The Intersection Product of Transverse Invariant Measures

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ABSTRACT. We show that the self-intersection homology class of a Ruelle-Sullivan foliation cycle is completely determined by the self-intersection classes of the compact leaves in the support of the cycle. For a cycle obtained from an invariant transverse measure with no atoms, we conclude that the self-intersection class vanishes.

1. Introduction and Main Theorems. Let μ denote a transverse, holonomy invariant measure for a C^1-foliation $\mathcal{F}$ of codimension q on a compact oriented manifold M of dimension m. We assume that the tangent bundle to the leaves of $\mathcal{F}$ is oriented and that μ is finite on compact transversals (i.e., μ is *locally finite*). Then the transverse measure defines an asymptotic homology class $[C_\mu] \in H_p(M;\mathbf{R})$, where p is the leaf dimension. For flows ($p = 1$), this class is called the Schwartzman asymptotic cycle [13]; for the general case $p \geq 1$, it was constructed by Ruelle and Sullivan [12].

The existence of a transverse invariant measure for a foliation has implications for its leaf dynamics (cf. [8, 11, 12, 14].) One approach to understanding the relation between transverse measures and dynamics is to study the values of cohomology invariants for the foliation, restricted to the support of the measure (cf. [3]). This program has been studied for the secondary invariants of foliations by the first author [4, 5], and the second author [8] considered it for two of the primary invariants of a measured foliation: the Euler class of the normal bundle, and the self-intersection of the Ruelle-Sullivan class,

$$(1) \qquad [C_\mu] \cap [C_\mu] \in H_{p-q}(M;\mathbf{R}).$$

In this paper, we address the question of how the self-intersection product (1) depends upon the geometry of the *support* of the measure μ. For example, if μ is defined by a closed q-form ω on M, then the wedge product $\omega \wedge \omega = 0$, and the intersection product also vanishes. The vanishing of the self-intersection

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product in this case corresponds to a well-known result in measure theory: for a Borel measure space $\{X,\mu\}$, the measure μ has no atoms if and only if the diagonal $\Delta \subset X \times X$ has measure zero for the product measure $\mu \times \mu$. We use this principle to give a complete evaluation of the self-intersection product (1) of a transverse invariant measure.

Our first result, Theorem 1, shows that this intersection reduces to the self-intersections of the *atoms* of the transverse measure. We then establish a second result, Proposition 1, giving a simple geometric criterion for the vanishing of these atomic self-intersections. One application of this result is Corollary 1, which shows that the atoms of the measure μ which have non-zero self-intersection are isolated. The self-intersection of an atomic measure equals the evaluation of the transverse Euler class on the atom by standard methods. These results are combined in Corollary 2, to obtain a complete evaluation of (1). Corollary 3 summarizes the criteria for the vanishing of (1) that we obtain.

We say that a leaf L of $\mathcal{F}$ is an *atom* for μ if there exists $\varepsilon_L > 0$ so that for every sufficiently small transversal disc to $\mathcal{F}$, $i : \mathbf{D}^q \to M$, which intersects L, then

$$\int_{\mathbf{D}^q} |i^*(d\mu)| \geq \varepsilon_L \tag{2}$$

where $|i^*(d\mu)| = i^*(d\mu)^+ - i^*(d\mu)^-$ denotes the positive, unsigned measure associated to the signed measure $i^*(d\mu)$. Otherwise, we say that L is a *leaf of continuity* for μ. We then have the following properties of the atoms of a transverse measure.

Remark 1. *If L is an atom for a locally finite transverse invariant measure μ, then L is a compact leaf.*

Proof. If L is non-compact, then it has a point of accumulation $x \in M$, and every open transversal through x intersects L infinitely often. The holonomy invariance of μ and Estimate (2) then imply that every open transversal through x has infinite μ mass, contradicting that μ is locally finite. □

Remark 2. *The set of leaves for $\mathcal{F}$ which are atoms for μ is countable.*

Let $A(\mu)$ denote the countable set of leaves in M which are atoms for μ. Define the *atomic part* of μ to be the transverse measure

$$\mu_a = \sum_{L \in A(\mu)} \mu(L)\mu_L \tag{3}$$

where μ_L is the "Dirac" transverse measure associated to the oriented compact leaf L (cf. Plante [11]). The term $\mu(L)$ is the μ-measure of L intersected with a small transversal. That is, choose a sequence of transverse embedded discs, $\{i_n : \mathbf{D} \to M\}$ whose image limits to a point $x \in L$, then

$$\mu(L) = \lim_{n\to\infty} \int_{\mathbf{D}^q} i_n^*(d\mu).$$

Define the *continuous part* of μ to be the difference $\mu_c = \mu - \mu_a$. We say that μ is a *continuous* measure if it has no atoms.

Remark 3.

1. *The measures μ_a and μ_c are holonomy invariant and locally finite;*
2. *For every transverse disc $\{i : \mathbf{D}^q \to M\}$, there is the estimate*

$$\int_{\mathbf{D}^q} |i^*(d\mu_a)| \leq \int_{\mathbf{D}^q} |i^*(d\mu)|. \tag{4}$$

3. *Every leaf of $\mathcal{F}$ is continuous for μ_c.*

Proof. Holonomy invariance of μ_a follows from that of each "Dirac measure" μ_L. The remaining assertions are standard properties of the decomposition of a measure into its continuous and atomic parts, in this case applied to $i^*(d\mu)$ on each transversal $\mathbf{D}^q$. □

We say that $\mathcal{F}$ is a C^1-foliation if the bundle $T\mathcal{F} \subset TM$ of vectors tangent to the leaves of $\mathcal{F}$ is a C^1-vector subbundle.

Theorem 1. *Let $\mathcal{F}$ be a C^1-foliation of the closed oriented manifold M, and suppose that the tangent bundle to the leaves $T\mathcal{F}$ is also oriented. Let μ be a locally-finite transverse invariant measure. Then the self-intersection of the Ruelle-Sullivan current for μ is given by the sum*

$$[C_\mu] \cap [C_\mu] = [C_{\mu_a}] \cap [C_{\mu_a}] \tag{5}$$

$$= \sum_{L \in A(\mu)} \mu(L)^2 \{[L] \cap [L]\}. \tag{6}$$

The proof of Theorem 1 is given in Section 2. The reduction from (5) to (6) follows from the interpretation of the intersection product as a geometric intersection, which for disjoint leaves is always zero. The evaluation of the self-intersection product is reduced by (6) to the study of the self-intersections $[L] \cap [L]$ of the compact leaves of F. We make an observation, which is the basis for Proposition 1 below, that further simplifies the sum in (6).

The orientation on $T\mathcal{F}$ orients the compact leaves of $\mathcal{F}$, so each compact leaf L determines a homology class $[L] \in H_p(M; \mathbf{R})$. We say that two leaves L and L' are *hp-equivalent* (for *homology projective*) if they determine the same ray in the real homology group $H_p(M; \mathbf{R})$. For example, if $H_p(M; \mathbf{R}) \cong \mathbf{R}$, then any two compact leaves of $\mathcal{F}$ which are non-zero in homology are hp-equivalent.

Proposition 1. *Let L be a compact leaf of $\mathcal{F}$. Suppose there exists a distinct compact leaf L' of $\mathcal{F}$ which is hp-equivalent to L. Then the self-intersection $[L] \cap [L] = 0$.*

Proof. The intersection product $[L] \cap [L'] \in H_{p-q}(M; \mathbf{R})$ of two p-cycles can be calculated geometrically for homology classes represented by closed oriented submanifolds: choose a perturbation of L' to a closed submanifold $L'' \subset M$, so that L'' is homologous to L' and is in general position with respect to L. Then the connected components of the geometric intersection, $L \cap L''$, consist of closed submanifolds of M of dimension $(p-q)$, and inherit orientations from L and L''. Each component therefore determines a homology class in $H_{p-q}(M; \mathbf{R})$, and their total homology class equals $[L] \cap [L']$.

Consider a compact submanifold $L' \subset M$ of dimension p which is hp-equivalent to the class $[L]$. Then either both classes $[L]$ and $[L']$ vanish, and $[L] \cap [L] = 0$ is obvious, or $[L'] = C \cdot [L]$ for a non-zero constant C. In the latter case, observe that $[L] \cap [L'] = C([L] \cap [L])$, so that if L and L' are disjoint (e.g., they are distinct leaves of $\mathcal{F}$), then they are in general position, and their cap product equals the homology class of their geometric intersection, which is zero. □

The evaluation of the self-intersection of a compact leaf has a traditional interpretation in terms of the normal Euler class of the embedding $L \hookrightarrow M$. Let $\nu \to M$ denote the oriented normal bundle to $\mathcal{F}$, and $\nu_L \to L$ the restriction of ν to an individual leaf L. The orientations of TM and $T\mathcal{F}$ endow ν and $\nu(L)$ with orientations. Let $E(\nu) \in H^q(M; \mathbf{R})$ be the Euler class for ν, and $E(\nu_L) \in H^q(L; \mathbf{R})$ the Euler class of ν_L.

Remark 4. *Let L be a compact leaf of $\mathcal{F}$.*

1. *$E(\nu)$ and $E(\nu_L)$ vanish if q is odd;*
2. *$E(\nu_L)$ is the Poincaré dual of the self-intersection class*

$$[L] \cap [L] = \{[L] \setminus E(\nu_L)\} \in H_{p-q}(L; \mathbf{R}).$$

Proof. (cf. [1], Section 11). □

Fix a Riemannian metric on TM. For a leaf L of $\mathcal{F}$, let $N(L) \to L$ denote the unit disc subbundle in ν_L, and let $L_0 \subset N(L)$ denote the embedding of L as the zero section.

Corollary 1. *Let L be a compact leaf of $\mathcal{F}$. Suppose there exists an embedding $f : N(L) \hookrightarrow M$ such that $f|_{L_0} : L_0 \to M$ is a diffeomorphism onto L, and there is a compact leaf $L' \neq L$ of $\mathcal{F}$ with $L' \subset f(N(L))$. Then $[L] \cap [L] = 0$.*

Proof. The orientation of $T\mathcal{F}$ orients the open set $U(L) = f(N(L))$, and L' has orientation compatible with the core L of $U(L)$, so the homology class $[L'] \in H_p(U(L);\mathbf{R})$ is a positive multiple of $[L]$. In particular, L and L' are hp-equivalent and disjoint, so we can apply Proposition 1. □

We combine Theorem 1 and Proposition 1 with Remark 4 to obtain the following forumla for (1). A compact leaf $L \subset M$ is said to be *isolated* if L admits an open tubular neighborhood $U(L)$, as in Corollary 1, such that there are no compact leaves of $\mathcal{F}$ completely contained in $U(L)$. Let $A_0(\mu)$ denote the subset of the atoms for μ which are isolated compact leaves.

Corollary 2. *Let $\mathcal{F}$ and μ be as in Theorem 1. Then*

$$[C_\mu] \cap [C_\mu] = \sum_{L \in A_0(\mu)} \mu(L)^2 \{[L] \setminus E(\nu)\}. \tag{7}$$

There is another natural invariant for measured foliations with even codimension, the slant-product of the classes $[C_\mu] \setminus E(\nu) \in H_{p-q}(M;\mathbf{R})$. For an atomic measure $\mu = \mu_L$, the above discussion shows that this invariant reduces to the self-intersection product of $[C_{\mu_L}]$. For more general measures, this need not be true. The second author has shown in [9] that the Milnor inequality (cf. [7]) extends to this measured Euler invariant for 2-dimensional oriented foliations on oriented closed 4-manifolds. More generally, the previous discussion suggests the general problem of determining what geometric hypotheses on $\mathcal{F}$ are sufficient for the vanishing of the measured Euler class $[C_\mu] \setminus E(\nu)$. The paper [6] addresses this question in detail.

We conclude with a summary of the conditions which are sufficient to force the vanishing of the self-intersection class (1):

Corollary 3. *Let $\mathcal{F}$ and μ be as in Theorem 1, and suppose that at least one of the following conditions is satisfied:*

1. *Every leaf of $\mathcal{F}$ is continuous for μ;*
2. *μ has no isolated atoms;*
3. *The normal Euler class $E(\nu) = 0$.*

Then the self-intersection product $[C_\mu] \cap [C_\mu] = 0$.

2. Vanishing for a Continuous Measure. Let $\mathcal{F}$ be a C^1-foliation of the compact oriented manifold M, and let μ_0 and μ_1 be holomomy invariant, locally-finite transverse measures for $\mathcal{F}$. We will assume also that μ_0 is continuous for $\mathcal{F}$; that is, μ_0 has no atoms. There are corresponding Ruelle-Sullivan closed foliation p-currents, C_{μ_0} and C_{μ_1}, with Poincaré dual cohomology classes $[\omega_{\mu_0}]$, $[\omega_{\mu_1}] \in H^q(M;\mathbf{R})$. The main result of this section is then:

Proposition 2. *Let μ_0 and μ_1 be transverse invariant measures for $\mathcal{F}$ which are finite on compact transversals, and assume that each leaf of $\mathcal{F}$ is continuous for μ_0. Let $[\omega_{\mu_i}]$, denote the Poincaré dual cohomology class to the Ruelle-Sullivan current $[C_{\mu_i}]$, for $i = 0,1$. Then the cohomology cup-product $[\omega_{\mu_0}] \cup [\omega_{\mu_1}] = 0$.*

Proposition 2 implies Theorem 1 by first noting that the intersection product on homology is a bi-linear pairing, and then we use the standard properties of the correspondence between cup and cap products under Poincaré duality.

The proof of Proposition 2 is modeled on a seemingly unrelated result in measure theory. We include the simple proof of the following lemma, as it provides a guide to the proof of the proposition.

Lemma 1. *Let X be a separable Borel space with two Borel probability measures $\mathbf{m}_0$ and $\mathbf{m}_1$. If $\mathbf{m}_0$ has no atoms, then the diagonal $\Delta = \{(x,x) \in X \times X \mid x \in X\}$ has measure 0 for the product measure $\mathbf{m}_0 \times \mathbf{m}_1$.*

Proof. For $\varepsilon > 0$, the hypothesis that $\mathbf{m}_0$ has no atoms implies there exists a finite partition of X into disjoint Borel subsets, $X = \bigsqcup_1^N X_i$ where each X_i has $\mathbf{m}_0$-measure less than ε. The collection of product sets $\{X_i \times X_i \mid 1 \le i \le N\}$ is a covering of the diagonal Δ, and we have the estimate

$$\mathbf{m}_0 \times \mathbf{m}_1(\Delta) \le \mathbf{m}_0 \times \mathbf{m}_1\Big(\bigsqcup_1^N X_i \times X_i\Big)$$

$$= \sum_1^N \{\mathbf{m}_0(X_i) \cdot \mathbf{m}_1(X_i)\} \le \varepsilon \cdot \mathbf{m}_1(X) = \varepsilon.$$

As ε is arbitrary, $\mathbf{m}_0 \times \mathbf{m}_1(\Delta) = 0$. □

The first step in the proof of Proposition 2 is to construct closed q-forms representing the dual classes $[\omega_{\mu_0}]$ and $[\omega_{\mu_1}]$. This follows the standard method for constructing the Poincaré dual of a cycle, using the Thom class of the normal bundle to the cycle. For foliation cycles, it is necessary to average this normal bundle Thom class over the transverse measure (cf. [12], Proposition 1). Evaluating the cup product of the foliation cycles obtained from μ_0 and μ_1 involves integrating a form over a neighborhood of the diagonal $\Delta_{\mathcal{F}}$ to the transverse space of $\mathcal{F}$ with respect to the product measure $\mu_0 \times \mu_1$. The key technical result, Proposition 4 below, states that this form can be localized to arbitrarily small neighborhoods of the diagonal, and stays uniformly bounded. The method of proof of Lemma 1 then shows that the cup product vanishes in cohomology. We begin with some technical preliminaries.

Fix a Riemannian metric on TM, and let $d\nu_{\mathcal{F}}$ denote the corresponding leafwise volume-form. Let $\nu_1 \subset \nu \to M$ denote the unit disc subbundle of the normal bundle ν to $\mathcal{F}$. The Riemannian metric defines a C^1-embedding $\nu \subset TM$, so that we can restrict the geodesic exponential map $\exp : TM \to M \times M$ to the subbundle ν. The exponential map exp is C^1 when restricted to a C^1-distribution (cf. [2], page 95, Theorem 3.1). For each $s > 0$, define a scaled, C^1-exponential map

$$\begin{aligned} \exp_s : \nu &\to M \times M, \\ (x, \vec{v}) &\to (x, \exp_x\{s \cdot \vec{v}\}). \end{aligned} \tag{8}$$

The product of the leafwise volume form $d\nu_{\mathcal{F}}$ with the transverse measure μ_i defines a locally finite measure on M, denoted by $\hat{\mu}_i$. Let us briefly recall its definition. For $f : M \to \mathbf{R}$ a continuous function with support in the domain of a local foliation coordinate chart,

$$\psi : U \to \mathbf{D}^p \times \mathbf{D}^q; \quad \mathbf{D}^i = (-1,1)^i,$$

we set

$$\int_M f \, d\hat{\mu}_i = \int_{\mathbf{D}^q} \left\{ \int_{\mathbf{D}^p} f \circ \psi^{-1} \, d\nu_{\mathcal{F}} \right\} d\mu_i. \tag{9}$$

The holonomy invariance of the transverse measure μ_i implies that the pairing (9) is independent of the choice of foliation chart (U, ψ). We then extend the definition (9) to all continuous functions which are finite sums of those supported in foliation charts. A partition-of-unit argument shows that these are all of $C^0(M)$, as M is compact. In particular, note that $\int_M 1 \, d\hat{\mu}_i$ is finite.

Our next step is to choose a covering of M by regular foliation charts which play the role of the sets X_i in the proof of Lemma 1 with respect to μ_0 and μ_1. Recall that a foliation chart $\psi : U \to \mathbf{D}^p \times \Omega$, for $\Omega \subset \mathbf{R}^q$ a bounded convex open set, is *regular* if there is an open neighborhood $\tilde{U} \supset \bar{U}$ containing the closure of U, and an extension of ψ to a foliation chart

$$\tilde{\psi} : \tilde{U} \to (-1-\delta, 1+\delta)^p \times \tilde{\Omega},$$

for some $\delta > 0$ and some open subset $\tilde{\Omega} \subset \mathbf{R}^q$ which contains the closure of Ω. For example, if $\Omega = \mathbf{D}^q$, then we can require that $\tilde{\Omega} = (-1-\delta, 1+\delta)^q$. The following construction of foliation coordinates makes essential use of the hypothesis that μ_0 has no atoms.

Lemma 2. *There exist constants K, $N > 0$ so that for $\varepsilon > 0$ sufficiently small, there exists a finite covering*

$$\{\psi_a : V_a \to \mathbf{D}^p \times \Omega_a \mid a \in \mathcal{A}\}$$

of M by regular foliation charts (where Ω_a is an open subset of $\mathbf{D}^q$ as determined in the proof below) with foliation chart extensions

$$\{\tilde{\psi}_a : \tilde{V}_a \to \mathbf{D}^p \times \tilde{\Omega}_a \mid a \in \mathcal{A}_\varepsilon\},$$

satisfying:

1. *The μ_0-mass of the transversal $\tilde{T}_a = \tilde{\psi}_a^{-1}(0 \times \tilde{\Omega}_a)$ is less than ε;*
2. *Let $\chi_{\tilde{V}_a}$ denote the characteristic function of the open set $\tilde{V}_a$, then*

$$\sum_{a \in \mathcal{A}_\varepsilon} \int_M \chi_{\tilde{V}_a} \, d\hat{\mu}_1 \leq K \cdot \int_M 1 \, d\hat{\mu}_1; \tag{10}$$

3. *For each $a \in \mathcal{A}_\varepsilon$, the cardinality of the set $\{b \in \mathcal{A}_\varepsilon \mid \tilde{V}_a \cap \tilde{V}_b \neq \emptyset\}$ is at most N.*

Proof. Choose a covering $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in \mathcal{A}\}$ of M by regular foliation charts. For an open convex set $W \subset (-1,1)^q$, we obtain a new foliation chart $\psi_{\alpha,W} \to \mathbf{D}^p \times W$, by restricting ψ_α to the inverse image open set $U_{\alpha,W} = \psi^{-1}(\mathbf{D}^p \times W)$.

Continuity of the measure μ_0 implies that there exists a $\delta(\varepsilon) > 0$ so that any Borel subset $W \subset \mathbf{D}^q$ with diameter less than $\delta(\varepsilon)$ has μ_0-mass at most ε (and this can be chosen so that it holds for all $\alpha \in \mathcal{A}$). Choose an integer $N_\varepsilon > \frac{2}{\varepsilon}$ so large that any cube of side $2/N_\varepsilon$ in $\mathbf{D}^q$ has diamter less than $\frac{\delta(\varepsilon)}{2}$. Make a regular subdivision of the cube $\mathbf{D}^q$ into closed subcubes with side lengths $2/N_\varepsilon$ labeled $\{\mathbf{D}_i^q \mid 1 \leq i \leq N_\varepsilon^q\}$. Note that this is not a disjoint subdivision of $\mathbf{D}^q$, as the faces of the cubes overlap. We need to also choose Borel subsets $\mathbf{E}_i \subset \mathbf{D}_i^q$ so that the interior of $\mathbf{E}_i$ equals the interior of $\mathbf{D}_i^q$, and the collection of Borel sets $\{\mathbf{E}_i \mid 1 \leq i \leq N_\varepsilon^q\}$ is a disjoint decomposition of $\mathbf{D}^q$.

For each cube $\mathbf{D}_i^q$ in this subdivision, let $\tilde{\Omega}_i$ denote an open convex neighborhood of the Borel subset $\mathbf{E}_i$, such that the μ_0-mass of $\tilde{\Omega}_i$ is at most twice the μ_0-mass of $\mathbf{E}_i$, and $\tilde{\Omega}_i$ is contained in the $\frac{\delta(\varepsilon)}{4}$-open neighborhood of $\mathbf{D}_i^q$. Then choose an open, convex neighborhood Ω_i of $\mathbf{E}_i$ whose closure is contained in $\tilde{\Omega}_i$. Note that the μ_0-masses of Ω_i and $\tilde{\Omega}_i$ are at most ε, as these sets have diameter less than $\delta(\varepsilon)$.

Define an open cover of M by the charts $\{U_{\alpha,\Omega_i} \mid \alpha \in \mathcal{A}, 1 \leq i \leq N_\varepsilon^q\}$. The corresponding foliation charts then satisfy (2.1). Property (2.2) also follows from this construction, where we take $K = 2\sum_{\alpha\in\mathcal{A}} \int_M \chi_{\tilde{U}_\alpha}\, d\hat{\mu}_\alpha$, using that the collection $\{\mathbf{E}_i \mid 1 \leq i \leq N_\varepsilon^q\}$ is a disjoint decomposition of $\mathbf{D}^q$ and the μ_0-mass of $\tilde{\Omega}_i$ is at most twice that of $\mathbf{E}_i$. Property (2.3) is a standard consequence of the combinatorics of regular subdivisions of a given fixed covering (cf. [15], Section 5).

We then rename the open sets $V_a = U_{\alpha,\Omega_i}$ where now $a \in \mathcal{A}_\varepsilon$, for $\mathcal{A}_\varepsilon = \mathcal{A} \times \{1,\ldots,N_\varepsilon^q\}$, with corresponding transverse factors Ω_a. The coordinate functions are similarly renamed $\varphi_a = \psi_{\alpha,i}$.

Fix a covering of M by regular foliation charts $\{(U_\alpha,\psi_\alpha)|\alpha \in \mathcal{A}\}$ as in the proof of Lemma 2. For $\varepsilon > 0$, we also assume fixed a covering $\{(V_a,\psi_a) \mid a \in \mathcal{A}_\varepsilon\}$ of M by regular foliation charts, constructed as in Lemma 2. Choose in addition a collection of smooth, non-negative functions $\{\lambda_a \mid a \in \mathcal{A}_\varepsilon\}$ which form a partition-of-unity for M subordinate to this covering.

We comment, for the reader's benefit, that in the following we are essentially working with both sets of foliation coverings of M simultaneously, and use essentially that the second is a refinement of the first. All estimates are obtained using the initial covering $\{(U_\alpha,\psi_\alpha) \mid \alpha \in \mathcal{A}\}$, while the calculation of the cup-product is made for the refined coverings $\{(V_a,\psi_a) \mid a \in \mathcal{A}_\varepsilon\}$ of M.

Let us next construct closed q-forms on M which are dual to the foliation cycles $[C_{\mu_i}]$. For each $a \in \mathcal{A}_\varepsilon$ and $u \in \Omega_a$, the *plaque* of $\mathcal{F}$ in V_a through the point $\psi_a^{-1}(0,y)$ is the set

$$P_a(y) = \psi_a^{-1}(\mathbf{D}^p \times \{y\}),$$

which is relatively open in a leaf of $\mathcal{F}$. The restriction of the unit normal disc subbundle ν_1 to the plaque $P_a(y)$ is denoted by

$$\nu_1(a,y) = \nu_1|_{P_a(y)} \to P_a(y).$$

For $s > 0$ sufficiently small, for all $a \in \mathcal{A}_\varepsilon$ and $y \in \Omega_a$, the scaled exponential map (8) restricts to a local diffeomorphism into,

$$\exp(a,y,s) = \exp_s : \nu_1(a,y) \to M. \tag{11}$$

Let Φ be a smooth closed q-form on the open manifold ν_1 which represents the Thom class of $\pi : \nu \to M$, and has support contained in the subdisc bundle $\nu_{1/2}$ of vectors with length at most $\frac{1}{2}$ (cf. [1], Section 12).

For $a \in \mathcal{A}_\varepsilon$, $y \in \Omega_a$ and $s > 0$ fixed as above, let $\tilde{\lambda}_a = \pi^*\{\lambda_a|_{P_a(y)}\}$ be the lift to the local bundle $\nu_1(a,s)$ of the partition function λ_a restricted to the plaque $P_a(y)$. Then define a q-form $\Phi(\varepsilon,a,y,s)$ on M:

$$\Phi(\varepsilon,a,y,s) = \exp(a,y,s)_*(\tilde{\lambda}_a \cdot \Phi). \tag{12}$$

Note that the support of $\tilde{\lambda}_a \cdot \Phi$ does not intersect the boundary of $\nu_1(a,y)$, so $\Phi(\varepsilon,a,y,s)$ is a smooth q-form on M. Moreover, there exists a constant $\varepsilon \geq s_\varepsilon > 0$, independent of y and a, so that for $0 < s < s_\varepsilon$, the support of $\Phi(\varepsilon,a,y,s)$ is contained in the open star of V_a:

$$\operatorname{star}\{V_a\} \stackrel{\text{def}}{=} \bigcup_{\substack{b \in \mathcal{A}_\varepsilon \\ V_b \cap V_a \neq \emptyset}} V_b.$$

Remark 5. Observe that there are at most N indices $b \in \mathcal{A}_\varepsilon$ which appear in the union in $\operatorname{star}\{V_a\}$ by Lemma 2. Therefore, there are at most N^2 indices $b \in \mathcal{A}_\varepsilon$ so that $\Phi(\varepsilon,a,y,s) \wedge \Phi(\varepsilon,b,y',s) \neq 0$.

Finally, we average these forms over the parameters a and y with respect to the measure μ_i for $i = 0,1$:

$$\Phi(i,\varepsilon,a,s) = \int_{y \in \Omega_a} \Phi(\varepsilon,a,y,s)\, d\mu_i(y), \tag{13}$$

$$\Phi(i,\varepsilon,s) = \sum_{a \in \mathcal{A}_\varepsilon} \Phi(i,\varepsilon,a,s). \tag{14}$$

The following is standard (cf. [10], Proposition 2.2 or [12], Proposition 1):

Proposition 3. *$\Phi(i,\varepsilon,s)$ is a closed q-form on M, and its deRham cohomology class is independent of ε and s, being equal to the Poincaré dual class $[\omega_{\mu_i}] \in H^q(M,\mathbf{R})$ of the Ruelle-Sullivan homology class $[C_{\mu_i}]$.*

We now come to the main idea of the proof of Proposition 2. We want to show that $\Phi(0,\varepsilon,s) \wedge \Phi(1,\varepsilon,s)$ has total mass on M estimated by a fixed multiple of ε. We then let ε tend to zero to obtain that this closed $2q$-form has cohomology class zero. If both forms are actually in the dual bundle ν^* to $\mathcal{F}$, then the wedge product vanishes identically. In the general case, the product need not vanish, and our idea is to use a measure-theoretic approach, as suggested by Lemma 1. The key to this approach is to construct the Poincaré dual forms via the refined covering of M. In this process, we must control the support of the forms $\Phi(i,\varepsilon,a,s)$, so that their overlaps correspond to the overlaps of the adjacent open sets in the covering. This allows us to make a uniform estimate of the number of non-vanishing overlaps, at most N^2 by Remark 5. However, to obtain this we require that $0 < s < s_\varepsilon$, so that the "diffusion parameter" s tends to zero, and the push-forward forms $\Phi(\varepsilon,\alpha,y,s)$ correspondingly diverge in the pointwise norm on forms. It is therefore critical to obtain an estimate on the mass of the wedge product which is independent of the parameter s. This is the content of the next result, which is the technical basis for the proof of Proposition 2, and hence of Theorem 1.

Proposition 4. *Let φ be a given closed $(p-q)$-form on M. Then there exists a constant $K(\varphi)$ so that for every $\varepsilon > 0$ sufficiently small, for all a, $b \in \mathcal{A}_\varepsilon$ and $0 < s < s_\varepsilon$, there is the uniform estimate in y, y':*

$$\int_M \left|\Phi(\varepsilon,a,y,s)\wedge\Phi(\varepsilon,b,y',s)\wedge\varphi\right| \leq K(\varphi)\{\mathrm{vol}_{\mathcal{F}}\{P_a(y)\}\}. \tag{15}$$

Corollary 4. *Let φ be a closed $(p-q)$-form on M. Then for ε sufficiently small, and with notation as above,*

$$\int_M \left|\Phi(0,\varepsilon,s)\wedge\Phi(1,\varepsilon,s)\right| \leq \varepsilon\cdot K\cdot K(\varphi)N^2\cdot\int_M 1\,d\hat{\mu}_1. \tag{16}$$

Hence, the cohomology pairing, which is independent of ε, satisfies

$$\int_M \Phi(0,\varepsilon,s)\wedge\Phi(1,\varepsilon,s)\wedge\varphi = 0.$$

Proof of Corollary 4. Let $\varepsilon > 0$ be sufficiently small so that Proposition 4 applies. Then make a calculation of (16), using the definitions and the estimates of Lemma 2 and (15):

$$\begin{aligned}
&\int_M \left|\Phi(0,\varepsilon,s_\varepsilon)\wedge\Phi(1,\varepsilon,s_\varepsilon)\wedge\varphi\right| \\
&\leq \sum_{a,b\in\mathcal{A}_\varepsilon}\int_{y\in\mathbf{D}^q}\int_{y'\in\mathbf{D}^q}\int_{V_a\cap V_b}\left|\Phi(\varepsilon,a,y,s_\varepsilon)\wedge\Phi(\varepsilon,b,y',s_\varepsilon)\wedge\varphi\right|\,d\mu_0(y)\,d\mu(y') \\
&\leq \sum_{\substack{a,b\in\mathcal{A}_\varepsilon\\ \mathrm{star}\{V_a\}\cap\mathrm{star}\{V_b\}\neq\emptyset}} K(\varphi)\cdot\int_{y\in\mathbf{D}^q}d\mu_0(y)\int_{y'\in\mathbf{D}^q}\mathrm{vol}_{\mathcal{F}}\{P_b(y')\}\,d\mu_1(y') \\
&\leq \varepsilon\cdot K(\varphi)N^2\cdot\sum_{b\in\mathcal{A}_\varepsilon}\int_M \chi_{\tilde{V}_b}\,d\hat{\mu}_1 \\
&\leq \varepsilon\cdot K(\varphi)KN^2\cdot\int_M 1\,d\hat{\mu}_1. \qquad\square
\end{aligned}$$

It remains to prove Proposition 4. Fix $\varepsilon > 0$, $a \in \mathcal{A}_\varepsilon$ and $y \in \Omega_a$. The integrand in (15) is non-zero only when the supports of the forms $\Phi(\varepsilon,a,y,s)$ and $\Phi(\varepsilon,b,y',s)$ overlap, so we can pull the integral back to the unit disc bundle

$$\pi : \nu_1(a,y) \to P_a(y).$$

Introduce a change-of-coordinates transformation, mapping an open subdomain of $\nu_1(a,y)$ to an open subdomain of $\nu_1(b,y')$:

$$E_{ba}(y,y',s) = \exp(b,y',s)^{-1} \circ \exp(a,y,s). \tag{17}$$

Also define

$$\begin{aligned} \Phi_{ba}(y,y',s) &= E_{ba}(y,y',s)^*(\tilde{\lambda}_b \cdot \Phi), \\ \varphi(a,y,s) &= \exp(a,y,s)^*(\varphi), \end{aligned} \tag{18}$$

and then observe that

$$\begin{aligned} &\int_M \left|\Phi(\varepsilon,a,y,s) \wedge \Phi(\varepsilon,b,y',s) \wedge \varphi\right| \\ &\qquad = \int_{\nu_1(a,y)} \left|\tilde{\lambda}_a \Phi \wedge \Phi_{ba}(y,y',s) \wedge \varphi(a,y,s)\right|. \end{aligned} \tag{19}$$

For φ fixed, the forms $\varphi(a,y,s)$ have uniform bounds independent of the parameters. Thus, to obtain a uniform estimate of the integral (19), it suffices to obtain pointwise estimates of the forms $\Phi_{ba}(y,y',s)$ independent of ε, s, y and y'. This will follow, in turn from a uniform estimate for the derivatives of the coordinate change $E_{ba}(y,y',s)$. We next introduce natural, well-adapted coordinates for the bundle ν_1 and calculate the derivatives of $E_{ba}(y,y',s)$ with respect to them.

For each $\alpha \in \mathcal{A}$, the restricted bundle $\nu|_{\tilde{U}_\alpha}$ is trivial, as $\tilde{U}_\alpha$ is contractible. Choose a C^1-orthonormal framing $\{e_\alpha^1,\ldots,e_\alpha^q\}$ of ν over $\tilde{U}_a$ which is parallel along the plaques $\tilde{P}_\alpha(y)$ in $\tilde{U}_\alpha$. The closure of U_α is compact and contained in $\tilde{U}_\alpha$, hence the restriction of this framing to U_α has uniformly bounded covariant derivative $\nabla(e_a^i)$. For $\varepsilon > 0$ given, the refinement $\{(V_a,\psi_a) \mid a \in \mathcal{A}_\varepsilon\}$ of the foliation cover $\{(U_\alpha,\varphi_\alpha) \mid \alpha \in \mathcal{A}\}$ is formed from partitions of the open sets U_α in the transverse direction; we restrict the framing of ν to the open subsets $\{\tilde{V}_a \mid a \in \mathcal{A}_\varepsilon\}$ to obtain framings $\{e_a^1,\ldots,e_a^q\}$ of the restricted bundles $\nu_1(a,y)$ which have uniformly bounded covariant derivatives.

Let $x_a \in \mathbf{D}^p$ denote the coordinates on the plaque $P_a(y)$ induced by the chart ψ_a. Let $\mathbf{B}^q$ denote the unit ball in $\mathbf{R}^q$, with coordinate vector $v \in \mathbf{B}^q$. Then the orthonormal framing $\{e_a^1(x),\ldots,e_a^q(x)\}$ of the fiber $\nu(x)$ over $\psi_a(x) \in P_a(y)$ gives an identification of $\mathbf{B}^q$ with the unit disc fiber $\nu_1(x)$. We combine these to obtain "bundle coordinates" (x,v) on $\nu_1(a,y)$. When there is more than one set of coordinates under consideration, we use the notation (x_a,v_a), (x_b,v_b), etc., to avoid confusion.

Assume that (a,y) and (b,y') are such that the supports of the forms $\Phi(\varepsilon,a,y,s)$ and $\Phi(\varepsilon,b,y',s)$ overlap. For s sufficiently small, we can assume that the plaque $P_b(y')$ intersects the coordinate chart $(\tilde{V}_a,\tilde{\psi}_a)$. Observe that this restriction on s depends only on the geometry of the cover $\{(U_\alpha,\psi_\alpha) \mid \alpha \in \mathcal{A}\}$ and its extension, so we will assume in the following that $0 < s < s_\varepsilon$ implies that this condition holds. Let $z' \in \tilde{\Omega}_a$ denote the transverse coordinate of the plaque $P = b(y')$ in the coordinate chart $(\tilde{V}_a,\tilde{\psi}_a)$. Then by the compactness of M and the regularity of the initial foliation cover, there exists a constant $K' > 0$, depending only on the initial choices of Riemannian metric on TM and foliation charts $\{(U_\alpha,\psi_\alpha) \mid \alpha \in \mathcal{A}\}$, such that the Euclidean distance from y to z' is at most $sK' \leq \varepsilon K'$.

Introduce matrix notation for the derivatives of $E_{ba}(y,y',s)$ with respect to the bundle coordinates (x_a,v_a) on the domain $\nu_1(a,y)$, and (x_b,v_b) on the range $\nu_1(b,y')$:

$$J_{ba}(y,y',x_a,v_a) = \frac{DE_{ba}(y,y',s)}{D(x,v)} = \begin{bmatrix} A(s,x_a,v_a) & B(s,x_a,v_a) \\ C(s,x_a,v_a) & D(s,x_a,v_a) \end{bmatrix}. \tag{20}$$

The foliation chart (V_a,ψ_a) induces coordinates on V_a denoted by $(\tilde{x},\tilde{y}) = (\tilde{x}_a,\tilde{y}_a)$, for $\tilde{x} \in \mathbf{D}^p$ and $\tilde{y} \in \Omega_a$. Given $b \in \mathcal{A}_\varepsilon$ and $y' \in \Omega_b$, we give $\nu_1(b,y')$ the bundle coordinates (x_b,v_b). We then have the "exponential" change of coordinates,

$$(\tilde{x}_a,\tilde{y}_a) = \exp(b,y',s)(x_b,v_b). \tag{21}$$

Introduce the derivative matrix for this change of coordinates:

$$M_{ab}(Y',s,x_b,v_b) = \frac{D\exp(b,y',s)}{D(x_b,v_b)} = \begin{bmatrix} M_x^{\tilde{x}}(s,x_b,v_b) & M_v^{\tilde{x}}(s,x_b,v_b) \\ M_x^{\tilde{y}}(s,x_b,v_b) & M_v^{\tilde{y}}(s,x_b,v_b) \end{bmatrix}. \tag{22}$$

Lemma 3.

1. *There exist matrices $M_v^{\tilde{x}}(x_b,v_b)$ and $M_v^{\tilde{y}}(x_b,v_b)$ which are continuous in (x_b,v_b) so that*

$$M_v^{\tilde{x}}(s,x_b,v_b) = s \cdot M_v^{\tilde{x}}(x_b,v_b) \tag{23}$$

$$M_v^{\tilde{y}}(s,x_b,v_b) = s \cdot M_v^{\tilde{y}}(x_b,v_b); \tag{24}$$

2. *There is a uniform estimate on the matrix norm*

$$\|M_x^{\tilde{y}}(s,x_b,v_b)\| = O(s|v_b|). \tag{25}$$

Proof. The first part is elementary, as $M_v^{\tilde{x}}(s, x_b, v_b)$ and $M_v^{\tilde{y}}(s, x_b, v_b)$ are the partial derivatives of $\exp(b, y', s) = \exp_s|_{\nu_1(b,y')}$ with respect to the coordinate v_b, and $\exp_s$ is simply the rescaling of the map $\exp_1$ by the factor s in the v-coordinate. Thus, we can take for $M_v^{\tilde{x}}(x_b, v_b)$ and $M_v^{\tilde{y}}(x_b, v_b)$ the matrix of partials of $\exp_1$ with respect to v_b, evaluated at the point (x_b, sv_b).

The estimate on the norm $\|M_x^{\tilde{y}}(s, x_b, v_b)\|$ is more geometric in origin. The framing $\{e_v^1, \ldots, e_b^q\}$ is parallel along the plaque $P_b(y')$, which implies that the matrix of partial derivatives of the $\tilde{y}_a$-coordinates of $\exp_1|_{\nu_1(b,y')}$ with respect to x_b vanishes identically for $v_b = 0$. Let $p_{\tilde{y}} : V_a \to \Omega_a$ denote the projection onto the second coordinate. The map exp is smooth on TM, and the coordinates (x_b, v_b) are C^1, so there is a uniform estimate

$$\left\| \frac{Dp_{\tilde{y}} \circ \exp_1|_{\nu_1(b,y')}}{Dx_b}(x_b, v_b) \right\| = O(|v_b|). \tag{26}$$

The maps $\exp_s$ are obtained by precomposing with the s-scaling map in the v_b-coordinate, which replaces (x_b, v_b) with $(x_b, s \cdot v_b)$, so the chain rule yields a uniform estimate

$$\left\| \frac{Dp_{\tilde{y}} \circ \exp(b, y', s)}{Dx_b}(x_b, v_b) \right\| = O(|s \cdot v_b|) \tag{27}$$

which is equivalent to (25). □

We can now finish the proof of Proposition 4, which has been reduced to obtaining a uniform estimate on the derivative matrix of

$$E_{ba}(y, y', s) = \exp(b, y', s)(x_b, v_b)^{-1} \circ \exp(a, y, s)(x_a, v_a). \tag{28}$$

In order for the integrand of (15) to be non-zero, we must have that y and z' be at most $\varepsilon K'$ apart in the transversal space $(-1-\delta, 1+\delta)^q$ for the chart $\tilde{\psi}_\alpha$, where $a = (\alpha, i)$. The exponential map exp is a quasi-isometry in the normal parameter v when restricted to the unit disc bundle ν_1, so there exists a second constant, K'', so that if the integrand (15) is non-zero, then $|s \cdot v_a| < \varepsilon K''$. Interchanging the rôles of y and y' gives the corresponding estimate $|s \cdot v_b| < \varepsilon K''$.

The derivative matrix of (28) has the matrix factorization

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{(s,x_a,v_a)} = \begin{bmatrix} M_x^{\tilde{x}} & M_v^{\tilde{x}} \\ M_x^{\tilde{y}} & M_v^{\tilde{y}} \end{bmatrix}^{-1}_{(s,x_b,v_b)} \cdot \begin{bmatrix} M_x^{\tilde{x}} & M_v^{\tilde{x}} \\ M_x^{\tilde{y}} & M_v^{\tilde{y}} \end{bmatrix}_{(s,x_a,v_a)}. \tag{29}$$

A cofactor calculation of the inverse matrix in (29) and the matrix estimates (23), (24) and (25), yields that the product in (29) is uniformly bounded by a factor proportional to $|s \cdot v_a|$ and $|s \cdot v_b|$. Then by the previous remarks, the estimate on the distance from y to z' forces the estimates $|s \cdot v_a| < \varepsilon K''$ and $|s \cdot v_b| < \varepsilon K''$, from which we get a uniform estimate on the product (29), independent of s.

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