)^2 \cdot a(\psi, \chi) \right\} \cdot \eta(D_M, \bar{\alpha}, \varrho) \\ &= \psi(e) \cdot \eta(D_M, \bar{\alpha}, \varrho) . \end{aligned}$$

The functional $\eta(D_M, \bar{\alpha}, \varrho)$ can be reformulated in terms of D_V, and the formal extension (7.39) is then seen to be surprisingly natural. Given a smooth map $u: V \rightarrow U_N$, we obtain a multiplication operator on $C^\infty(E \otimes \varepsilon^N)$, denoted again by u, by letting u act pointwise on the left on the coefficients ε^N. Let u^* be the multiplication operator by the pointwise adjoint matrix to u. Extend D_V to $D_V \otimes I_N$ on $C^\infty(E \otimes \varepsilon^N)$, then set

(7.40) $$D^u = u \circ (D_V \otimes I_N) \circ u^*$$

(7.41) $$D_t^u = t \cdot D^u + (1-t) \cdot D_V \otimes I_N .$$

The representation ϱ induces a continuous map $u(\varrho)$, defined as the composition

(7.42) $$V \xrightarrow{\theta} M \times G \longrightarrow G \xrightarrow{\varrho} U_N .$$

Then set

(7.43) $$D^\varrho \equiv D^{u(\varrho)} ; \quad D_t^\varrho = D_t^{u(\varrho)} .$$

Proposition 7.10. *D^ϱ is right G-invariant, with essentially self-adjoint restrictions $D^\varrho(\chi)$ to the invariant subspaces $C^\infty(E \otimes \varepsilon^N, \chi)$. There is an identification*

(7.44) $$L^2(E \otimes \varepsilon^N, \chi) \cong L^2(E_M \otimes E(\lambda_\chi \circ \alpha) \otimes \varepsilon^N)$$

under which $D^\varrho(\chi)$ is unitarily conjugate to $D_M \otimes \nabla^{\lambda_\chi \circ \alpha} \otimes \nabla^{\overline{\varrho \circ \alpha}}$, hence they have the same spectrum.

Proof. The differential operator $D_V \otimes I_N$ commutes with the constant matrix multiplier $\varrho(g)$ for all $g \in G$, so

$$\begin{aligned} R_g^*(D^\varrho) &= u(\varrho) \cdot \varrho(g) \cdot R_g^*(D_V) \cdot \varrho(g)^* u(\varrho)^* \\ &= u(\varrho) \cdot D_V \cdot \varrho(g)\varrho(g)^* u(\varrho)^* \\ &= D^\varrho . \end{aligned}$$

The projection onto the closed subspace $L^2(E \otimes \varepsilon^N, \chi)$ commutes with D^ϱ so the restrictions are essentially self-adjoint. The isomorphism (7.44) is a consequence of (7.23), as G acts trivially on the coefficients ε^N. Finally, to identify $D^\varrho(\chi)$, we first identify the lift $\tilde{D}^\varrho$ to $C^\infty(\tilde{E} \otimes \varepsilon^N)$ on $\tilde{M} \times G \to V$. Our method will basically be a repeat of the idea of Theorem 4.4.

The lift of Θ to coverings gives a right G-map $\tilde{\Theta}$, commuting with the two Γ-actions in the diagram

(7.45)
$$\begin{array}{ccc} \Gamma & & \Gamma \\ \downarrow & & \downarrow \\ \tilde{M} \times G & \xrightarrow{\tilde{\Theta}} & \tilde{M} \times G \\ \downarrow & & \downarrow \\ V & \xrightarrow{\Theta} & M \times G \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & M & \end{array}$$

The identity $\pi_1 \circ \Theta(x) = \pi(x)$ for all $x \in V$ implies that

(7.46) $$\tilde{\Theta}(\tilde{x}, g) = (\tilde{x}, \varphi(\tilde{x} \cdot g)) ; \qquad (\tilde{x}, g) \in \tilde{M} \times G$$

where $\varphi : \tilde{M} \to G$ satisfies

(7.47) $$\varphi(\tilde{x} \cdot \gamma) = \varphi(\tilde{x}) \cdot \alpha(\gamma) .$$

The lift $\tilde{u}(\varrho)$ to $\tilde{M} \times G$ of $u(\varrho) : V \to U_N$ is given by $\tilde{u}(\varrho) = \varrho \circ \pi_2 \circ \tilde{\Theta}$, so that

(7.48) $$\tilde{u}(\varrho(\tilde{x}, g)) = \varrho(\varphi(\tilde{x}) g) = \varrho(\varphi(\tilde{x})) \cdot \varrho(g) .$$

Thus, D^ϱ lifts to

$$\tilde{u}(\varrho) \cdot (\tilde{D}_V \otimes I_N) \cdot \tilde{u}(\varrho)^* = \varrho(\varphi(\tilde{x})) \cdot (\tilde{D}_M \otimes I_{C^\infty(G)} \otimes I_N) \cdot \varrho(\varphi(\tilde{x}))^*$$

and the restriction $\tilde{D}^\varrho(\chi)$ is given by

(7.49) $$\tilde{D}(\chi) = \varrho \circ \varphi \cdot (\tilde{D}_M \otimes I_{M(\tilde{\chi})} \otimes I_N) \cdot \varrho \circ \varphi^*$$

acting on $C^\infty(\tilde{E}_M \otimes M(\chi) \otimes \varepsilon^N)$. Introduce an automorphism A of $C^\infty(\tilde{E}_M \otimes M(\chi) \otimes \varepsilon^N)$, defined as pointwise multiplication on the left on ε^N by $\varrho(\varphi(\tilde{x}))$. Thus

(7.50) $$\tilde{D}^\varrho(\chi) = A \circ (\tilde{D}_M \otimes I_{M(\chi)} \otimes I_N) \circ A^*$$

induces a unitary conjugation between $\tilde{D}^\varrho(\chi)$ and a Γ-invariant operator acting on sections of the form $A^* \tilde{s}$, $\tilde{s} \in C^\infty(\tilde{E}_M \otimes M(\chi) \otimes \varepsilon^N)$. A section $A^* \tilde{s}$ transforms under the Γ-action as:

$$(7.51)\qquad \begin{aligned} A^*\tilde{s}(\tilde{x}\cdot\gamma) &= \varrho(\varphi(\tilde{x}\cdot\gamma))^*\cdot\tilde{s}(\tilde{x}\cdot\gamma) \\ &= \varrho\circ\alpha(\gamma^{-1})\cdot\varrho(\varphi(\tilde{x}))\cdot\lambda_\chi\circ\alpha(\gamma^{-1})\cdot\tilde{s}(\tilde{x}) \\ &= (\varrho\otimes\lambda_\chi)\circ\alpha(\gamma^{-1})\cdot A^*(\tilde{s})(\tilde{x}). \end{aligned}$$

Hence, $A^*\tilde{s}$ is the lift of a section of the flat bundle $E((\varrho\otimes\lambda_\chi)\circ\alpha)\to M$. Conversely, A induces a map from lifts of sections of this flat bundle to sections of $C^\infty(\tilde{E}_M\otimes M(\chi)\otimes\varepsilon^N)$. Thus, A^* is a unitary map from $C^\infty(E\otimes\varepsilon^N,\chi)$ to $C^\infty(E_M\otimes E((\varrho\otimes\lambda_\chi)\circ\alpha))$ which conjugates $D^\varrho(\chi)$ to $D_M\otimes\nabla^{\lambda_\chi\circ\alpha}\otimes\nabla^{\varrho\circ\alpha}$. □

Define $(D_t^u)^{-s}$ acting on $C^\infty(E\otimes\varepsilon^N)$ as in (7.27).

Corollary 7.11. *For each* $\chi\in\hat{G}$ *and* $0\leqq t\leqq 1$, $s\in\mathbf{C}$ *with* $\mathrm{Re}(s)>-2$,

$$(7.52)\qquad \eta(t\cdot D_M\otimes\nabla^{\lambda_\chi\circ\alpha}\otimes\nabla^{\overline{\varrho\circ\alpha}}+(1-t)D\otimes I_{M(\chi)}\otimes I_N, s) = \mathrm{Tr}_{E^N}\{(D_t^u)^{-s}\circ e(\chi)\}.$$

Thus, for $\psi\in R_\infty(G)$,

$$(7.53)\qquad \eta(D_M,\bar{\alpha},\varrho,\psi)=\int_0^1 \frac{d}{dt}\{\mathrm{Tr}_{E^N}[(D_t^u)^{-s}\circ k_\psi^*]\}_{s=0}\cdot dt. \quad □$$

8. The Fõlner condition and spectral flow

In this section we relate the distributional relative eta-invariant of Sect. 7 to the values of the renormalized transverse cocycle $\hat{c}$ of Theorem 6.11. More precisely, we show that:

Theorem 8.1. *Let M be a compact oriented manifold of odd dimension, and D_M a geometric operator. For $\bar{\alpha}=(\alpha,\Theta)$, $\varrho: G\to U_N$ a representation of G and $u(\varrho): V\to U_N$ defined as in Sect.* 7, *we have*

$$(8.1)\qquad \hat{c}(u(\varrho)) = -\eta(D_M,\bar{\alpha},\varrho).$$

Remark 8.2. The values of $c_\Phi(u(\varrho))$, as defined by (3.32), and $\hat{c}(u(\varrho))$ agree by Theorem 6.11, so Theorem 1.1 of the Introduction follows from the definition (7.35) of $\eta(D_M,\bar{\alpha},\varrho)$ and Theorem 8.1. □

The proof of (8.1) will occupy the rest of this section. The first step will be to introduce the spectral flow of a family of elliptic self-adjoint operators. While this spectral flow is infinite, it decomposes into a distributional sum of finite spectral flows over $\hat{G}$ and so can be renormalized. The basic idea of the proof is that the Fõlner property of the heat kernel F.A.I. of Sect. 6 can be used to relate $\eta(D_M,\bar{\alpha},\varrho)$ to the renormalized spectral flow. On the other hand, the finite spectral flows associated to characters of G are equal to the indices of certain Toeplitz operators, which are homotopic to Fredholm operators obtained from a sharp parametrix as in Sect. 5. Their indices are thus equal, which relates the renormalized flow to the values of the renormalized cocycle $\hat{c}$, completing the proof of (8.1).

Let $\{D_t | 0 \leqq t \leqq 1\}$ be a smooth family of first-order, self-adjoint elliptic differential operators on M. The *spectral flow* of the family is defined to be the sum

(8.2) $$sf(\{D_t\}) = \frac{1}{2}\left\{\int_0^1 \dot{\eta}(D_t)dt - \eta(D_1) + \eta(D_0)\right\}.$$

When an eigenvalue of D_t changes sign as t varies, the eta-invariant $\eta(D_t)$ changes by ± 2, so that (8.2) is always an integer. In fact, it agrees with the more customary definition of spectral flow, which is the net flow of eigenvalues through the origin, counted with multiplicity (cf. page 93, [6]).

Let us recall, for the reader's convenience, the notation from Sect. 7 that for a class function $\psi \in R_\infty(G)$:

k_ψ denotes the operator in $L^2(G)$ with kernel $k_\psi(h, g) = \psi(hg^{-1}) = \psi(g^{-1}h)$

k_ψ^* denotes the operator on $L^2(V)$ with kernel $k_\psi^*(y \cdot g, y) = \psi(g)$.

We also use this notation for diagonal extensions to operators on bundles over G and V.

Our first step towards the proof of Theorem 8.1 is to generalize (8.2) to a distributional spectral flow, using $\tilde{\eta}$ as in (7.29). For $\psi \in R_\infty(G)$ and $u = u(\varrho)$, set

(8.3) $$\begin{aligned}\tilde{\eta}(D_M, \alpha, u^*\psi u) &\equiv \mathrm{Tr}\{D_V^{-s} \circ u(\varrho)^* \circ k_\psi^* \circ u(\varrho)\}|_{s=0} \\ &= \mathrm{Tr}_E\{(D^\varrho)^{-s} \circ k_\psi^*\}|_{s=0}\end{aligned}$$

where the last equality follows from (7.44) and Proposition 7.10. Note that $u(\varrho)$ acts via pointwise multiplication on $C^\infty(E \otimes \varepsilon^N)$, and hence does not commute with the convolution operator k_ψ^*. The distributional spectral flow is given by

(8.4) $$sf(D_M, \bar{\alpha}, \varrho, \psi) = \tilde{\eta}(D_M, \bar{\alpha}, \varrho, \psi) - \tilde{\eta}(D_M, \alpha, u^*\psi u) + \tilde{\eta}(D_M, \alpha, \psi).$$

Recall that the compact group G has a bi-invariant Riemannian metric chosen on it, with total volume 1. The Laplacian Δ for this metric has heat operators $e^{-t\Delta}$ with kernel functions $\{\psi_t \in R_\infty(G) | t > 0\}$. As each $e^{-t\Delta}$ is strictly positive, the coefficients $a(\psi_t, \chi)$ in the expansion (7.13) for ψ_t are real and positive. (An explicit formula for ψ_t is given by Fegan [46].) Consequently, the operator $e^{-t\Delta}$ on $L^2(G)$ has trace given by

(8.5) $$\mathrm{Tr}\{e^{-t\Delta}\} = \psi_t(e).$$

The first step in the proof of (8.1) is provided by:

Proposition 8.3.

(8.6) $$\eta(D_M, \bar{\alpha}, \varrho) = \lim_{t \to 0} \psi_t(e)^{-1} \cdot sf(D_M, \bar{\alpha}, \varrho, \psi_t).$$

Proof. By (7.39) and (8.4) it will suffice to prove

(8.7) $$\psi_t(e)^{-1} \cdot |\tilde{\eta}(D_M, \alpha, u^*\psi_t u) - \tilde{\eta}(D_M, \alpha, \psi_t)|$$

tends to zero with t. We consider the kernels $k_{\psi_t}^*$ and $u^* \circ k_{\psi_t}^* \circ u$ acting on $C^\infty(E \otimes \varepsilon^N)$ via convolution with the right G-action on V. By (7.22), for $y \in V$ and $g \in G$,

(8.8) $$k_{\psi_t}^*(y, y \cdot g) = \sum_\chi a(\psi_t, \chi) \cdot N(\chi) \cdot \chi(g^{-1}),$$

$$(8.9)\quad u^*k^*_{\psi_t}u(y,y\cdot g)=u(\varrho(y))^*\cdot\sum_\chi a(\psi_t,\chi)\cdot N(\chi)\cdot\chi(g^{-1})\cdot u(\varrho(y))^*\cdot u(\varrho(yg))$$
$$=\sum_\chi a(\psi_t,\chi)\cdot N(\chi)\cdot\chi(g^{-1})\cdot u(\varrho(g))$$

as $u(\varrho)$ is a right G-map. The calculation (8.9) is the manifestation of (8.3) for the operator D_V, and implies that the expression (8.7) is the scaled difference between the eta-invariants of D_M coupled to two families of flat bundles, so that (8.7) is equal to

$$(8.10)\quad \psi_t(e)^{-1}\cdot\left|\sum_\chi a(\psi_t,\chi)\{\eta(D_M\otimes V^{\chi\circ\alpha}\otimes V^{\varrho\circ\alpha})-\eta(D_M\otimes V^{\chi\circ\alpha}\otimes V^{\mathrm{Id}})\}\right|.$$

By the Cheeger-Gromov estimate (7.2), the expression (8.10) is estimated by

$$(8.11)\qquad \psi_t(e)^{-1}\cdot c(D_M)\cdot|\mathrm{Tr}\{\varrho^*\cdot k_{\psi_t}\cdot\varrho-k_{\psi_t}\}|$$

where k_{ψ_t} acts via convolution on $L^2(G,\mathbf{C}^N)$, and ϱ is left multiplication on $\mathbf{C}^N$ by $\varrho(g)$ for $g\in G$. The Fölner condition (6.5) implies that

$$(8.12)\qquad \mathrm{Tr}\{\varrho^*\cdot\mathfrak{k}_{\psi_t}\cdot\varrho-\mathfrak{k}_{\psi_t}\}=\|[\mathfrak{k}_{\psi_t},\varrho]\|_1=o(\psi_t(e))$$

so that (8.10) tends to zero, which completes the proof of (8.6). □

Remark 8.4. A novel point about (8.6) is that the left-hand side is a real number, while the right-hand side is a scaled (renormalized) integer-valued spectral flow of eigenvalues of D_V through the origin, with the scaling set according to the $\hat{G}$-decomposition. □

Remark 8.5. Let us now justify our notation that (6.5) is a "Fölner condition on $L^2(\hat{G})$". For $t>0$ small, the operator $e^{-t\Delta}$ is almost a projection onto the "lower energy" eigenstates of Δ. Introduce the decomposition of $\hat{G}$ into sets

$$\hat{E}(t,\varepsilon)=\{\chi\in\hat{G}|e^{-t\Delta}\circ P_\chi\geqq(1-\varepsilon)P_\chi\}$$
$$\hat{P}(t,\varepsilon)=\{\chi\in\hat{G}|\varepsilon\cdot P_\chi<e^{-t\Delta}\cdot P_\chi<(1-\varepsilon)P_\chi\}$$
$$\hat{K}(t,\varepsilon)=\{\chi\in\hat{G}|e^{-t\Delta}\circ P_\chi\leqq\varepsilon\cdot P_\chi\}.$$

There is a corresponding decomposition

$$L^2(G)=E(t,\varepsilon)\oplus P(t,\varepsilon)\oplus K(t,\varepsilon)$$

and the operator gives "weights" to each of these subspaces, so that we can think of $e^{-t\Delta}$ as determining a "tile" $\hat{E}(t,\varepsilon)\subseteq\hat{G}$ with penumbra $\hat{P}(t,\varepsilon)$ in the sense of Roe [74]. Then the formulas (6.5) and (8.12) imply that the action of the unitary multiplier, ϱ, on $L^2(G)\cong L^2(\hat{G})$ moves the weighted "tile" $e^{-t\Delta}$ by an amount which is negligible relative to the total mass of $e^{-t\Delta}$ as $t\to 0$. This is exactly parallel to the idea of the Fölner condition for foliations (cf. [48], [79]), and should be compared to Ocneanu's construction of Fölner sets for $\hat{G}$ when $L^2(\hat{G})$ is given a co-H-space structure [68, 69]. □

Recall that the densely defined operator D_V on the Hilbert space $\mathscr{H}=L^2(E)$ extends to $\mathscr{H}^N=L^2(E\otimes\varepsilon^N)$ as an essentially self-adjoint operator with pure-point spectrum by Proposition 7.1. Let P^+, respectively P^-, denote the projection onto the non-negative, respectively negative, eigenspaces of D_V in $\mathscr{H}^N$. Set

(8.13) $$\mathscr{E}=P^+-P^-,$$

an involution on $\mathscr{H}^N$. These operators commute with the right G-module action on $\mathscr{H}^N$, and hence commute with the operators k_ψ^* for $\psi\in R_\infty(G)$.

Let $q=2r-1>m=\dim M$. For $u=u(\varrho): V\to U_N$ considered as a multiplication operator on $\mathscr{H}^N$ and $\psi\in R_\infty(G)$, set

(8.14) $$\hat{c}_V(u(\varrho),\psi)=-(-4)^{-r}\cdot\mathscr{E}\cdot\{[\mathscr{E},u][\mathscr{E},u^*]\}^r\circ k_\psi^*.$$

Proposition 8.6. *The operator* (8.14) *is trace-class on* $\mathscr{H}^N$, *and*

(8.15) $$\operatorname{Tr}[\hat{c}_V(u(\varrho),\psi)]=sf(D_M,\bar{\alpha},\varrho,\psi).$$

Proof. We first prove that (8.14) is trace-class by restricting to k_χ^* for $\chi\in\hat{G}$, and then show that (8.14) is equivalent to a ψDO of order $-q$ on M, from which the claim follows from the Sobolev Lemma and standard methods.

With notation as in Sect. 7, the operator D_V restricted to $L^2(E\otimes\varepsilon^N,\chi)$ is isomorphic to

(8.16) $$D(\chi)\otimes I_N\cong D_M\otimes\nabla^{\lambda_\chi\circ\alpha}\otimes\nabla^0$$

acting on $C^\infty(E_M\otimes E(\lambda_\chi\circ\alpha)\otimes\varepsilon^N)$ by Proposition 7.5. The conjugate operator $u\circ D_V\circ u^*$ restricted to $L^2(E\otimes\varepsilon^N,\chi)$ is isomorphic to

(8.17) $$D^\varrho(\chi)\cong D_M\otimes\nabla^{\lambda_\chi\circ\alpha}\otimes\nabla^{\overline{\varrho\circ\alpha}}$$

by Proposition 7.10. Both operators (8.16) and (8.17) are elliptic with the same principal symbols, as they differ only by the choice of a flat connection acting on ε^N. By ellipticity, all of the opeerators

(8.18) $$\begin{cases}\mathscr{E}(\chi)=\mathscr{E}\circ k_\chi^*\\ \mathscr{E}_u(\chi)=u\circ\mathscr{E}\circ u^*\circ k_\chi^*\\ P^+(\chi)=P^+\circ k_\chi^*\\ P^-(\chi)=P^-\circ k_\chi^*\end{cases}$$

are pseudo-differential of order 0 on M. Note that $\mathscr{E}(\chi)$ is the involution associated to $D(\chi)$, and $\mathscr{E}_u(\chi)$ is the involution associated to $D^\varrho(\chi)$. Thus, $\mathscr{E}_u(\chi)-\mathscr{E}(\chi)$ has order -1 on M, and the same holds for

(8.19) $$P^+(\chi)\circ(\mathscr{E}_u(\chi)-\mathscr{E}(\chi))\circ P^+(\chi)$$

(8.20) $$P^-(\chi)\circ(\mathscr{E}_u(\chi)-\mathscr{E}(\chi))\circ P^-(\chi).$$

For the grading of $\mathscr{H}^N$ induced by $\mathscr{E}$, introduce the matrix notation

(8.21) $$u=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\quad u^*=\begin{bmatrix}A^* & C^*\\ B^* & D^*\end{bmatrix}\quad \mathscr{E}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

The identities $uu^*=\mathrm{Id}=u^*u$ imply that

(8.22) $$AA^*+BB^*=\mathrm{Id}^+\ ;\quad CC^*+DD^*=\mathrm{Id}^-$$

(8.23) $$A^*A+C^*C=\mathrm{Id}^+\ ;\quad B^*B+D^*D=\mathrm{Id}^-.$$

In this notation, using (8.22) and (8.23) the operators (8.19) and (8.20) become

$$-2BB^* \circ k_\chi^* \tag{8.24}$$

$$2CC^* \circ k_\chi^* \tag{8.25}$$

respectively. A symbol calculation as used in the proof of Lemma 6.10 shows that the Schatten $(m+1)/2$-norms of (8.24) and (8.25) are estimated by a polynomial in the weight of the character χ.

Using the identies (8.22) and (8.23), the operator (8.14) reduces to

$$-\{(BB^*)^r - (CC^*)^r\} \circ k_\psi^* \tag{8.26}$$

from which the trace class assertion follows using the previous remarks.

Define the difference operator

$$(u \circ P^+ \circ u^* - P^+) \circ k_\chi^* \equiv K_u(\chi). \tag{8.27}$$

By the previous remarks, this is a pseudo-differential operator of order -1 on $C^\infty(E_M \otimes E(\lambda_\chi \circ \alpha) \otimes \varepsilon^N)$, hence is compact.

It remains to identify the trace of (8.14) with a discrete spectral flow. Introduce subspaces

$$\begin{cases} SF^+(u) = (\mathscr{H}^N)^+ \cap u^*(\mathscr{H}^N)^- \\ SF^-(u) = (\mathscr{H}^N)^- \cap u^*(\mathscr{H}^N)^+ \end{cases} \tag{8.28}$$

with corresponding orthogonal projections

$$\begin{cases} P_u^+ : \mathscr{H}^N \to SF^+(u) \\ P_u^- : \mathscr{H}^N \to SF^-(u) \end{cases} \tag{8.29}$$

Lemma 8.7.

$$\operatorname{Tr}[\tilde{c}_V(u(\varrho), \psi)] = \operatorname{Tr}\{P_u^+ - P_u^-\} \circ k_\psi^* \tag{8.30}$$

Proof. Note that the operator

$$A^* = P^+ \circ u \circ P^+ \circ u^* \circ P^+$$

is self-adjoint, is G-invariant as $u = u(\varrho)$ is multiplication by a character of G, and the restrictions $AA^* \circ k_\chi^*$ to the range of $P^+(\chi)$ are elliptic. Thus, AA^* has pure-point spectrum, and by (8.22) the same holds for BB^*. Similarly, CC^* and DD^* have pure-point spectrum.

Let $0 \neq v \in \mathscr{H}^N$ satisfy $BB^*v = \lambda v$. For $\lambda = 1$,

$$v = BB^*v = P^+ \circ u \circ P^- \circ u^* \circ P^+(v) \tag{8.31}$$

implies that $v \in (\mathscr{H}^N)^+$ and $u^*v \in (\mathscr{H}^N)^-$, or $v \in SF^-(u)$. For $0 < \lambda < 1$, calculate using (8.22) and (8.23),

$$\begin{aligned} CC^*(CA^*v) &= C(\mathrm{id}^* - A^*A)A^*v \\ &= \lambda \cdot (CA^*v) \end{aligned} \tag{8.32}$$

so that CA^* maps the range of BB^* corresponding to $\lambda < 1$ onto the range of CC^* with $\lambda < 1$, with kernel $SF^-(u)$. The adjoint AC^* has kernel $SF^+(u)$, so that the trace of (8.26) is the difference of the traces of $P_u^+ \circ k_\psi^*$ and $P_u^- \circ k_\psi^*$, proving (8.30). □

To conclude the proof of Proposition 8.6, we recall the definition of spectral flow from (§ 7, [6]) and show that it is equivalent to the right-hand side of (8.30). Let

$$\tilde{D}_t(\chi) = t \cdot D^{\varrho}(\chi) + (1-t) D(\chi) \otimes I_N = D(\chi) \otimes I_N + t \cdot K_u(\chi) \tag{8.33}$$

be the family of self-adjoint operators used to define $sf(D_M, \bar{\alpha}, \varrho, \chi)$ in (8.4). This family is unitarily conjugate to a smooth family of elliptic differential operators on $C^\infty(E_M \otimes E(\lambda_\chi \circ \alpha) \otimes \varepsilon^N)$, so we can *continuously* label the eigenvalues

$$\{\ldots, \lambda_{-1}(t), \lambda_0(t), \lambda_1(t), \ldots\} \tag{8.34}$$

where

$$\begin{cases} \lambda_{-1}(0) < 0 \leqq \lambda_0(0) \\ \lambda_i(0) \quad \leqq \lambda_{i+1}(0) \end{cases} \tag{8.35}$$

and the corresponding orthonormal eigenvectors $\{\ldots, v_{-1}(t), v_0(t), v_1(t), \ldots\}$ *also* depend continuously on t.

Define subsets of the integers,

$$I^+(\chi) = \{i \geqq 0 \mid \lambda_i(1) < 0\}$$
$$I^-(\chi) = \{i < 0 \mid \lambda_i(1) \geqq 0\} \tag{8.36}$$

The difference in cardinalities,

$$\# I^+(\chi) - \# I^-(\chi) \tag{8.37}$$

counts the number of negative eigenvalues of $D(\chi)$ which turn positive in $D^{\varrho}(\chi)$, minus the number of positive eigenvalues which become negative. The sets $\{v_i(0) \mid i \in I^{\pm}(\chi)\}$ are clearly bases for the ranges of $P_u^{\pm} \circ k_\chi^*$. Thus, we obtain

$$\mathrm{Tr}\,[\tilde{c}_V(u(\varrho), \psi)] = \sum_\chi a(\psi, \chi) \{\# I^+(\chi) - \# I^-(\chi)\} \tag{8.38}$$

from (8.30). On the other hand, the discussion of (§ 7, [6]) identifies the negative of (8.37) with $sf(D_M, \bar{\alpha}, \varrho, \chi)$, which concludes the proof of (8.15). □

For $\psi = \chi \in G$, the right-hand side of (8.15) is an integer called the *essential codimension* of $u \circ \mathscr{E} \circ u^* \circ k_\chi^*$ relative to $\mathscr{E} \circ k_\chi^*$. This invariant was introduced by Brown, Douglas and Filmore [14], and rediscovered in a context similar to the present discussion by Wojciechowski (§ 4, [90]). Finally, the essential codimension can be formulated as the integer-valued index for appropriate Kasparov bimodules, which is the key to the final step in the proof of (8.1).

Let $\{k_l\}$ be the F.A.I. for G constructed from the heat kernels $h_l = \exp(-l^{-1} \cdot \Delta)$ by multiplying with a symmetric cut-off function. Then there exist central functions $\psi_l \in R_\infty(G)$ so that $k_l = k_{\psi_l}$. Combining Propositions 8.3 and 8.6, and Corollary 6.12, formula (8.1) follows from:

Lemma 8.8.

$$\lim_{l \to \infty} \psi_l(e)^{-1} \cdot \mathrm{Tr}_{\mathscr{H}^N}\{\mathscr{E}([\mathscr{E}, u][\mathscr{E}, u^*])^r \circ k_{\psi_l}^*\} = \lim_{l \to \infty} \psi_l(e)^{-1} \cdot \mathrm{Tr}_{\mathscr{H}^{4N}}\{\Pi_V([\Pi_V, u][\Pi_V, u^*])^r \circ k_{\psi_l^*}\}. \tag{8.39}$$

This is a consequence, in turn, of

Lemma 8.9. *For* $\chi \in \hat{G}$,

$$\text{(8.40)} \quad \mathrm{Tr}_{\mathscr{H}^N}\{\mathscr{E}([\mathscr{E},u][\mathscr{E},u^*])^r \circ k_\chi^*\} = \mathrm{Tr}_{\mathscr{H}^{4N}}\{\Pi_V([\Pi_V,u][\Pi_V,u^*])^r \circ k_\chi^*\}.$$

Proof. The two sides of (8.40) are algebraically identical, but differ in the Hilbert module structure and the choice of involution. The idea of the proof is to identify each side of (8.40) with the index of a Kasparov bimodule, then show the defining data for the two bimodules agree modulo compacts, so that they have the same indices.

Recall the definition of the Hilbert modules over $C^\infty(V)$:

$$\text{(8.41)} \quad \begin{cases} \mathscr{H}(\lambda_\chi \otimes I_N) \equiv L^2(E_M \otimes \varepsilon^4 \otimes E((\lambda_\chi \otimes I_N) \circ \alpha)) \\ \mathscr{H}(\lambda_\chi \otimes \varrho) \equiv L^2(E_M \otimes \varepsilon^4 \otimes E((\lambda_\chi \otimes \varrho) \circ \alpha)) \end{cases}$$

In the following, let "$\cdot$" denote either the representation $(\lambda_\chi \otimes I_N) \circ \alpha$ or $(\lambda_\chi \otimes \varrho) \circ \alpha$. The module action of $C^\infty(M)$ on $\mathscr{H}(\cdot)$ is the natural fiberwise diagonal action on $L^2(E_M \otimes E_M(\cdot))$ extended to $\mathscr{H}(\cdot)$ by the rule (3.19) with respect to the factor ε^4.

The operators D_M and its "phase" Φ_M acting on $C^\infty(E_M)$ extend to essentially self-adjoint operators on $C^\infty(E_M \otimes E(\cdot))$, denoted respectively by $D(\cdot)$ and $\Phi(\cdot)$. To define these extensions, note that it suffices to do so locally. For a section s supported in an open set $C \subset M$ of diameter $\frac{1}{4} \cdot c_1$, we can choose a contractible open set U containing C so that U contains a $\frac{1}{4} \cdot c_1$-open neighborhood of C. The flat bundle $E(\cdot)$ has an Hermitian trivialization over U, so that D_M and Φ_M extend diagonally to define $D(\cdot)(s)$ and $\Phi(\cdot)(s)$. Their support is again contained in U by our requirement on the supports of the distributional kernels of D_M and Φ_M (cf. the detailed construction in Appendix B.) We leave it to the reader to check that via the isomorphisms of Proposition 7.10, the operators D_V and Φ_V restrict to $C^\infty(E_M \otimes E(\cdot))$ to give $D(\cdot)$ and $\Phi(\cdot)$, respectively.

Let $\Pi(\cdot)$ denote the involution of $\mathscr{H}(\cdot)$ constructed from $\Phi(\cdot)$ via the 4×4 trick of Sect. 3, with corresponding eigenspace projections $\Pi^\pm(\cdot)$.

Let $P^\pm(\cdot)$ denote the projections onto the non-negative (respectively, negative) eigenspaces of $D(\cdot)$, acting densely in $L^2(E_M \otimes E(\cdot))$. Define projections on $\mathscr{H}(\cdot)$ by

$$\text{(8.42)} \quad \mathscr{E}^\pm = \begin{bmatrix} P^\pm(\cdot) & 0 & 0 & 0 \\ 0 & P^\mp(\cdot) & 0 & 0 \\ 0 & 0 & P^\mp(\cdot) & 0 \\ 0 & 0 & 0 & P^\pm(\cdot) \end{bmatrix}$$

and set $\mathscr{E}(\cdot) = \mathscr{E}^+(\cdot) - \mathscr{E}^-(\cdot)$.

Note that both $\Phi(\cdot)$ and the involution $P^+(\cdot) - P^-(\cdot)$ are ψDO's of order 0 with the same principal symbol. Thus, by formula (3.20) the difference $\Pi(\cdot) - \mathscr{E}(\cdot)$ is a ψDO on $\mathscr{H}(\cdot)$ of order -1.

Introduce the $C^0(M)$-module

$$\text{(8.43)} \quad \mathscr{H}(\chi, \varrho) = \mathscr{H}(\lambda_\chi \otimes I_N) \oplus \mathscr{H}(\lambda_\chi \otimes \varrho).$$

Proposition 7.10 implies that multiplication by $u = u(\varrho)$ defines an isomorphism from $\mathscr{H}(\lambda_\chi \otimes I_N)$ to $\mathscr{H}(\lambda_\chi \otimes \varrho)$. Extend the projections $\mathscr{E}^\pm(\cdot)$ and $\Pi^\pm(\cdot)$ to act on the appropriate summands in $\mathscr{H}(\chi, \varrho)$. Then define

$$T_0 = \mathscr{E}^+(\lambda_\chi \otimes \varrho) \circ M_u \circ \mathscr{E}^+(\lambda_\chi \otimes I_N) \tag{8.44}$$

$$T_1 = \Pi^+(\lambda_\chi \otimes \varrho) \circ M_u \circ \Pi^+(\lambda_\chi \otimes I_N). \tag{8.45}$$

Use the grading ε_0 (respectively ε_1) of $\mathscr{H}(\chi, \varrho)$ induced by $\mathscr{E}$ (respectively Π) to define for $i=0$ (respectively $i=1$)

$$F_i = \begin{bmatrix} 0 & T_i^* \\ T_i & 0 \end{bmatrix} \tag{8.46}$$

Lemma 8.10. *For $i=0$ and 1, the pair $(\mathscr{H}(\chi, \varrho), F_i)$ determines a Kasparov cycle*

$$[\mathscr{H}(\chi, \varrho), F_i] \in E_0(\mathbf{C}, \mathbf{C}) \tag{8.47}$$

and their equivalence classes are equal in $KK(\mathbf{C}, \mathbf{C}) \cong \mathbf{Z}$.

Proof. The self-adjoint involutions $\mathscr{E}$ and Π are ψDO, and their principal symbols commute with the unitary M_u. By standard methods [62] it follows that (8.47) is a cycle. Moreover, the principal symbols of $\mathscr{E}$ and Π agree so that $\mathscr{E} - \Pi$ is a compact operator, hence $(F_1 - F_0) \in \mathscr{K}(\mathscr{H}(\chi, \varrho))$ which implies the equality of these cycles in $KK(\mathbf{C}, \mathbf{C})$. □

Finally, to establish (8.40) we remark that the method of proof for Proposition 8.6 also yields (cf. Proposition 5, page 91 and Theorem 5d), page 75 of [30]):

Lemma 8.11. *The identification $KK(\mathbf{C}, \mathbf{C}) \cong \mathbf{Z}$ maps the class of $[\mathscr{H}(\chi, \varrho), F_0]$ to the left-hand side of* (8.40), *and maps the class of $[\mathscr{H}(\chi, \varrho), F_1]$ to the right-hand side of* (8.40). □

Appendix A: Classifying spaces and the *KK*-eta-invariant

In this appendix we discuss the bordism invariance of the longitudinal cyclic Chern character, and the remark of A. Connes that our main theorem implies there is a Kasparov *KK*-class representing the relative eta-invariant. We begin with a discussion of the topological aspects of flat-bundle classifying spaces. This is standard material in the foliation literature (cf. pages 58 to 60, [63]). We then discuss an unstabilized version of the geometric *K*-homology groups, which has been further investigated in [55]. Finally, using the Baum-Connes μ-map [10] and *KK*-pairings, we obtain the promised *KK*-eta-invariant.

Let G be a connected Lie group, and let G^δ denote the same group equipped with the discrete topology. The Milnor join construction for G (cf. [65], [50]) defines a connected space BG which classifies principal G-bundles. The same construction applied to G^δ yields a connected topological space BG^δ which is a $K(G, 1)$. That is, $\pi_1(BG) \cong G$ and $\pi_i(BG) = 0$ for $i > 1$. The inclusion $i: G^\delta \to G$ induces a continuous map $Bi: BG^\delta \to BG$. As sets, both of these spaces are the same, with the source having a finer topology than the range. One says that this is a space with two topologies (cf. [66]). The difference in the two topologies is measured by introducing the homotopy fiber $B\bar{G}$. This is defined by first replacing Bi with a homotopy-equivalent weak fibration over BG, then take for $B\bar{G}$ (the homotopy class of) the fiber. The description below of this space is just the first step of the construction of the Puppe Sequence for Bi (cf. Chapter III, § 6, [88]).

Choose a basepoint $* \in BG^\delta$, and also let $* \in BG$ denote its image. The space of continuous paths in BG with initial point $*$ is denoted by $P(BG)$, where for a path γ the endpoint is denoted $e(\gamma) \in BG$. There is a fibration

(A1) $$\Omega(BG) \longrightarrow P(BG) \xrightarrow{e} BG$$

where $\Omega(BG)$ is the space of $*$-based loops and has the weak-homotopy-type of G. Define $B\bar{G}$ via the homotopy pull-back diagram

(A2) $$\begin{array}{ccc} \Omega(BG) & \longrightarrow & \Omega(BG) \\ \downarrow & & \downarrow \\ B\bar{G} & \longrightarrow & P(BG) \\ \downarrow & & \downarrow e \\ BG^\delta & \xrightarrow[Bi]{} & BG \end{array}$$

A principal G-bundle, $G \longrightarrow P \xrightarrow{\pi} M$, over a manifold M is equivalent to giving an open covering $\mathscr{U} \equiv \{U_i | i \in \mathfrak{I}\}$ of M and for each non-empty intersection, a continuous map $g_{ij} : U_i \cap U_j \to G$ satisfying the cocycle law

(A3) $$g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x) \quad \text{for} \quad x \in U_i \cap U_j \cap U_k .$$

This data defines a continuous map $g_P : M \simeq |\mathscr{U}| \to BG$ (cf. [50]). If the functions g_{ij} are locally constant, then g_P can be factored through BG^δ as a *continuous* map.

A choice of transition functions $\{g_{ij}\}$ which are locally constant is equivalent to specifying a flat G-structure on P. Each product $U_i \times G$ has a G-invariant foliation by the horizontal slices $U_i \times \{g\}$. The bundle P is obtained from the disjoint union $\coprod_{\mathfrak{I}} U_i \times G$ by identifying pairs $(x_i, g) \sim (x_j, g_{ij}(x_i) g)$ where $x_i \in U_i$ and $x_j \in U_j$ correspond to the same point $x \in M$. When the functions g_{ij} are locally constant, the foliations on $U_i \times G$ are preserved under this identification to yield a foliation $\mathscr{F}_\alpha$ on P whose holonomy map $\alpha : \pi_1(M) \to G$ defines the classifying map $B\alpha : M \to BG^\delta$.

Conversely, given a continuous map $B\alpha : M \to BG^\delta$, there is induced a representation $\alpha : \pi_1(M) \to G$ and a corresponding flat principal G-bundle, $P_\alpha = \tilde{M} \times_\Gamma G$, as in Sect. 2.

The topological type of the G-bundle $P_\alpha \to M$ is determined by the composition

(A4) $$g_\alpha : M \xrightarrow{B\alpha} BG^\delta \xrightarrow{Bi} BG .$$

The principal bundle is trivial if and only if g_α is homotopic to the constant map $M \to *$. The choice of such a homotopy, say

(A5) $$\{g_{\alpha,t} | 0 \leqq t \leqq 1 ; \, g_{\alpha,0}(x) \equiv * ; \, g_{\alpha,1} = g_\alpha\}$$

defines a map $g_\alpha^\# : M \to P(BG)$,

(A6) $$g_\alpha^\#(x)(t) = g_{\alpha,t}(x) ,$$

and the definition of $B\bar{G}$ as a pull-back implies there is an induced lift

(A7) $$\begin{array}{ccc} & B\bar{\alpha} & B\bar{G} \\ & \nearrow & \downarrow \\ M & \xrightarrow[B\alpha]{} & BG^\delta \end{array}$$

Conversely, a lift $B\bar{\alpha}$ of $B\alpha$ in (A7) determines a homotopy $\{g_{\alpha,t} | 0 \leqq t \leqq 1\}$ from g_α to the constant map (directly by the definition of $B\bar{G}$!)

One final remark about $B\bar{G}$ is that a choice of homotopy $\{g_{\alpha,t}\}$ is equivalent to specifying a global section of P_α. Given a homotopy, for each t there is a canonical bundle $P_t \to M$ with $P_0 \equiv M \times G$ and $P_1 \equiv P$. Choose a connection on the total bundle $\{P_t\} \to M \times I$, and use the parallel transport over the curves $\{x\} \times I$ to define $s(x) \in P_1$ as the endpoint at $t=1$ of the horizontal curve starting at $\{(x,e)\} \times \{0\} \in P_0$. The homotopy class of the section $s: M \to P_1 \equiv P$ is independent of the connection chosen.

The use of the connection in the above construction avoids the homotopy-theoretic approach necessary for showing the converse: the space $\Omega(BG)$ of based loops is a homotopyequivalent retract of G considered as the canonical paths in the inclusion of the suspension $\Sigma G \subset BG$. A choice of retraction defines a weak-homotopy equivalence of bundles

$$\text{(A8)} \qquad \begin{array}{ccc} G & \longrightarrow & \Omega(BG) \\ \downarrow & & \downarrow \\ EG & \longrightarrow & P(BG) \\ \downarrow & & \downarrow \\ BG & = & BG \end{array}$$

A section e_α of $P_\alpha \to M$ is equivalent to specifying $e_\alpha : M \to EG$, as P_α is the pull-back of EG via g_α. The composition

$$M \xrightarrow{e_\alpha} EG \longrightarrow P(BG)$$

induces a section $B\bar{\alpha}: M \to B\bar{G}$ of the pull-back (A2).

The above discussion is the homotopy theoretic basis for writing $\bar{\alpha} = (\alpha, \Theta)$ where $\Theta: P_\alpha \cong M \times G$ is the trivialization that induces e_α above and hence $B\bar{\alpha}$.

The based loops $\Omega(BG)$ act naturally on $B\bar{G}$ via their action on the fiber of $B\bar{G} \to BG^\delta$. Given

$$\text{(A9)} \qquad B\bar{\alpha}: M \to B\bar{G} \quad \text{and} \quad f: M \to G \simeq \Omega(BG)$$

we can define their product

$$\text{(A10)} \qquad B\bar{\alpha}^f : M \to B\bar{G} \times \Omega(BG) \to B\bar{G}.$$

In terms of the product structure Θ, we are composing Θ with the gauge automorphism $f_*: M \times G \to M \times G$ induced by f. The space of maps $\{M, G\}$ is called the *gauge group*. The countable group

$$\pi_0(\{M, G\}) \equiv [M, G]$$

of pointed homotopy classes then acts on the equivalence classes of trivialized foliated G-bundles over M. This action has a non-trivial effect on the analytic invariants constructed in this paper via spectral flow in the classical sense of [8]. This is illustrated in an example from [43], which we briefly recall.

Let $M = S^1$ with $\Gamma = \mathbf{Z}$, choose $\alpha \in \mathbf{R}$, and let $\alpha: \Gamma \to SO(2)$ also denote the representation obtained by sending 1 to the rotation by α. Then $V = \mathbf{R} \rtimes S$ is abstractly a two-torus, and the role of Θ is to provide an explicit realization $V \cong S^1 \times S^1$, under which $\mathscr{F}_\alpha$ can be assumed sent to the foliation of $S^1 \times S^1$ by lines of slope β. The numbers α and β are related by $\beta = \alpha + b$, for some integer b. The fiber preserving automorphism group

$$[S^1, S^1] \cong \mathbf{Z}$$

is realized in its action on V by the diagonal matrices

$$\left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbf{Z} \right\}$$

acting on $\mathbf{R}^2$ in the natural way, then passing to the quotient $S^1 \times S^1$. Without fixing Θ, the "slope" β is only well-defined modulo $\mathbf{Z}$, while a choice of Θ specifies α. All other choices are obtained from the action of the gauge group. This is the typical situation for all examples $P_\alpha \to M$.

For a topological space X, let $\Omega_m^U(X)$ denote the bordism group of oriented cycles $f: M \to X$, where M is a closed manifold of dimension m with an almost complex structure on TM. This determines a $\mathrm{Spin}^{\mathbf{C}}$-structure on TM and hence a (graded, for m even). Dirac operator, $\partial\!\!\!/$ acting on the spinors over M. The stabilization of $\Omega_m^U(X)$ under "vector-bundle modification" (cf. § 11, [11]) yields the Baum-Douglas topological K-homology group $K_{(m)}^t(X)$.

A cycle in $\Omega_m^U(B\bar{G})$ is equivalent to specifying a flat principal G-bundle $P_\alpha \to M$, up to bordism of flat G-bundles. The foliation index of a leafwise Dirac operator is a bordism invariant, using the cohomological formulation of the index. Thus, by Theorem 1.1 the relative eta-invariant gives a well-defined pairing for m odd:

(A11) $$\eta : \Omega_m^U(B\bar{G}) \times \mathrm{Rep}(G) \to \mathbf{R}\,.$$

The invariance of the (odd) chern character under vector bundle modification of a cycle (M, f) and the cohomological form of the index theorem implies that (A11) is preserved by stabilization to the group $K_1^t(B\bar{G})$.

One application of Theorem 1.1 is that the pairing (A11) is calculated via a von Neumann index [44], and generalizing this, A. Connes suggested reformulating (A11) in terms of analytic K-homology (cf. § 4, [9]). We first consider a particular form of this remark. Fix a cycle $G \to P_\alpha \to M$ for $\Omega_m^U(B\bar{G})$, with m odd and $\mathscr{F}_\alpha$ the flat foliation on P_α. The Dirac operator on M lifts to a leafwise geometric operator, $\partial\!\!\!/_\alpha$, along the leaves of $\mathscr{F}_\alpha$. The construction of Connes-Skandalis [34] yields an odd KK-class

(A12) $$[\partial\!\!\!/_\alpha] \in KK^1(C^0(P_\alpha), C^*(P_\alpha/\mathscr{F}_\alpha))\,.$$

A unitary $[u] \in K^1(G)$ is pulled-back by the trivialization $P_\alpha \xrightarrow{\Theta} M \times G \to U_N$ to give a class

(A13) $$[u \circ \Theta] \in KK^1(\mathbf{C}, C^0(P_\alpha))\,.$$

The Kasparov Pairing [62] (cf. Appendix B) defines a class

(A14) $$\begin{aligned}[u \circ \Theta] \boxtimes [\partial\!\!\!/_\alpha] &\in KK^0(\mathbf{C}, C^*(P_\alpha)) \\ &\cong K_0(C^*(P_\alpha/\mathscr{F}_\alpha))\,.\end{aligned}$$

Definition A1. The KK-eta-invariant of the cycle $(P_\alpha \to M, \Theta, D)$ is the element

(A15) $$\begin{aligned}E(\bar{\alpha}, u) &\equiv [u \circ \Theta] \boxtimes [\partial\!\!\!/_\alpha] \\ &\in K_0(C^*(P_\alpha/\mathscr{F}_\alpha))\,. \quad \square\end{aligned}$$

The justification for this definition is that for the trace Tr_μ on $C^*(P_\alpha/\mathscr{F}_\alpha)$ induced from Haar measure on G, we can calculate the value of $\mathrm{Tr}_\mu(E(\bar{\alpha}))$ via the Connes-Skandalis Index Theorem (cf. § 4, [34]). It is seen to be the topological index side of (1.1), and hence equal to the relative eta-invariant $\eta(\not{\partial}, \bar{\alpha}, \varrho)$ for the trivialized flat bundle $(P_\alpha \to M, \Theta, \not{\partial})$ and representation ϱ.

The particular case discussed above generalizes to a universal construction. The foliation algebra $C^*(P_\alpha/\mathscr{F}_\alpha)$ of a cycle in $\Omega_m^U(B\bar{G})$ is stably isomorphic to the cross-product algebra $C^0(G) \rtimes \Gamma$, and composing with the holonomy induces a map

(A16) $$h_\alpha : C^*(P_\alpha/\mathscr{F}_\alpha) \to (C^0(G) \rtimes G^\delta) \otimes \mathscr{K}\,.$$

The induced map on analytic K-homology is denoted $h_\alpha^!$, and functoriality yields

$$\mathrm{Tr}_\mu\{h_\alpha^!(E(\bar{\alpha}, \varrho))\} = \eta(\not{\partial}, \bar{\alpha}, \varrho)\,.$$

We thus obtain a universal map, for m odd:

(A17) $$\begin{cases} h_\varrho \Omega\, \Omega_m^U(B\bar{G}) & \to K_0(C^0(G) \rtimes G^\delta) \\ h_\varrho(P_\alpha \to M, \Theta, \not{\partial}) = h_\alpha^!(E(\bar{\alpha}, \varrho)) & \end{cases}$$

which composes with Tr_μ to yield the relative eta-invariants of cycles.

Appendix B: Longitudinal index and evaluation of odd degree cocycles

Let c_Φ be the odd degree longitudinal cyclic cocycle over $C^\infty(V)$ of Sect. 3. For a smooth unitary $u: V \to U_N$, set

(B1) $$c_\Phi(u) = c_\Phi(u, u^*, \ldots, u, u^*)\,.$$

We evaluate $c_\Phi(u)$ in terms of the foliation index theorem of Connes, and show that $c_\Phi(u)$ equals the Breuer index of an appropriate Breuer Fredholm operator for the foliation von Neumann algebra. This will complete the proof of Theorem 4.3.

The first step is to characterize (B1) as the "index" associated to the boundary map of an extension of C^*-algebras. Introduce the field of Hilbert spaces

$$\mathscr{H}^{4N} = \mathscr{H}^4 \otimes \mathbf{C}^N \equiv L^2(E \otimes \varepsilon^4 \otimes \varepsilon^N)$$

equipped with a scalar action of $C^0(V)$ on $\mathscr{H}^4$ by (3.18), extended via the diagonal action on $\mathbf{C}^N$. The leafwise $\not{\psi}$DO, Π, of (3.20) extends diagonally to $\mathscr{H}^{4N}$.

Define an extension of C^*-algebras

(B2) $$0 \to C^*(V/\mathscr{F}) \to \mathscr{T}_\Pi^+ \xrightarrow{\sigma} M(N, C^\infty(V)) \to 0$$

where $\mathscr{T}_\Pi^+$ is the subalgebra of bounded operators on the field of Hilbert spaces $\mathscr{H}^{4N}$, generated by $C^*(V/\mathscr{F})$ and the leafwise operators of order 0 of the form

(B3) $$\{T_\varphi^+ = \Pi^+ \circ M_\varphi \circ \Pi^+ \,|\, \varphi: V \to M(N, \mathbf{C})\}\,.$$

Of course, $\Pi^+ = \frac{1}{2}(1 + \Pi)$ denotes the leafwise projections onto the non-negative eigenspaces of the leafwise operators Π, which forms a projection on $\mathscr{H}^N$.

The extension (B2) determines a K-theory boundary map

(B4) $$\partial^+ : K^1(V) \to K_0(C^*(V/\mathscr{F}))\,.$$

Compose (B4) with the foliation trace to obtain

(B5) $$\gamma_\mu^+ = \mathrm{Tr}_\mu \circ \partial^+ : K^1(V) \to \mathbf{R}.$$

There is a similar index map, γ_μ^-, defined using the extension $\mathcal{T}_\Pi^-$ generated by $\Pi^- = \frac{1}{2}(1-\Pi)$.

Proposition B1. *For* $u: V \to U(N)$,

(B6) $$\gamma_\mu^+([u]) = -\gamma_\mu^-([u]) = c_\Phi(u).$$

Proof. Decompose $\mathcal{H}^{4N}$ into its $+/-$ eigenspaces for Π, then write the multiplication operators u and u^* in corresponding 2×2 matrix form

(B7) $$u = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad u^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}.$$

Note that $uu^* = u^*u \equiv \tilde{I}$ is not the identity on $\mathcal{H}^{4N}$, due to the peculiar action (3.18) of functions on $\mathcal{H}^4$, but it is an idempotent. Set

$$\tilde{I}^\pm = \Pi^\pm \circ \tilde{I} \circ \Pi^\pm.$$

From the definition (3.23) we obtain for $q = 2l-1$,

(B8) $$\begin{cases} c_\Phi(u) = -\mathrm{Tr}_\mu\{(BB^*)^l - (CC^*)^l\} \\ c_\Phi(u^*) = -\mathrm{Tr}_\mu\{(C^*C)^l - (B^*B)^l\} \end{cases}$$

so that subtracting yields

(B9) $$2c_\Phi(u) = -\mathrm{Tr}_\mu\{(BB^*)^l + (B^*B)^l - (C^*C)^l - (CC^*)^l\}.$$

A modification of Connes' proof of (Proposition 6, page 88, [30]) shows that the indices $\gamma_\mu^\pm([u])$ are calculated by

(B10) $$\begin{cases} \gamma_\mu^+([u]) = \mathrm{Tr}_\mu\{(\tilde{I}^+ - A^*A)^l - (\tilde{I}^+ - AA^*)^l\} \\ \gamma_\mu^-([u]) = \mathrm{Tr}_\mu\{(\tilde{I}^- - D^*D)^l - (\tilde{I}^- - DD^*)^l\}. \end{cases}$$

From $u^*u = \tilde{I} = uu^*$, we obtain the identities

$$A^*A + C^*C = \tilde{I}^+ = AA^* + BB^*$$

$$D^*D + B^*B = \tilde{I}^- = DD^* + CC^*$$

so that (B10) becomes

(B11) $$\begin{cases} \gamma_\mu^+([u]) = \mathrm{Tr}_\mu\{(C^*C)^l - (BB^*)^l\} \\ \gamma_\mu^-([u]) = \mathrm{Tr}_\mu\{(B^*B)^l - (CC^*)^l\} \end{cases}$$

The extensions $\mathcal{T}_\Pi^+$ and $\mathcal{T}_\Pi^-$ are inverses in the Kasparov Extension group (§7, [62]), $E^1(C(V), C^*(V/\mathcal{F}))$, as $\tilde{I}$ is an idempotent commuting with the action of the continuous functions, $C(V)$, on $\mathcal{H}^{4N}$ so we may apply the inverse construction (cf. page 21, [38]) to $\mathcal{T}_\Pi^+$ to obtain $\mathcal{T}_\Pi^-$. Thus,

$$\partial^+([u]) = -\partial^-([u]) \in K_0(C^*(V/\mathcal{F}))$$

and so $\gamma_\mu^+([u]) + \gamma_\mu^-([u]) = 0$. Subtracting the two lines of (B11) and comparing the result to (B9) yields (B6). □

In the above proof we considered the extension (B2) constructed from Φ as an element of Kasparov's Ext group, $E^1(C(V), C^*(V/\mathscr{F}))$. There is an isomorphism of this group with $KK^1(C(V), C^*(V/\mathscr{F}))$, given by Kasparov (Theorem 1, § 7, [62]). Let $[\Phi^+]$ denote the bivariant KK-class corresponding to (B2). Given the leafwise elliptic order 0 operator Φ, the construction of Connes and Skandalis (§ 4, [34]) yields a bivariant class

$$[\Phi] \in KK^1(C(V), C^*(V/\mathscr{F})).$$

A calculation involving Hilbert bimodules shows that $[\Phi^+]+[\Phi^-] \sim 0$ in bivariant KK-theory, and that $[\Phi] \sim [\Phi^+]-[\Phi^-]$. For details of these identifications, see (§ 2, [45]).

The Kasparov external product defines a pairing

(B12) $$K^1(V) \boxtimes KK^1(C(V), C^*(V/\mathscr{F})) \to K_0(C^*(V/\mathscr{F})).$$

In particular, we define via this pairing a KK-theory boundary map

(B13) $$\tilde{\partial}^+([u]) \stackrel{\text{def}}{=} [u] \boxtimes [\Phi^+] \in K_0(C^*(V/\mathscr{F})).$$

A basic result (Theorem 3, § 7, [62]) identifies the pairing map $\tilde{\partial}^+$ with the boundary map for the extension (B2), so that $\tilde{\partial}^+([u]) = \partial^+([u])$. Then by the KK-foliation index theorem of Connes and Skandalis, we can identify

(B14) $$\gamma_\mu^+([u]) = \mathrm{Tr}_\mu\{\tilde{\partial}([u])\}$$

with the topological index on the right-hand side of (4.9).

This conclusion can be shown by a direct approach, based on the method of (§ 20, [11]). We need only to identify the index of the operator $\Pi^+ \circ u \circ \Pi^+$ in terms of an element in $K_0(C^*(V/\mathscr{F}))$.

Let $S\mathscr{F}$ denote the unit cosphere bundle to the leaves of $\mathscr{F}$, a sphere bundle over V with fiber dimension $(m-1)$. Let $\tilde{E} \to S\mathscr{F}$ denote the lift of $E \to V$ to $S\mathscr{F}$. For each $(x, \xi) \in S\mathscr{F}$, the principal symbol $\sigma_\Phi(x, \xi)$ of Φ is an involution on the fiber $\tilde{E}_{(x,\xi)}$ which varies continuously, so defines a continuous direct sum decomposition $\tilde{E} = \tilde{E}^+ \oplus \tilde{E}^-$ into $+/-$ eigenspaces. Use the map u to define a new symbol map $\sigma_u^+ : S\mathscr{F} \to GL(E \otimes \varepsilon^N)$ by:

(B15) $$\sigma_u^+(x, \xi)(v, w) = \begin{cases} v \otimes u(x) \cdot w, & v \in \tilde{E}^+_{(x,\xi)} \\ v \otimes w, & v \in \tilde{E}^-_{(x,\xi)} \end{cases}$$

Let P_u^+ be a leafwise ψDO of order 0 as in Sect. 3 whose principal symbol is given by σ_u^+. The leafwise operator P_u^+ is invertible modulo the foliation C^*-algebra, $C^*(V/\mathscr{F})$, (cf. Proposition 7.12, [67]). It thus has an "index" defined as an element

$$\mathrm{Ind}(P_u^+) \in K_0(C^*(V/\mathscr{F})).$$

By abuse of notation, we let $[u] \boxtimes [\Phi^+]$ also denote $\mathrm{Ind}(P_u^+)$. This is justified by:

Proposition B2.

(B16) $$\mathrm{Ind}(P_u^+) = \partial^+([u]).$$

Proof. Introduce the extension

(B17) $$0 \to C^*(V/\mathscr{F}) \to \bar{\mathscr{P}}_0 \xrightarrow{\sigma} C(S\mathscr{F}) \to 0$$

where $\bar{\mathscr{P}}_0$ denotes the closure of the leafwise, order 0, ψDO's for $\mathscr{F}$ (cf. Proposition 7.16, [67]). The operator P_u^+ is a lift to $\bar{\mathscr{P}}_0$ of the symbol class $[\sigma_u^+] \in K^1(S\mathscr{F})$, so $\operatorname{Ind}(P_u^+) = \partial[\sigma_u^+]$. This index class can be calculated using any lift of the symbol class $[\sigma_u^+]$.

Consider the leafwise, order 0, ψDO for $\mathscr{F}$ defined by

(B18) $$\tilde{\Pi}^{\pm} = \tfrac{1}{2}(I \pm \Phi)$$

whose principal symbol defines the projection onto $\tilde{E}^{\pm}$. By the multiplicative property of symbols, the operators P_u^+ and

(B19) $$\tilde{T}_u^+ = \tilde{\Pi}^+ \circ u \circ \tilde{\Pi}^+ + \tilde{\Pi}^-$$

both represent lifts of σ_u^+, so

(B20) $$\operatorname{Ind}(P_u^+) = \operatorname{Ind}(\tilde{T}_u^+).$$

Next, note that $\partial^+([u]) = \operatorname{Ind}(T_u^+)$, where

(B21) $$\begin{cases} T_u^+ = \Pi^+ \circ u \circ \Pi^+ \\ \quad\;\; = (\Pi^+ \circ \tilde{I}) \circ u \circ (\tilde{I} \circ \Pi^+). \end{cases}$$

With the matrix notation of (3.20), consider the extension of Φ to

(B22) $$\tilde{\Phi} = \begin{bmatrix} \Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and let $\tilde{\Pi}^{\pm} = \frac{1}{2}(\tilde{I} \pm \tilde{\Phi})$ also denote the extension of $\tilde{\Pi}^{\pm}$. Then by (3.21) and (3.22), we have

$$(\Pi^+ \circ \tilde{I}) = (\tilde{I} \circ \tilde{\Pi}) = \tilde{\Pi}^+$$

modulo the operators $\mathscr{D}_{\psi}^{-1}(E^{4N}, \mathscr{F}, 3\delta, \infty)$ using the notation of Sect. 3. As these leafwise operators are in $C^*(V/\mathscr{F})$, we have from (B21) that

(B23) $$\begin{aligned} \operatorname{Ind}(T_u^+) &= \operatorname{Ind}(\tilde{\Pi}^+ \circ u \circ \tilde{\Pi}^+) \\ &= \operatorname{Ind}(\tilde{T}_u^+), \end{aligned}$$

completing the proof of (B16). □

Observe that (4.8) follows by combining (B6) and (B16). To derive (4.9), we apply Connes' measured foliation index theorem (cf. [27, 28] or §4, [76]) to the operator P_u^+ constructed for the proof of (B16). Recall that the symbol of P_u^+ is a smooth map

(B24) $$\sigma_u^+ : S\mathscr{F} \to GL(\tilde{E} \otimes \varepsilon^N).$$

Theorem. (Connes).

(B25) $$\begin{aligned} \operatorname{Ind}_\mu(P_u^+) &= \operatorname{Tr}_\mu(\operatorname{Ind}(P_u^+) \\ &= (-1)^m \langle \psi^{-1}(\operatorname{ch}^*(\sigma_u^+)) \cup \operatorname{Td}(F), [C_\mu] \rangle. \quad \square \end{aligned}$$

The Thom isomorphism

(B26) $$\psi : H^*(V) \to H^{*+m-1}(S\mathscr{F})$$

is a module map over $H^*(V)$ (cf. §2, [7]), so that

(B27) $$\begin{aligned}\psi^{-1}(\mathrm{ch}^*(\sigma_u^+)) &= \psi^{-1}(\mathrm{ch}^*(\tilde{E}^+) \cup \pi^* \mathrm{ch}^*[u]) \\ &= \psi^{-1}(\mathrm{ch}^*(\tilde{E})) \cup \mathrm{ch}^*[u].\end{aligned}$$

Substituting (B27) into (B25) then using the intermediate step (B16) we obtain (4.9).

The last result of this appendix equates $c_\Phi(u)$ with the Breuer index of a leafwise "Toeplitz" operator. For $P=D$ a first order geometric operator on leaves, this is discussed in detail in (§4, [45]). For our discussion here, we require the mild hypothesis that P is a leafwise, essentially self-adjoint ψDO of positive order. The idea of the following is to observe that for an operator $X_u \in \mathscr{T}_\Phi$ with $\sigma(X_u) = \sigma_u^+$, then the proof of Proposition B2 shows that

(B28) $$c_\Phi(u) = \mathrm{Tr}_\mu[\mathrm{Ind}(X_u)].$$

We then use the spectral theorem (leafwise for $\mathscr{T}$) and standard local properties of elliptic ψDO's to construct a family $\{X_u(t) | 0 < t \leqq \infty\}$ such that

(B29) $X_u(t)$ is leafwise ψDO for $t>0$,

(B30) $\sigma(X_u(t)) = \sigma_u^+$ for $t>0$, and

(B31) $X_u(t)$ converges strongly to $X_u(\infty)$, which is the Toeplitz operator associated to the leafwise positive spaces of P and the multiplier u.

By the normality of the foliation trace Tr_μ and an estimate on the differences $X_u(t) - X_u(t')$, we conclude from (B30) and (B31) that

$$c_\Phi(u) = \mathrm{Ind}_\mu(X(\infty)),$$

the latter being the Breuer index of $X(\infty)$ with respect to the state Tr_μ on $W^*(V/\mathscr{F}, \mu)$.

We require two facts from the theory of elliptic ψDO's. Let $\mathscr{B}_b(\mathbf{R})$ denote the algebra of bounded Borel functions and $\mathscr{B}_{bc}(\mathbf{R})$ the subalgebra of functions with compact support. For $f \in \mathscr{B}_b(\mathbf{R})$, define $f(P)$ via the spectral theorem.

(B32) For $f \in \mathscr{B}_{bc}(\mathbf{R})$, $f(P)$ is represented by a smooth distributional kernel k_f on the leaves of $\mathscr{F}$. (Caution: the leafwise kernels k_f will a priori only be measurable as a function of the transverse parameter, and within each leaf k_f need not have compact support around the diagonal.)

(B33) Let f be a smooth function on $\mathbf{R}$ such that $\lim_{x \to \pm\infty} f(x) = \pm 1$. Then $f(P)$ is a leafwise ψDO of order 0 with $\sigma(\mathrm{f(P)}) = \sigma_\Phi$.

Choose a smooth function ψ such that

$$\psi(x) = \begin{cases} +1 & \text{for } x \geqq 1 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x \leqq -1 \end{cases}$$

and ψ is monotone increasing for $-1 < x < 1$. For $\delta > 0$ let $\varphi(\delta)$ be the leafwise "kernel cut-off" function constructed in the proof of Lemma 3.3. Then for $s, t > 0$ set

(B34) $$\Phi(s,t)=\varphi(s)\cdot\psi(t\cdot P).$$

From (B33) and the proof of Lemma 3.3, $\Phi(s,t)$ is a leafwise ψDO with $\sigma(\Phi(s,t))=\sigma_\Phi$ and the distributional kernel of $\Phi(s,t)$ is supported in an s-diameter tube about the diagonals in the leaves of $\mathscr{F}$. Introduce

(B35) $$\tilde{\Pi}^{\pm}(s,t)=\tfrac{1}{2}(I\pm\Phi(s,t))$$

(B36) $$X_u(s,t)=\tilde{\Pi}^{+}(s,t)\circ u\circ\tilde{\Pi}^{+}(s,t)+\tilde{\Pi}^{-}(s,t).$$

Then each $\tilde{\Pi}^{+}(s,t)\in\mathscr{T}_\Phi^{+}$ and $\sigma(X_u(s,t))=\sigma_\Phi^{+}$, so that

(B.37) $$c_\Phi(u)=\mathrm{Tr}_\mu[\mathrm{Ind}(X_u(s,t))].$$

It remains to show that

(B38) $$\lim_{s,t\to\infty} X_u(s,t)\equiv X(\infty)$$

and the convergence is Cauchy in the Tr_μ norm. We gather into a Lemma the facts needed to prove these claims.

Lemma B3.

a) For $f\in\mathscr{B}_{bc}(\mathbf{R})$, $|\mathrm{Tr}_\mu f(P)|<\infty$

b) For $f, g\in\mathscr{B}_{bc}(\mathbf{R})$ with $0\leqq f\leqq g$ and μ is a non-negative transverse measure, then

$$0\leqq \mathrm{Tr}_\mu f(P)\leqq \mathrm{Tr}_\mu g(P).$$

c) Let $\{f_n|n=1,2,\ldots\}\subset\mathscr{B}_{bc}(\mathbf{R})$ satisfy $0\leqq f_{n+1}\leqq f_n$ and $\lim_{n\to\infty} f_n(x)=0$ for all x. Then

$$\lim_{n\to\infty}\mathrm{Tr}_\mu f_m(P)=0.$$

Proof. a) We can assume that $f\geqq 0$, so that $f(P)=[f^{1/2}(P)]^2$. Then by (Theorem 1.10, [67]) and (B32) we conclude that $f(P)$ is locally traceable. As V is compact and P depends continuously on the transverse parameter, the local traces are bounded Borel functions of the transverse parameter and hence $\mathrm{Tr}_\mu f(P)$ exists and is finite.

b) $f\geqq 0$ implies that $f(P)$ is a non-negative operator by the spectral theorem, and hence has positive local traces. Apply this remark to f, g and $(g-f)$ to obtain b).

c) Let k_n denote the leafwise kernel for $f_n(P)$. As the f_n are non-negative, the local traces are determined by the restrictions of the k_n to the diagonal, hence they are monotonically decreasing. Using the Gärdings inequality and the Sobolev lemma, we can conclude $k_n\to 0$ so that the result follows from the Dominated Convergence Theorem. □

Returning to the proof of (B38), note that the family

$$\psi(t,t')(x)=\psi(t\cdot x)-\psi(t'\cdot x)$$

satisfies Lemma B3 c) for t fixed and $t'\to\infty$. As u is a bounded multiplication operator, we obtain that the family $\{X_u(s,t)|t>0\}_s$ is Cauchy in the strong topology and in the Tr_μ-norm (cf. Lemma 1.2, [75]).

Finally, for $t>T$ uniformly large, the differences $\Phi(s,t)-\Phi(s',t)$ for $s,s'>S$ large are represented by smooth kernels on leaves, which vanish in the diameter

S-tubes around the leaf diagonals. The spectral theorem and the Sobolev lemma, along with (Proposition 5.8, [78]) imply that these smooth kernels are uniformly locally trace class, hence Cauchy in the Tr_μ-norm for S large. (As a hint to the reader, we remark that we must allow the value of S to depend upon t, since P is not differential, its support increases in size as we take powers P^l.)

It is then standard, given (B38), that $\mathrm{Ind}_\mu(X(\infty))$ exists and is the limit of the right-hand side of (B37). More details for the case $P = \not\partial$ are given in [45].

Appendix C: Some uniform eigenvalue estimates

Let M denote a compact Riemannian manifold. For this appendix, we consider D, a first order, essentially self-adjoint elliptic ψDO acting on the smooth sections $C^\infty(E)$ of an Hermitian vector bundle $E \to M$. Under a mild hypothesis on the support of D, for each representation $\beta : \Gamma = \pi_1(M) \to U_N$ we can form the extended operator $D \underset{\beta}{\otimes} I_N$ on $E \otimes E(\beta)$. Introduce the spectral counting function

$$\text{(C1)} \qquad E(D, \beta, \lambda) = \dim \operatorname{span} \{\text{eigenvectors of } D \underset{\beta}{\otimes} I_N \text{ with eigenvalue of modulus } \leqq \lambda\}.$$

In this appendix, we adopt the method of Allard (cf. pages 78–79, [47]) to give an elementary uniform estimate on this function which is linear in N, and then derive two important consequences.

Proposition C1. *There exists a constant $c(D)$, depending only on D and the Hermitian structure on E and TM, such that*

$$\text{(C2)} \qquad E(D, \beta, \lambda) \leqq N \cdot c(D) \cdot (1+\lambda)^{2d}$$

where d is the least integer with $d > \dim M/2$.

Proof. First we explain the mild hypothesis on D that is required. Fix a good, finite cover of M by open coordinate charts which are geodesically convex, and thus their multiple intersections are all contractible. Let ε_0 be the Lebesgue number for this cover. Then we assume that D is represented by a distributional kernel whose support is contained in an $\varepsilon_0/4d$-neighborhood of the diagonal in $M \times M$.

The method of the proof is to adapt the estimate in Lemma 1.6.3 of [47], based on Allard's Trick, to an operator coupled to a flat bundle. For completeness, we reprove this result in our context, and indicate the one nuance needed to obtain a bound linear in the dimension N.

Replace D with the power $P = D^d$. This will have support in an $\varepsilon_0/4$-neighborhood of the diagonal. Choose a parametrix Q for P as in Sect. 3 such that Q has kernel supported in an $\varepsilon_0/2$-neighborhood, and $(PQ - I)$, $(QP - I)$ are smoothing.

For a flat bundle $E(\beta) \to M$, both P and Q admit canonical extensions to operators on the sections of $E \otimes E(\beta) \to M$. There are two ways to define these extensions, and they are easily seen to be equivalent. First, a section $\varphi \in C^\infty(E \otimes E(\beta))$ can be written $\varphi = \sum_{j=1}^{l} \varphi_j$, where each φ_j has support in a closed set of diameter at most $\varepsilon_0/4$. For each j choose an open set in the cover, U_{i_j}, containing an $\varepsilon_0/2$-neighborhood of the support of φ_j, and a trivialization

$$E(\beta)|_{U_{i_j}} \cong U_{i_j} \times \mathbf{C}^N$$

compatible with the flat Hermitian structure. Then define $P \underset{\beta}{\otimes} I_N(\varphi_j)$ and $Q \underset{\beta}{\otimes} I_N(\varphi_j)$ to be the sections over U_{i_j} obtained by applying the kernels of P and Q to the components of $\varphi_j = (\varphi_j^1, \ldots, \varphi_j^N)$.

For the second definition of $P \underset{\beta}{\otimes} I_N$ and $Q \underset{\beta}{\otimes} I_N$, identify the lift of $E(\beta)$ to $\tilde{M}$

$$\widetilde{E(\beta)} \cong \tilde{M} \times \mathbf{C}^N \tag{C3}$$

so that the Γ-covering action on $\widetilde{E(\beta)}$ is transformed into the diagonal action on $\tilde{M} \times \mathbf{C}^N$. The contractibility of ε_0-balls in M implies there is a covering map

$$N(\varepsilon_0/2, \tilde{\Delta}) \xrightarrow{\pi} N(\varepsilon_0/2, \Delta)$$

from the $\varepsilon_0/2$-neighborhood of the diagonal $\tilde{\Delta} \subset \tilde{M} \times \tilde{M}$ to the $\varepsilon_0/2$-neighborhood of $\Delta \subset M \times M$. Use this covering map to lift the distributional kernels of P and Q to kernels on $\tilde{M} \times \tilde{M}$, defining operators $\tilde{P}$ and $\tilde{Q}$ on $C^\infty(\tilde{E})$, where $\tilde{E} \to \tilde{M}$ is the lift of $E \to M$. These operators extend naturally to $C^\infty(\tilde{E} \otimes \mathbf{C}^N)$, and via the isomorphism (C3) define Γ-equivariant operators on $C^\infty(\tilde{E} \otimes \widetilde{E(\beta)})$. For a section $\varphi \in C^\infty(E \otimes E(\beta))$, we lift it to a section $\tilde{\varphi} \in C^\infty(\tilde{E} \otimes \widetilde{E(\beta)})$, and observe that $\tilde{P} \otimes I_N(\tilde{\varphi})$ and $\tilde{Q} \otimes I_N(\varphi)$ are Γ-invariant, so descend to sections denoted by $P \underset{\beta}{\otimes} I_N(\varphi)$ and $Q \underset{\beta}{\otimes} I_N(\varphi)$. The reader can easily check that this definition agrees with the previous one.

Let $\phi \in C^\infty(E \otimes E(\beta))$ be an eigenfunction of $P \underset{\beta}{\otimes} I_N$. The lift $\tilde{\phi}$ is an eigenfunction for $\tilde{P} \otimes I_N$ on $\tilde{M}$, so in particular each of the component functions $\tilde{\phi}^i$ in $\tilde{\phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^N)$ is an eigenfunction of $\tilde{P}$. Thus, the second description of $P \underset{\beta}{\otimes} I_N$ realizes eigenfunctions of this operator as N (generalized) eigenfunctions for the lift of P to $\tilde{M}$.

As the bundle E is considered as part of the "fixed" data, we can without loss reduce to the case where P and Q act on $C^\infty(M)$ via the method of (page 45, [47]).

For $f \in C^\infty(M)$, let

$$|f|_{\infty,0} = \sup_{x \in M} |f(x)|$$

$$|f|_d \quad = d-\text{Sobolev } L^2\text{-norm}$$

For our choice of d, we have

Sobolev Lemma. *There exists a constant c_1 such that for all $f \in C^\infty(M)$,*

$$|f|_{\infty,0} \leqq c_1 \cdot |f|_d . \tag{C4}$$

From general elliptic theory we obtain a constant c_2 such that for $f \in C^\infty(M)$,

$$|Qf|_d \leqq c_2 \cdot |f|_0 \tag{C5}$$

$$|(QP - \mathrm{Id})f|_d \leqq c_2 \cdot |f|_0 \tag{C6}$$

where we use that $(QP - \mathrm{Id})$ and Q are ψDO's with order $\leqq d$. Finally, we need:

Gärdings Inequality. *There exists a constant c_3 such that for all $f \in C^\infty(M)$*

$$|f|_0 + |Pf|_0 \leqq c_3 \cdot |f|_d . \tag{C7}$$

Also note that (C5) and (C6) yield a converse estimate to (C7),

$$\begin{aligned} |f|_d &\leqq |(QP - \mathrm{Id})f|_d + |QPf|_d \\ &\leqq c_2 \cdot (|f|_0 + |Pf|_0) . \end{aligned} \tag{C8}$$

With these preliminaries, we can now give the proof of the proposition. Fix $a \geqq 0$, and let $\{\phi_1, \ldots, \phi_{n(a)}\}$ be an orthonormal set in $C^\infty(M, E(\beta))$ which are eigenfunctions for $P \underset{\beta}{\otimes} I_N$ having eigenvalues

$$-a \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_{n(a)} \leqq a .$$

For $f \in \mathrm{Span}\{\phi_1, \ldots, \phi_{n(a)}\}$, the estimates (C4) to (C8) yield

$$|f|_{\infty,0} \leqq c_1 \cdot |f|_d \leqq c_4(1+a)|f|_0 \tag{C9}$$

where $c_4 = c_1 c_2$. For any set of constants $\{c_j^l \mid 1 \leqq j \leqq n(a)\}$, where the dependence on l will be made explicit in a moment, (C9) implies

$$\begin{aligned} \left\| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j(x) \right\| &\leqq \left| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j \right|_{\infty,0} \\ &\leqq c_4(1+a) \left| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j \right|_0 \\ &= c_4(1+a) \left[\sum_{j=1}^{n(a)} (c_j^l)^2 \right]^{1/2} . \end{aligned} \tag{C10}$$

Let $x \in U_i$, and use the Hermitian trivialization of $E(\beta)|_{U_i}$ to express in coordinates

$$\phi_j = (\phi_j^1, \ldots, \phi_j^N) .$$

As the local trivialization is fiberwise Hermitian, we obtain by (C10) the pointwise estimate, for each $1 \leqq l \leqq N$,

$$\left| \sum_{j=1}^{n(a)} c_j^l \cdot \phi_j^l(x) \right| \leqq c_4(1+a) \left[\sum_{j=1}^{n(a)} (c_j^l)^2 \right]^{1/2} . \tag{C11}$$

Fix x and set $c_j^l = \overline{\phi_j^l(x)}$, so that (C11) yields the pointwise estimate

$$\sum_{j=1}^{n(a)} |\phi_j^l(x)|^2 \leqq c_4(1+a) \cdot \left[\sum_{j=1}^{n(a)} |\phi_j^l(x)|^2 \right]^{1/2} \tag{C12}$$

hence

$$\sum_{j=1}^{n(a)} |\phi_j^l(x)|^2 \leqq c_4^2(1+a)^2 . \tag{C13}$$

Now sum (C13) over l and integrate over x to obtain

$$\text{(C14)} \qquad n(a) = \int_M \sum_{j=1}^{n(a)} \|\phi_j(x)\|^2 \leqq N \cdot c_4^2 (1+a)^2 \cdot \operatorname{vol}(M).$$

Substitute $a = \lambda^{m+1}$ to obtain

$$E(D, \beta, \lambda) = N \cdot c(D) \cdot (1+\lambda)^{2d}$$

for a new constant $c(D)$ depending only on c_4, $\operatorname{vol}(M)$ and the dimension m of M. □

We draw two corollaries of the estimate (C2). First, introduce the "estimated eigenvalues" for $D \underset{\beta}{\otimes} I_N$:

$$\text{(C15)} \qquad \begin{cases} \lambda_n^* = f_N^{-1}(n), \ n > N \cdot c(D) \\ \lambda_n^* = 0 \ \text{ for } \ 0 \leqq n \leqq N \cdot c(D) \end{cases}$$

where $f_N(x) = N \cdot c(D) \cdot (1+x)^{2d}$.

Lemma C2. *Order the eigenvalues of* $D \underset{\beta}{\otimes} I_N$:

$$\leqq \lambda_{-2} \leqq \lambda_{-1} < 0 \leqq \lambda_0 \leqq \lambda_1 \leqq \dots .$$

Then

$$\text{(C16)} \qquad \begin{cases} \lambda_n \geqq \lambda_n^*, & n = 0, 1, 2, \dots \\ \lambda_{-n} \leqq -\lambda_n^*, & n = 1, 2, \dots \end{cases}$$

Proof. The function $f_N(x)$ is monotone increasing for $x \geqq 0$, so that (C16) for $n > N \cdot c(D)$ follows from the estimate based on (C2),

$$n \leqq E(D, \beta, \lambda_n) \leqq f_N(\lambda_n).$$

For $n \leqq -N \cdot c(D)$ a similar estimate holds. (C16) is evident for $|n| \leqq N \cdot c(D)$. □

Taking $N = 1$, we define

$$\text{(C17)} \qquad \begin{cases} \mu_n = f_1^{-1}(n), & n > c(D) \\ \mu_n = 0, & n \leqq c(D). \end{cases}$$

Introduce the heat operator $\exp\left(-t \cdot \left\{D \underset{\beta}{\otimes} I_N\right\}^2\right)$ on the Hilbert space $\mathscr{H}_\beta = L^2(E \otimes E(\beta))$.

Corollary C3. *For any representation* $\beta : \Gamma \to U_N$,

$$\text{(C18)} \qquad \operatorname{Tr}_{\mathscr{H}_\beta} \left\{ \exp\left(-t \cdot \left\{D \underset{\beta}{\otimes} I_N\right\}^2\right)\right\} \leqq 2N \cdot \sum_{n=0}^{\infty} e^{-t(\mu_n)^2}.$$

Proof. The left-hand side of (C18) is estimated by

$$\text{(C19)} \qquad \sum_{n \in Z} e^{-t\lambda_n^2} \leqq 2 \cdot \sum_{n=0}^{\infty} e^{-t(\lambda_n^*)^2},$$

so that (C18) follows from (C19) and the observations that $f_1(x)$ is monotone for $x>0$, and

$$\lambda_n^* = f_1^{-1}(n/N) \quad \text{for} \quad n > N\cdot c(D). \quad \square$$

The estimate (C18) is a weak (global) form of Kato's Inequality for the heat operators of the ψDO's $D \underset{\beta}{\otimes} I_N$. The standard Kato Inequality compares the pointwise trace of the kernel for $\exp\left(-t\cdot\left\{D \underset{\beta}{\otimes} I_N\right\}^2\right)$ with $c\cdot N\cdot \mathrm{Tr}_x \exp(-tD^2)$, where the trace is pointwise and c is a geometric constant. Integrating the pointwise trace over M yields (C18), as the $\{\mu_n\}$ dominate the eigenvalues of D. The pointwise estimate depends upon the existence of a "Weizenbock Formula" for $D \underset{\beta}{\otimes} I_N$, which requires that D be a geometric operator. The point of the corollary is that more standard elliptic estimates suffice to prove the weaker global estimate (C18).

The proof of the Cheeger-Gromov estimate (7.19) depends upon two estimates. The first is a pointwise estimate for the heat kernel for small t. As discussed in (§2, [71]), this is a consequence of the Bismut-Freed condition

$$\text{(C20)} \qquad \mathrm{Tr}_{\mathscr{H}_\beta} \exp\left(-t\cdot\left\{D \underset{\beta}{\otimes} I_N\right\}\right) = o(t^{1/2})$$

for $t\to 0$, and the uniformity of the coefficients in an asymptotic expansion. The second is an estimate of the trace of the heat kernel of $D \underset{\beta}{\otimes} I_N$ for $t \geqq 1$ which is linear in N. This second estimate can be derived from Kato's inequality, as in [71], or in the context of flat bundles, where we apply the Cheeger estimate, from (C18).

We next derive the second Corollary of Proposition C1, which was cited in Sect. 8. Recall that D_V is the densely-defined self-adjoint operator on $L^2(V, E)$ considered in Sect. 8. For each $\chi \in \hat{G}$, convolution with the kernel k_χ^* defines projection onto the χ-subspace of $L^2(V, E)$. Moreover, it was proven in Proposition 7.5 that the restriction $D_V \circ k_\chi^*$ is unitarily conjugate to $D \underset{\beta}{\otimes} I_N$ where $\beta = \lambda_\chi \circ \alpha$. For a bounded measurable real function ψ on $\mathbf{R}$, the analysis of $\psi(D_B)\circ k_\chi^*$ thus reduces to that of $\psi\left(D_M \underset{\beta}{\otimes} I_N\right)$.

Introduce the functions

$$\mathrm{sign}(x) = \begin{cases} +1, & x \geqq 0 \\ -1, & x < 0 \end{cases}$$

$$\varphi(x) = \begin{cases} +1, & x \geqq 1 \\ -1, & x \leqq -1 \\ & \text{monotone increasing} \\ & \text{for } -1 < x < 1 \end{cases}$$

$$\varphi_\varepsilon(t) = \varphi(t/\varepsilon)$$

$$\psi(\xi) = \begin{cases} 1, & \xi \leqq 1/2 \\ 0, & \xi \geqq 1 \\ & \text{monotone decreasing} \\ & \text{for } 1/2 < \xi < 1 \end{cases}$$

and introduce $\varphi_{\varepsilon,s}$ characterized by its Fourier transform

$$\hat{\varphi}_{\varepsilon,s}(\xi)=\hat{\varphi}_\varepsilon(\xi)\cdot\psi(\xi/s).$$

The corresponding operators are

$$\mathscr{E}=\operatorname{sign}(D_V)$$
$$\Phi_\varepsilon=\phi_\varepsilon(D_V)$$
$$\Phi_{\varepsilon,s}=\varphi_{\varepsilon,s}(D_V).$$

Lemma C4. *There exist estimates*

(C21) $$\operatorname{Tr}_{\mathscr{H}}|(\mathscr{E}-\Phi_\varepsilon)\circ k_\chi^*|\leqq N(\chi)\cdot c(D)\cdot(1+\varepsilon)^{2d}$$

(C22) $$\operatorname{Tr}_{\mathscr{H}}|(\Phi_\varepsilon-\Phi_{\varepsilon,s})\circ k_\chi^*|=N(\chi)\cdot c(D)\cdot o(1/s).$$

Proof. (C21) follows from noting that $|\mathscr{E}-\Phi_\varepsilon|$ is dominated by the spectral projection of D_V in the interval $(-\varepsilon,\varepsilon)$ and then applying (C2). The second estimate follows from observing that $|\varphi_\varepsilon-\varphi_{\varepsilon,s}|$ tends to zero in the Schwartz space topology as $s\to\infty$, so the polynomial bound of (C2) implies the uniform bound (C22). □

Corollary C5. *For G compact,*

(C23) $$\psi(e)^{-1}\cdot\operatorname{Tr}_{\mathscr{H}}|(\mathscr{E}-\Phi_\varepsilon)\circ k_\psi^*|\leqq c(D)\cdot(1+\varepsilon)^{2d}$$

(C24) $$\psi(e)^{-1}\cdot\operatorname{Tr}_{\mathscr{H}}|(\Phi_\varepsilon-\Phi_{\varepsilon,s})\circ k_\psi^*|=o(1/s)$$

uniformly in $\psi\in R_\infty(G)$.

Appendix D. Remarks on the development of von Neumann index and eta invariants

In the mid 1970's, Atiyah, Patodi and Singer introduced and studied, in a series of papers [3, 4, 5, 6], a new invariant for a self-adjoint elliptic differential operator D on an odd dimensional manifold M, the eta-invariant $\eta(D)$. This real-valued invariant measures, in principle, the positive versus negative spectral asymmetry of D. It represents a regularized signature which generalizes the signature of a self-adjoint linear transformation on a finite vector space.

Among the many remarkable properties of $\eta(D)$ is that if one is given a smooth family of operators, $\{D_t|0\leqq t\leqq 1\}$, the values $\eta(D_t)$ depend piecewise smoothly on t, and there is a well-defined continuous derivative $\dot{\eta}(D_t)$. The relative eta-invariant of the family is defined to be the integral

(D1) $$\eta(\{D_t|0\leqq t\leqq 1\})=\int_0^1\dot{\eta}(D_t)\,dt,$$

The original philosophy of Atiyah, Patodi and Singer was that there exists a closed 1-form, $\dot{\eta}$, on the space of self-adjoint Fredholm operators, and the invariant (D1) is just the path integral of $\dot{\eta}$ over the path determined by $\{D_t|0\leqq t\leqq 1\}$. Recently, this has been made precise by the work of Bismut and Freed [13].

The derivative, $\dot{\eta}(D_t)$, has a local expression in terms of the complete symbol of D_t (cf. Lemma 1.10.2, [47]). For example, given a complex vector bubdle $E\to M$ with two flat Hermitian connections ∇^0, ∇^1, we can use these to define essentially self-adjoint differential operators

$$D_0 = D \otimes \nabla^0, \quad D_1 = D \otimes \nabla^1$$

which are homotopic, and let $\eta(D, \nabla^0, \nabla^1)$ denote the corresponding relative eta-invariant. Then the local expression for $\dot{\eta}(D_t)$ involves the transgression forms comparing ∇^0 and ∇^1, and the resulting topological formula (cf. Theorem 4.4.6, [47]):

$$\text{(D2)} \qquad \eta(D, \nabla^0, \nabla^1) = (-1)^m \cdot \int_{S(T^*M)} \mathrm{Td}(M) \cdot \mathrm{ch}(D) \cdot \mathrm{Tch}(\nabla^1, \nabla^0)$$

resembles the usual Chern character of an elliptic operator. However, there is an additional factor in the integrand, $\mathrm{Tch}(\nabla^1, \nabla^0)$, which is a transgression form representing the "real-K^1-theory" data defined by a bundle with two flat Hermitian connections (cf. §7, [6]).

A self-adjoint Fredholm operator on a Hilbert space has index zero. However, it was realized by Baum and Douglas [11] and Kasparov [62] in the late 1970's that if this operator arises from an elliptic differential operator on a compact manifold, then it still can yield analytic index invariants by appropriately coupling it to unitary multipliers to obtain nonself-adjoint Fredholm operators, whose indices are topological invariants. The pairing is realized by taking the integer index of the compression of the multiplier to the positive eigenspan of the operator, a construction that directly generalizes the classical construction of Toeplitz operators on the circle. The Kasparov reformulation of this pairing, which was used in Sect. 8 and Appendix B above, is not quite as explicit, but allows much greater freedom in the choice of the defining data (cf. also [51]).

Also in the late 1970's, the L^2-index theorem for infinite coverings of Atiyah [2] and Singer [82] was extended by A. Connes to the far more general case of a foliated manifold $(V, \mathscr{F})$ with a holonomy invariant, transverse measure μ. A leafwise-elliptic, leafwise-ψDO for $\mathscr{F}$ then has a μ-index, and the measured foliation index theorem (cf. [27, 28]) gives a topological formula for the index.

Connes' foliation index theorem is a very broad extension of the remarkable theory for quasi-periodic operators developed by Coburn, Douglas, Moyer, Singer and Schaffer [23, 24, 25]. In part of their work, a real-valued Breuer index in a von Neumann algebra was defined for these operators. This index theory was subsequently developed extensively by Šubin (cf. [85]). The other part of their work concerned how the index problem can be represented on the Bohr compactification of the quasi-periodic symbol data. Here, the index is realized in terms of generalized (i.e., not L^2) eigenfunctions of the operator. This second part of their program was not considered by Connes, but manifests itself in our study of the transverse index theory in Sect. 5 and Sect. 6 above.

In the context of a foliated manifold $(V, \mathscr{F})$, a leafwise-elliptic, self-adjoint, leafwise ψDO for $\mathscr{F}$ determines on each leaf of $\mathscr{F}$ a closed subspace of the Hilbert space domain of the operator, consisting of the non-negative range of the operator. A smooth map $u: V \to U_N$ determines a unitary multiplier on each of these leaf Hilbert spaces, and its compression to the non-negative subspace yields a Breuer Fredholm operator, which has real-valued Breuer index calculated by Connes' foliation index theorem. This construction, which is behind the development of Sect. 3 and Appendix B, generalizes the Toeplitz operators obtained by compressing a quasi-periodic unitary character on $\mathbf{R}$ to the Hardy space in $L^2(\mathbf{R})$. The topological index of these foliation Toeplitz operators are characteristic classes of the pair $(\mathscr{F}, \mu)$, of the type called "μ-classes" in [52].

For a foliation defined by a minimal flow, Curto, Muhly and Xia [35] made a comprehensive study of the operator and index theoretic aspects of the above Toeplitz construction. For foliations with higher dimensional leaves, a study of the "extension problem" arising from the compressed operators was made in [45], while [53, 55] investigated further aspects of their index theory.

From the above discussion, the reader will see that many aspects of index theory from the 1970's have been incorporated into the present monograph. The original version of our main Theorem 1.2, announced in [42], required that $\Gamma = \pi_1(M)$ be an amenable group. This reflected the authors' dedication to the program of [23, 24]. The subsequent work of J. Roe [76] provided an alternate approach that is heavily represented in the present treatment.

To conclude our brief history of the index developments from the 1970's which are incorporated in this work, we must mention the transverse index theory of Atiyah and Singer. This theory, introduced in [1] and [81], was extended by Connes to be a special case of cyclic cocycles constructed from transversally elliptic operators (cf. § 8, [30]). The self-adjoint version of this theory is developed in [54]. Finally, we note that the outstanding open problem in transverse index theory is to obtain an "index theorem" for transversely elliptic operators that equates the distributional index to an explicit topological formula.

Note: This has recently been solved by M. Vergne [9].

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