Annals of Mathematics, **135** (1992), 361–410

Rigidity for Anosov actions of higher rank lattices

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1. Introduction

The natural action of the determinant-one, integer $n \times n$ matrices $\mathrm{SL}(n, \mathbf{Z})$ on $\mathbf{R}^n$ preserves the integer lattice $\mathbf{Z}^n$; hence for each subgroup $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ there is an induced "standard action" on the quotient n-torus,

$$\varphi: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n.$$

This is the simplest example of a large class of analytic "standard" actions of lattices in semisimple Lie groups on locally homogeneous spaces. A basic problem is to understand the differentiable actions near to such a standard action in terms of their geometry and dynamics (cf. [13], [50], [51]).

A C^r-action $\varphi: \Gamma \times X \to X$ of a group Γ on a compact manifold X is said to be *Anosov* if there exists at least one element, $\gamma_h \in \Gamma$, such that $\varphi(\gamma_h)$ is an Anosov diffeomorphism of X.

We begin in this paper to study the Anosov differentiable actions of lattices, including many standard algebraic examples, and especially to study their stability properties. Our main theme is that either the C^r-rigidity or the C^r-deformation rigidity of an Anosov action (for $1 \leq r \leq \infty$, or even for the real analytic case) can be shown just by studying the behavior of the periodic orbits for the action.

There are two notions of "structural stability" that appear in this paper, *rigidity* and *deformation rigidity*. A C^1*-perturbation* of a C^r-action φ is simply another C^r-action φ_1 such that for a finite set of generators $\{\delta_1, \ldots, \delta_d\}$ of Γ, the C^r-diffeomorphisms $\varphi(\delta_i)$ and $\varphi_1(\delta_i)$ are C^1-close for all i. An action φ is said to be C^r*-rigid* (or *topologically rigid* if $r = 0$) if every sufficiently small C^1-perturbation of φ is C^r-conjugate to φ, for $0 \leq r \leq \infty$, or $r = \omega$ in the case of real analytic actions.

A C^1*-deformation* of an action φ is a continuous path of C^r-actions φ_t defined for some $0 \leq t \leq \varepsilon$ with $\varphi_0 = \varphi$. An action φ is said to be C^r*-deform-*

*Supported in part by NSF Grant DMS 89-02960.

ation rigid (or *topologically deformation rigid* if $r = 0$) if every C^1-deformation of φ, with φ_t contained in a sufficiently small C^1-neighborhood of φ, is C^r-conjugate to φ by a continuous path of C^r-diffeomorphisms.

Our strategy for studying the C^r-rigidity and C^r-deformation rigidity properties of Anosov actions has two steps. The first is to focus on the intermediate task of showing the topological rigidity and topological deformation rigidity of the given action (the cases where $r = 0$). For an Anosov action with dense periodic orbits we observe that it suffices to control the behavior on the periodic orbits to obtain a topological conjugacy of the full group action.

A higher rank lattice group always contains a maximal rank abelian subgroup generated by semisimple elements (cf. [37], and see Theorem 7.2). The second step in proving rigidity is to use both the restricted actions of these abelian subgroups and the associated concept of a *trellis* structure for the abelian action (see Section 2) to prove that a topological conjugacy between trellised actions must be as smooth as the actions involved. The dynamical data obtained from restricting a given action to an action of a maximal abelian subgroup are formalized in the definition of a *Cartan* action, Definition 2.13. We prove that a volume-preserving Cartan action is smoothly determined by its exponents at periodic points, Theorem 2.19, and that a Cartan action with *constant exponents* is necessarily affine, Theorem 2.21.

Our rigidity results for Anosov actions, as detailed in Section 2, are illustrated in this Introduction by applying them to the case of the standard action of a subgroup $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ on the torus $\mathbf{T}^n$. These and other applications are discussed in detail in Section 7.

A finitely generated group Γ is said to be a *higher rank lattice* if Γ is a discrete subgroup of a connected semisimple algebraic $\mathbf{R}$-group G, with the $\mathbf{R}$-split rank of each factor of G at least 2, G having finite center and $G^0_{\mathbf{R}}$ having no compact factors, and such that G/Γ has finite volume.

THEOREM 1.1. *Let $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ be isomorphic to a subgroup of finite index of a higher rank lattice (hence $n \geq 3$) and suppose that Γ contains a hyperbolic matrix. Then the standard action of Γ on $\mathbf{T}^n$ is topologically deformation rigid under continuous deformations in the C^1-topology on differentiable actions.*

The conclusion of Theorem 1.1 is false for $\mathrm{SL}(2, \mathbf{Z})$: Example 7.21 and Theorem 7.22 show that the standard action of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{T}^2$ can be smoothly deformed through volume-preserving C^ω-actions that are not topologically conjugate to linear actions.

The approach to geometric rigidity developed in this paper is based on ideas from dynamical systems and especially follows the philosophy that an Anosov dynamical system is determined by its behavior at periodic orbits. We use the Anosov hypothesis repeatedly to reduce proofs to questions about the behavior at periodic orbits. This is a standard method for the study of Anosov diffeomorphisms (cf. especially [2], [23], [8], [28], [18], [39], [40]), and we show that similar techniques also work for group actions.

For example, our approach to topological deformation rigidity is to study the behavior of the periodic orbits for the system under deformation. The Anosov hypothesis guarantees that these periodic orbits are always isolated, with a unique fixed point for their associated linear isotropy actions. We impose an additional hypothesis on the first cohomology of the group Γ (the "strong vanishing cohomology" condition SVC(N) of Definition 2.7) so that by the stability theorem of D. Stowe ([41], [42]), the periodic orbits are stable under perturbation. For example, the hypothesis that the group has higher rank implies the condition SVC(N), as a consequence of a deep result of G. A. Margulis (Theorem 2.1 of [33]; see also Theorem 2.8 below). These ideas lead to the proof of Theorem 2.9, which together with Theorem 2.8 implies Theorem 1.1.

The central problem for Anosov dynamical systems with one generator is to find conditions under which topological conjugacy implies smooth conjugacy ([9], [10], [11], [18], [26], [27], [29], [28], [31], [32], [36], [35]). A second theme of this paper is to develop criteria when topological conjugacy of Anosov group actions implies smooth conjugacy. This leads to the notion of a *trellised action*, Definition 2.11. Briefly this is an Anosov action with sufficiently many hyperbolic elements that preserve a maximally transverse system of "sufficiently regular" one-dimensional foliations of X. These foliations yield a dynamically defined affine structure on X, which is stable under perturbations. Theorems 2.12 and 2.15 formulate hypotheses on a trellised group action, sufficient to prove the differentiable regularity of topological conjugacies between Anosov actions. These two theorems, applied to the case of the standard action on the torus, yield:

THEOREM 1.2. *Let $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ be isomorphic to a subgroup of finite index of a higher rank lattice and suppose there is a linear trellis $\mathcal{T}_0$ for which the standard action of Γ on $\mathbf{T}^n$ is trellised, with associated hyperbolic elements $\Delta = \{\gamma_1, \ldots, \gamma_n\}$.*

(1) *Let $\{\varphi_t | 0 \leq t \leq 1\}$ be a one-parameter family of C^r-actions, which lie in a sufficiently small C^1-neighborhood of φ_0, and let $\{H_t | 0 \leq t \leq 1\}$ be a continuous family of homeomorphisms conjugating each φ_t to φ. Then the maps H_t are C^r-diffeomorphisms, and the family is continuous in t for the C^r-topology on maps, for $r = 1$ or ∞.*

(2) *Let $H_1 \colon \mathbf{T}^n \to \mathbf{T}^n$ be a topological conjugacy between the standard action φ and a C^r-action φ_1 of Γ on $\mathbf{T}^n$ such that H maps $\mathcal{T}_0$ to another trellis*

$\mathcal{T}_1$ on $\mathbf{T}^n$ with the same associated hyperbolic elements Δ. Suppose that the group elements Δ commute, the subgroup $\mathcal{A} \subset \Gamma$ generated by Δ is a cocompact lattice in a maximal $\mathbf{R}$-split torus of G, and the restriction of φ_0 to Δ defines an abelian Cartan action; *then H is a C^r-diffeomorphism, for $r = 1$ or ∞. Moreover, if φ_1 is a real analytic action and $\mathcal{T}_1$ is an analytic trellis, then the conjugacy H_1 is real analytic.*

The proof of Theorem 1.2 follows from Theorems 2.8, 2.12, 2.15 and 5.1, Propositions 4.1 and 5.11 and the remarks at the beginning of Section 7.

The results of this paper developed from the study of rigidity and stability of the standard action for subgroups of finite index in $\mathrm{SL}(n, \mathbf{Z})$. R. Zimmer obtained the first rigidity result for higher rank lattice actions on compact manifolds: he proved that an ergodic, volume-preserving C^1-perturbation of an isometric action of a higher rank lattice is again isometric (see [49], [51]). Later he proved [52] the *infinitesimal rigidity* for ergodic actions on locally homogeneous spaces by higher rank cocompact lattices (cf. also [50], [51]). J. Lewis [22] showed in his Thesis that for $n \geq 7$, the standard action of $\mathrm{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n$ is infinitesimally rigid. (Note that the Weil approach [45] to deducing differentiable stability from infinitesimal rigidity encounters serious difficulties when applied to deformations of lattice actions, as one needs tame estimates on the coboundaries produced, which for lattice actions are notoriously difficult to establish.

There have been a number of subsequent developments since the main results of this paper were announced in a preliminary form in August, 1989, and appeared in [14]. (Portions of the manuscript circulated in Fall, 1989, and the first version appeared in June, 1990.)

A. Katok and J. Lewis proved in [21] that the standard action of a subgroup of finite index, $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$, is topologically rigid for $n \geq 4$. Their method continues the approach of this paper, in that they construct the topological conjugacy on the periodic orbits. However, in place of the repeated application of Stowe's theorem made in this paper (cf. Remark 3.9), they require only one application to ensure the existence of a fixed point for the perturbation. Katok and Lewis then construct the conjugacy on the full set of periodic points by making use of the combinatorial structure of a subgroup Γ of finite index in $\mathrm{SL}(n, \mathbf{Z})$ and the additional structure provided by the abelian Cartan subaction for Γ.

J. Lewis and R. Zimmer announced [53], among other rigidity results, that Zimmer's cocycle super-rigidity theory ([46], and also Theorem 5.2.5 of [48]) and techniques of Anosov diffeomorphisms yield the C^∞-rigidity of the standard action of $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n$ for $n \geq 3$.

Zimmer's cocycle super-rigidity theorem is a very deep dynamical extension of the Margulis super-rigidity theorem for lattices. Its application, in combination with Theorem 2.21 of this paper, has yielded numerous definitive results on the rigidity of volume-preserving Anosov actions of higher rank lattices ([19], [17], [20]). We formulate one of these applications for the case of the standard action, based on Theorem 2.22:

THEOREM 1.3 (see [19]). *Let* $\varphi\colon \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ *be a standard action and suppose that*:

(1) $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ *is a subgroup of finite index for* $n \geq 3$; *or*

(2) $\Gamma \subset \mathrm{Sp}(n, \mathbf{Z}) \subset \mathrm{SL}(2n, \mathbf{Z})$ *is a subgroup of finite index of the group of integer symplectic matrices* $\mathrm{Sp}(n, \mathbf{Z})$, *for* $n \geq 2$; *or*

(3) $\Gamma \subseteq \Gamma_0 \times \cdots \times \Gamma_d \subset \mathrm{SL}(n, \mathbf{R})$ *is a subgroup of finite index, where each factor group* Γ_i *satisfies one of the two above cases, and* Γ *contains a hyperbolic element.*

Then φ *is* C^r*-rigid for* $r = 1, \infty$ *and for* $r = \omega$.

We conclude this Introduction with a conjecture from [16], supported by the available results to date:

Conjecture 1.4 (Anosov Rigidity). Let Γ be a lattice of higher rank and $\varphi\colon \Gamma \times X \to X$ be a C^r-Anosov action on a compact smooth manifold X of dimension n, for $r \geq 1$. Then

(1) there is a finite covering of X by a nilmanifold $\tilde{X}$;

(2) there is a subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index so that the action $\varphi | \tilde{\Gamma}$ lifts to an action $\tilde{\varphi}$ on $\tilde{X}$;

(3) the C^r-conjugacy class of $\tilde{\varphi}$ is determined by the homotopy type of the action. That is, $\tilde{\varphi}$ is topologically conjugate to the standard algebraic action induced on the nilmanifold $\hat{\pi}_1(\tilde{X})/\pi_1(\tilde{X})$, where $\hat{\pi}_1(\tilde{X})$ denotes the Malcev completion of the fundamental group of $\tilde{X}$.

Acknowledgement. The author is indebted to A. Katok, R. de la Llavé, R. Spatzier, D. Stowe, T. N. Venkataramanan and R. Zimmer for helpful conversations during the development of this work. The support of the NSF and the University of Colorado at Boulder for the special Conference on Geometric Rigidity, May, 1989, is also gratefully acknowledged.

2. Statement of results

In this section we will formulate the definitions and notions used throughout the paper. We then state the precise form of our theorems, some of whose applications were discussed in the Introduction. Proofs of the theorems are in following sections, and further applications are discussed in Section 7.

Let Γ be a finitely generated group and choose a set of generators $\{\delta_1, \ldots, \delta_d\}$. Let X be a compact Riemannian manifold of dimension n without boundary. Here $\varphi\colon \Gamma \times X \to X$ will denote a C^r-action of Γ on X. We will either assume that $r = 1$ or ∞ for a differentiable action or set $r = \omega$ if the action is real analytic. All of the results of this paper have counterparts for C^r-actions with $1 < r < \infty$; however there is some loss in regularity in applying the Sobolev lemma for finite degrees of differentiability (cf. Theorem 2.6, [18]). For reasons of exposition we omit discussion of the intermediate differentiability cases.

Recall first the definition of C^l-topology on the space of C^r-actions on X. Given $\varepsilon > 0$, two C^r-actions $\varphi_0, \varphi_1\colon \Gamma \times X \to X$ are *ε-C^l-close* if for each generator δ_i of Γ, the diffeomorphism $\varphi_0(\delta_i)$ of X is ε-close to $\varphi_1(\delta_i)$ in the uniform C^l-topology on maps. The ε-C^l-ball about φ is the set of all C^r-actions φ_1 that are ε-C^l-close to φ.

Given $\varepsilon > 0$, an *ε-C^l-perturbation* of φ is a C^r-action $\varphi_1\colon \Gamma \times X \to X$ with φ_1 contained in the ε-C^l-ball about φ.

An *ε-$C^{k,l}$-deformation* of an action φ is a one-parameter family of C^r-actions, $\{\varphi_t\colon \Gamma \times X \to X \mid 0 \le t \le 1\}$ so that $\varphi_0 = \varphi$ and

(1) φ_t is in the ε-C^1-ball about φ for each $0 \le t \le 1$ (note that ε refers to the C^1-topology);

(2) for each $\gamma \in \Gamma$ the map $\varphi_t(\gamma)$ varies C^k on the parameter t in the C^l-topology on maps.

A "sufficiently small C^1-deformation" of an action φ, as in the Introduction, is simply an ε-$C^{0,1}$-deformation for $\varepsilon > 0$ appropriately chosen.

Note that evaluating an ε-$C^{k,l}$-deformation of φ at a particular value of t yields an ε-C^l-perturbation of φ. However not every ε-C^l-perturbation is a priori obtained from an ε-$C^{k,l}$-deformation.

An ε-C^l-perturbation $\{\varphi_1\}$ of φ is *differentiably trivial* if there is a C^r-diffeomorphism, $H_1\colon X \to X$, such that for each $\gamma \in \Gamma$ we have

$$H_1^{-1} \circ \varphi_1(\gamma) \circ H_1 = \varphi(\gamma). \tag{1}$$

When φ is an analytic action, there is the corresponding notion of *analytically trivial* perturbation, where we require that H_1 be an analytic diffeomorphism. The perturbation is *topologically trivial* if there exists a homeomorphism $\{H_1\}$ satisfying (1).

An ε-$C^{k,l}$-deformation $\{\varphi_t\}$ is *differentiably trivial* if there is a one-parameter family of C^r-diffeomorphisms, $H_t\colon X \to X$, varying C^k with the parameter t in the C^l-topology on diffeomorphisms, and if for each $\gamma \in \Gamma$ and $0 \le t \le 1$ we have

$$\begin{aligned} H_t^{-1} \circ \varphi_t(\gamma) \circ H_t &= \varphi(\gamma), \\ H_0 &= Id_X. \end{aligned} \tag{2}$$

When φ is an analytic action and the deformation is through analytic actions, there is the corresponding notion of *analytically trivial* deformation, where we require that each H_t be an analytic diffeomorphism varying C^k with t in the C^l-topology on diffeomorphisms. The deformation is *C^k-topologically trivial* if there exists a C^k-family of homeomorphisms $\{H_t\}$ satisfying (2).

Definition 2.1. Let φ be a C^r-action of Γ on X, for $r = 1, \infty, \omega$.

• φ is *C^r-rigid* (respectively, *topologically rigid*) if there exists $\varepsilon > 0$ so that every ε-C^1-perturbation of φ is differentiably trivial (respectively, topologically trivial).

• φ is *$C^{k,l}$-deformation rigid* (respectively, *C^k-topologically deformation rigid*) if there exits $\varepsilon > 0$ so that every ε-$C^{k,l}$-deformation is differentiably trivial (respectively, every ε-$C^{k,1}$-deformation is C^k-topologically trivial).

We summarize the differing roles that the indices "r, k, l" play in this work:

• "r" is always the differentiability of the action;

• "k" is the differentiability of the path involved, if any;

• "l" indicates the topology on the space of actions, which is usually taken to be $l = 1$, although $l = \infty$ is possible for C^∞-actions when a path of diffeomorphisms is C^k for the C^∞-topology on actions;

• finally all of this takes place in an ε-ball about the given action in the C^1-topology on actions.

Remark 2.2. These definitions have natural interpretations in terms of the representation "variety" of Γ into the manifold $\mathscr{G} = \mathrm{Diff}^r(X)$ equipped with the C^r-Frechet topology, for $1 \le r \le \infty$. Let $\mathscr{R}(\Gamma, \mathscr{G})$ denote the set of representations, where each action $\varphi\colon \Gamma \times X \to X$ determines $\hat{\varphi} \in \mathscr{R}(\Gamma, \mathscr{G})$. For each $0 \le l \le r$ we can also consider the C^l-Frechet topology on $\mathscr{R}(\Gamma, \mathscr{G})$.

• An ε-C^l-perturbation of φ is a "point" $\hat{\varphi}_1 \in \mathscr{R}(\Gamma, \mathscr{G})$, which is ε-C^l-close to $\hat{\varphi}$.

• An ε-$C^{k,l}$-deformation $\{\varphi_t\}$ of an action φ corresponds to a path $\hat{\Phi}$: $[0,1] \to \mathscr{R}(\Gamma, \mathscr{G})$, which is C^k-differentiable (in t) in the C^l-topology on maps, $\hat{\Phi}(0) = \hat{\varphi}$, and $\hat{\Phi}(t)$ lies within the ε-C^1-ball about $\hat{\varphi}$ for all t.

The group $\mathscr{G}$ acts on $\mathscr{R}(\Gamma, \mathscr{G})$ via conjugation, and the action is continuous in the C^r-topology. Introduce the quotient topological space

$$\tilde{\mathscr{R}}(\Gamma, \mathscr{G}) = \mathscr{R}(\Gamma, \mathscr{G})/\mathscr{G}.$$

• φ is C^1-differentiably rigid implies that $\tilde{\varphi} \in \tilde{\mathscr{R}}(\Gamma, \mathscr{G})$ is isolated in the quotient C^r-topology.

• φ is $C^{k,1}$-deformation rigid implies that the point $\tilde{\varphi} \in \tilde{\mathscr{R}}(\Gamma, \mathscr{G})$ is an isolated C^k-path component in the quotient C^r-topology on $\tilde{\mathscr{R}}(\Gamma, \mathscr{G})$.

Let us now introduce three ideas which are central to the methods of this paper. A point $x \in X$ is *periodic* for φ if the set

$$\Gamma(x) \overset{\text{def}}{=} \{\varphi(\gamma)(x) | \gamma \in \Gamma\}$$

is finite. Let $\Lambda = \Lambda(\varphi) \subset X$ denote the set of periodic points for φ. For each $x \in \Lambda$ let $\Gamma_x \subset \Gamma$ denote the isotropy subgroup of x. Note that the index $[\Gamma_x : \Gamma] \leq o(x)!$, where $o(x) = |\Gamma(x)|$ is the order of the orbit of x (cf. Lemma 3.3).

A C^1-diffeomorphism $f\colon X \to X$ is said to be *Anosov* (cf. [2], [40]) if there exist

- a Finsler on TX,
- a continuous splitting of the tangent bundle into Df-invariant subbundles, $TX \cong E^+ \oplus E^-$,
- constants $\lambda > 1 > \mu > 0$ and $c > 0$

such that for all positive integers m,

$$\text{(3)}\qquad \begin{aligned} \|D(f^m)(v)\| &> c\lambda^m \cdot \|v\|; \qquad 0 \neq v \in E^+, \\ \|D(f^m)(v)\| &< c^{-1}\mu^m \cdot \|v\|; \qquad 0 \neq v \in E^-. \end{aligned}$$

The property that a diffeomorphism f is Anosov is independent of the choice of a Finsler on TX. One can also "re-norm" the bundle TX so that $c = 1$ and $\lambda = 1/\mu$ (cf. Mather, Appendix to [40]).

We say that $\gamma \in \Gamma$ is *φ-hyperbolic* if $\varphi(\gamma)$ is an Anosov diffeomorphism of X. Recall from the Introduction that φ is said to be an *Anosov action* if there is at least one φ-hyperbolic element in Γ.

The stable bundle E^- of an Anosov C^r-diffeomorphism f does not usually have any invariant (proper) subbundles. We single out one case where there is such a subbundle, which is dynamically determined. An Anosov diffeomorphism f has a *one-dimensional*, *strongest stable* distribution if there exists a Df-invariant, one-dimensional vector subbundle $E^{ss} \subset E^-$, which satisfies an exponential dichotomy; that is, there exist

- a Finsler on TX,
- a continuous splitting of the tangent bundle into Df-invariant subbundles, $TX \cong E^{cs} \oplus E^{ss}$,
- constants $\lambda > 1$ and $1 > \varepsilon > 0$

such that for all positive integers m,

$$\text{(4)}\qquad \begin{aligned} \|D(f^m)(v)\| &> (\lambda - \varepsilon)^{-m} \cdot \|v\|; \qquad 0 \neq v \in E^{cs}, \\ \|D(f^m)(v)\| &< (\lambda + \varepsilon)^{-m} \cdot \|v\|; \qquad 0 \neq v \in E^{ss}. \end{aligned}$$

The strongest stable distribution E^{ss} is necessarily integrable, and the leaves of the resulting foliation $\mathscr{F}^{ss}$ are C^r-immersed, one-dimensional submanifolds (cf. [12], [5], Chapter 6, [39]).

Finally we introduce the important notion of infinitesimal local stability of fixed points. This requires several preliminary notions to be set. Let $\mathbf{E}$ denote a finite-dimensional, real vector space, Γ a finitely generated group and $\tilde{\Gamma}$, generically, a subgroup of finite index of Γ.

Definition 2.3. A representation $\rho_{\mathbf{E}}$: $\tilde{\Gamma} \to \mathrm{GL}(\mathbf{E})$ is said to be

• *compact* if its image is contained in a compact subgroup of $\mathrm{GL}(\mathbf{E})$;

• *noncompact* if for every invariant subspace $\mathbf{F} \subset \mathbf{E}$, the restriction $\tilde{\rho}_{\mathbf{F}}$: $\tilde{\Gamma} \to \mathrm{GL}(\mathbf{F})$ is not compact;

• *hyperbolic* if there exists $\gamma_h \in \tilde{\Gamma}$ such that $\tilde{\rho}(\gamma_h)$ is a hyperbolic matrix.

Note that for a hyperbolic representation ρ of Γ, the restriction of ρ to every subgroup $\tilde{\Gamma}$ of finite index in Γ is noncompact, as $\gamma_h^k \in \tilde{\Gamma}$ for some $k > 0$.

Definition 2.4. A representation ρ: $\tilde{\Gamma} \to \mathrm{GL}(\mathbf{E})$ is *infinitesimally rigid* if

(1) the linear action of $\rho(\tilde{\Gamma})$ on $\mathbf{E}$ has 0 as the unique fixed point;

(2) the first cohomology group of $\tilde{\Gamma}$ with coefficients in the $\tilde{\Gamma}$-module $\mathbf{E}$ is trivial: $H^1(\tilde{\Gamma}; \mathbf{E}_\rho) = 0$.

For this work we introduce a strengthening of the above notion of rigidity, so that the hypotheses of the new definition are themselves stable under perturbation (cf. the proof of Proposition 3.6).

Definition 2.5. A representation ρ: $\tilde{\Gamma} \to \mathrm{GL}(\mathbf{E})$ is *strong-infinitesimally rigid* if

(1) ρ is hyperbolic;

(2) for all noncompact representations $\tilde{\rho}$: $\tilde{\Gamma} \to \mathrm{GL}(\mathbf{E})$, the first cohomology group of $\tilde{\Gamma}$ with coefficients in the $\tilde{\Gamma}$-module $\mathbf{E}$ is trivial: $H^1(\tilde{\Gamma}; \mathbf{E}_{\tilde{\rho}}) = 0$.

Note that a strong-infinitesimally rigid representation is infinitesimally rigid.

The standard definition of infinitesimal rigidity at a fixed point is formulated in terms of the linear isotropy representation. The following is a very useful extension of this idea:

Definition 2.6. An action φ of Γ of X is *(strong)-infinitesimally rigid* at a periodic point $x \in \Lambda$ if the isotropy representation $\rho_x = D_x\varphi$: $\Gamma_x \to \mathrm{GL}(T_xX)$ is (strong)-infinitesimally rigid.

We formulate a condition on the group Γ, which ensures that the hypotheses of Definition 2.6 are satisfied for every Anosov action of Γ.

Definition 2.7. A group Γ satisfies the *strong vanishing cohomology condition* if

SVC(N) $H^1(\tilde{\Gamma}; \mathbf{R}^N_{\tilde{\rho}}) = \{0\}$ for every subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index and representation $\tilde{\rho}\colon \tilde{\Gamma} \to \mathrm{GL}(N, \mathbf{R})$.

Observe that if Γ satisfies SVC(N), then every subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index also satisfies SVC(N).

A *lattice* is a discrete subgroup $\Gamma \subset G$ of a Lie group G such that the quotient G/Γ has finite volume. By a very remarkable result of G. A. Margulis, condition SVC(N) for arbitrary N holds for any subgroup Γ of finite index in $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 3$, as well as for many other lattices in higher rank semisimple Lie groups. The following is a special case of Theorem 2.1 [33]:

THEOREM 2.8 (Margulis). *Let $\Gamma \subset G$ be an irreducible lattice in a connected semisimple algebraic* $\mathbf{R}$*-group G. Assume that the* $\mathbf{R}$*-split rank of each factor of G is at least 2 and that $G^0_{\mathbf{R}}$ has no compact factors. Then Γ satisfies condition* SVC(N) *for every $N > 0$.* □

The Kunneth formula in cohomology implies that a product of groups satisfying condition SVC(N) will also satisfy SVC(N), so that Margulis' theorem implies that SVC(N) holds for products of lattices, as in Theorem 2.8.

Here is our main theorem on topological rigidity:

THEOREM 2.9 (Topological Deformation Rigidity). *Let φ be an Anosov C^1-action on a compact manifold X such that*

(1) *the periodic points Λ are dense in X;*

(2) *φ is strong-infinitesimally rigid at each periodic point $x \in \Lambda$.*

Then for all $k \geq 0$, φ is C^k-topologically rigid; that is, there exists $\varepsilon > 0$ such that every ε-$C^{k,1}$-deformation is C^k-topologically trivial.

The existence of the conjugating homeomorphisms $\{H_t | 0 \leq t \leq 1\}$ is proven in Section 3. The $C^{k,0}$-dependence of H_t on the parameter t is a consequence of Theorem A.1 of [28] and our method of proof, which exhibits H_t as the conjugating map between the Anosov diffeomorphisms $\varphi_t(\gamma_h)$ and $\varphi_0(\gamma_h)$.

We next consider the problem of showing that topologically conjugate actions are smoothly conjugate. Our methods depend upon the action preserving an additional structure, a "trellis" on X. The name is chosen to suggest the intuitive parallel with the cross-thatching of a vine trellis.

Definition 2.10 (Trellis). Let X be a compact smooth n-manifold without boundary. Let $1 \leq r \leq \infty$, or $r = \omega$ for the real analytic case. A *C^r-trellis* $\mathcal{T}$ on

X is a collection of one-dimensional, pairwise-transverse foliations $\{\mathcal{F}_i | 1 \leq i \leq n\}$ of X such that

(1) the tangential distributions have internal direct sum $T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_n \cong TX$;

(2) for each $x \in X$ and $1 \leq i \leq n$, the leaf $L_i(x)$ of $\mathcal{F}_i$ through x is a C^r-immersed submanifold of X;

(3) the C^r-immersions $L_i(x) \hookrightarrow X$ depend *uniform-Hölder* continuously on the basepoint x in the C^r-topology on immersions.

$\mathcal{T}$ is a *regular C^r-trellis* if it also satisfies the additional condition:

(4) each foliation $\mathcal{F}_i$ is transverse absolute-continuous, with a quasi-invariant transverse volume form that depends smoothly on the leaf coordinates.

The relation between a group action and a trellis on X is formulated in the definition of a *trellised action*. All methods of proving regularity of a topological conjugacy between Anosov actions seem to require, in some form, this additional structure.

Definition 2.11 (Trellised Action). A C^r-action $\varphi\colon \Gamma \times X^n \to X^n$ is *trellised* if there exist:

(1) a *regular C^r-trellis* $\mathcal{T} = \{\mathcal{F}_i | 1 \leq i \leq n\}$ on X;

(2) hyperbolic elements $\Delta = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ such that $\mathcal{F}_i$ is invariant under the Anosov diffeomorphism $\varphi(\gamma_i)$. That is, $\varphi(\gamma_i)$ maps each leaf of $\mathcal{F}_i$ to a leaf of $\mathcal{F}_i$.

We say that φ is an *oriented trellised action* if (2.11.1) and (2.11.2) hold, and in addition:

(3) each of the tangential distributions $T\mathcal{F}_i$ is oriented and the Anosov diffeomorphism $\varphi(\gamma_i)$ preserves the orientation of $T\mathcal{F}_i$.

We say that φ is a *volume-preserving trellised action* if (2.11.1) and (2.11.2) hold, and in addition:

(4) there is a C^r-volume form on X, which is invariant under the action of the hyperbolic elements $\gamma_i \in \Delta$.

The elements γ_i are not required to commute in the definition of a trellised action. Moreover we do not require that $\mathcal{F}_i$ be the stable, or even the strongest stable foliation of $\varphi(\gamma_i)$. The present definition allows, for example, that there is one fixed $\gamma \in \Gamma$ such that $\gamma_i = \gamma$ for all $1 \leq i \leq n$; such a γ would then be a "dynamical-regular" semisimple element for Γ.

Our first regularity result is formulated for deformations in the generality of Theorem 2.9:

THEOREM 2.12 (Deformation Regularity). *For a closed n-manifold X suppose that*:

(1) $\varphi_0\colon \Gamma \times X \to X$ *is a* C^r*-action with dense periodic orbits, for* $r = 1, \infty$ *or* ω;

(2) Γ *is a finitely generated group that satisfies the cohomology condition* $\mathrm{SVC}(n^2 - 1)$;

(3) φ_0 *is trellised by a regular trellis* $\mathscr{T}_0$, *with associated hyperbolic elements* $\Delta = \{\gamma_1, \dots, \gamma_n\}$;

(4) $\{\varphi_t | 0 \le t \le 1\}$ *is a* $C^{0,1}$*-deformation of* φ_0 *such that* $\varphi_t(\gamma_i)$ *is Anosov for all* $1 \le i \le n$ *and* $0 \le t \le 1$;

(5) φ_t *is conjugate to* φ_0 *by a continuous family of homeomorphisms* $\{H_t\colon X \to X | 0 \le t \le 1\}$;

(6) *there is a* C^r*-trellis* $\mathscr{T}_t$ *on X such that* H_t *maps the leaves of* $\mathscr{F}_{0,i}$ *to those of* $\mathscr{F}_{t,i}$.

Then H_t *is a* C^r*-diffeomorphism for all* $0 \le t \le 1$.

Suppose, in addition, that $\{\varphi_t | 0 \le t \le 1\}$ *is a* $C^{0,l}$*-deformation for* $l = 1$, *or* $l = \infty$ *if* $r = \infty$ *or* ω, *and the leaves of the foliations* $\{\mathscr{F}_{t,i}\}$ *depend continuously on the parameter t in the* C^l*-topology on immersions. Then the diffeomorphisms* H_t *depend continuously on t in the* C^l*-topology on maps.*

The cohomology hypothesis on Γ in Theorem 2.12 is used to control the type of linear isotropy representations at fixed points for the action under the deformation. For a topological conjugacy between two trellised actions we can prove the regularity of the conjugacy, given that the corresponding linear isotropy representations are conjugate. The conjugacy of the linear isotropy representations follows, for example, if we require that the hyperbolic elements γ_i commute. This suggests the following definitions:

Definition 2.13 (Cartan Action). Let $\mathscr{A}$ be a free abelian group with a given set of generators $\Delta = \{\gamma_1, \dots, \gamma_n\}$. Then (φ, Δ) is a *Cartan* C^r*-action* on the n-manifold X if

- $\varphi\colon \mathscr{A} \times X \to X$ a C^r-action on X;
- each $\gamma_i \in \Delta$ is φ-hyperbolic and $\varphi(\gamma_i)$ has a one-dimensional, strongest stable foliation $\mathscr{F}_i^{ss}$;
- the tangential distributions $E_i^{ss} = T\mathscr{F}_i^{ss}$ are pairwise transverse with their internal direct sum $E_1^{ss} \oplus \cdots \oplus E_n^{ss} \cong TX$.

We say that (φ, Δ) is a *maximal Cartan action* if φ is a Cartan action and, for each $1 \le i \le n$, the *stable* foliation $\mathscr{F}_i$ of the Anosov diffeomorphism $\varphi(\gamma_i)$ is one dimensional; hence $\mathscr{F}_i = \mathscr{F}_i^{ss}$.

We will call a Cartan action of an abelian group $\mathscr{A}$ an *abelian Cartan action* to distinguish it from the full action of a lattice group, which possesses an abelian Cartan subaction:

Definition 2.14 (Cartan action for lattices). Let $\varphi\colon \Gamma \times X \to X$ be an Anosov C^r-action on a manifold X. We say that φ is a *Cartan* (*lattice*) *action* if there is a subset of *commuting* hyperbolic elements $\Delta = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$, which generate an abelian subgroup $\mathscr{A}$, such that the restriction of $\varphi|\mathscr{A}$ is an abelian Cartan C^r-action on X.

The existence of an abelian Cartan subaction for a standard (algebraic) lattice action is in many cases a consequence of the work of Prasad and Raghunathan [37]. (See Theorem 7.2 below.)

A well-known theorem of J. Franks [7] states that if a compact n-dimensional manifold X admits an Anosov diffeomorphism with a one-dimensional, orientable, stable foliation, then it is diffeomorphic to the standard torus $\mathbf{T}^n$. Therefore, if one of the Anosov diffeomorphisms $\varphi(\gamma_i)$ in a Cartan action has $\mathscr{F}_i^{ss}$ as its stable foliation, then $X \cong \mathbf{T}^n$. In particular, $X \cong \mathbf{T}^n$ for a maximal abelian Cartan action. Also note that for a maximal abelian Cartan action, $\mathscr{A}$ must have rank at least $n - 1$.

Our approach to regularity then yields the following general result about smoothness of a topological conjugacy between trellised Cartan actions:

THEOREM 2.15 (Regularity). *For a closed n-manifold X suppose that*

(1) φ_0: $\Gamma \times X \to X$ *is a* C^r*-action of* Γ *with dense periodic orbits, for* $r = 1, \infty$ *or* ω;

(2) φ_0 *is trellised by a regular trellis* $\mathscr{T}_0$ *whose associated hyperbolic elements* $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ *determine an abelian Cartan subaction* $\varphi_0|\mathscr{A}$;

(3) Γ *is a higher rank lattice in a Lie group G, and the subgroup* $\mathscr{A} \subset \Gamma$ *generated by* Δ *is a cocompact lattice in a maximal* $\mathbf{R}$*-split torus of G*;

(4) φ_1: $\Gamma \times X \to X$ *is a* C^r*-action such that* $\varphi_1|\mathscr{A}$ *is an abelian Cartan subaction*;

(5) φ_1 *is conjugate to* φ_0 *by a homeomorphisms* H: $X \to X$.
Then H is a C^r*-diffeomorphism.*

Note that the action φ_0 in Theorem 2.15 preserves a dense set of atomic measures on X rather than an absolutely continuous measure.

The notion of an abelian Cartan action is the analogue in topological dynamics to the concept of a Cartan subgroup for a lattice (cf. [3]). The smooth classification of abelian Cartan actions is a developing topic (cf. [18], [26], [29], [36], [35]). Let us first note three preliminary results about Cartan actions:

THEOREM 2.16. (1) *For a Cartan C^r-action* (φ, Δ) *the collection of strongest stable foliations* $\mathcal{T} = \{\mathcal{F}_1^{ss}, \ldots, \mathcal{F}_n^{ss}\}$ *is a C^r-trellis on X.*

(2) *For a maximal Cartan C^r-action* (φ, Δ) *the collection of stable foliations* $\mathcal{T} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ *is a regular C^r-trellis on X.*

(3) *For a volume-preserving, maximal Cartan C^r-action* (φ, Δ), *with* $r \geq 3$, *each stable foliation* $\mathcal{F}_i$ *is transversally* $C^{1+\alpha}$ *for some* $0 < \alpha < 1$.

The content of (2.16.1) and (2.16.2) is the regularity of the trellis foliations, which is a consequence of the stable manifold theory of Hirsch and Pugh [12], and its subsequent embellishments (cf. [39]). Our (2.16.3) follows from Lemma 5.2 and the regularity theory of Hasselblatt [11]. Theorem 2.16 is proved in Section 6.

PROPOSITION 2.17 (C^1-Stability). *Let* (φ, Δ) *be a Cartan C^r-action on the closed, n-dimensional, infra-nilmanifold X for* $r \geq 1$. *There exists* $\varepsilon > 0$ *such that if* $\varphi_1 \colon \mathcal{A} \times X \to X$ *is* ε-C^1-*close to* φ, *then* (φ_1, Δ) *is again a Cartan action.*

Proof. The Anosov condition given in (3) and the strongest stable condition in (4) are both stable under C^1-perturbations, as they are equivalent to a contraction principle on an appropriate Banach space of sections of a Grassmann bundle over X (cf. Mather, Appendix A of [40]; and Appendix A of [28]). The splitting of TX is determined by an invariant section in this Banach space, and this section depends continuously on the action. Therefore $\varepsilon > 0$ can be chosen sufficiently small so that the perturbed, strongest stable foliations $\{\mathcal{F}_1^{ss}, \ldots, \mathcal{F}_n^{ss}\}$ are pairwise transverse and the internal direct sum $T\mathcal{F}_1^{ss} \oplus \cdots \oplus T\mathcal{F}_n^{ss} \cong TX$. $\square$

It is folklore that a transitive Anosov action of an abelian group on a torus with a common fixed point is topologically equivalent to an algebraic action. A slightly stronger result is possible:

PROPOSITION 2.18 (Topological Rigidity for Cartan Actions). *Let* (φ, Δ) *be a Cartan C^r-action on the closed, n-dimensional, infra-nilmanifold X for* $r \geq 1$. *Then*

(1) φ *has periodic orbit* x_0, *and therefore there is a positive integer p so that the action of the* p^{th}-*powers* $\Delta^p = \{\gamma_1^p, \ldots, \gamma_m^p\}$ *is topologically conjugate to a standard (algebraic) Cartan action induced by the map on homotopy,* $\varphi_{\#} \colon \Delta^p \times \pi_1(X; x_0) \to \pi_1(X; x_0)$.

(2) φ *is C^k-topologically deformation rigid for all* $k \geq 0$.

(3) *A topological conjugacy H between* φ *and a standard Cartan action on X maps the trellis* $\mathcal{T}$ *for* (φ, Δ^p) *to a linear trellis for the standard action.*

Proof. The results of Franks [7] and Manning [30] imply that an Anosov diffeomorphism of an infra-nilmanifold X is topologically conjugate to the linear action induced from the action on first homology. We can thus assume that γ_1 acts as a linear hyperbolic matrix, which therefore has a finite set of fixed points. The action of the generators Δ must permute the fixed points of γ_1 so there exists an exponent $p > 0$ in order that the action of each γ_i^p fixes this set. We then have a common fixed point $x_0 \in X$ for the action of Δ^p, and hence there is an induced action of $\mathscr{A}$ on $\pi_1(X, x_0)$. The same argument as the one used in Proposition 0 of [35], with $\mathrm{Aut}(\pi_1(X))$ in place of $\mathrm{GL}(n, \mathbf{Z}) = \mathrm{Aut}(H_1(X; \mathbf{Z}))$, shows that a family of commuting homeomorphisms of a compact nilmanifold, with one element algebraic Anosov, must be an algebraic action.

The conclusion of deformation rigidity follows immediately from (2.18.2) and Theorem A.1 of Appendix A, [28]. The topological invariance of the trellis follows from Theorem 1.1, [54]. □

Note that there are examples of abelian Cartan actions that do not have a fixed point for the full action [15] so that the reduction to the subgroup generated by Δ^p is necessary.

The next two results, Theorems 2.19 and 2.21, generalize to higher dimensions a combination of theorems of R. de la Llavé, J. M. Marco and R. Moriyon ([26], [27], [29], [28], [31], [32]) for $X = \mathbf{T}^2$.

THEOREM 2.19 (Differential Rigidity for Cartan Actions). *Let $\mathscr{A}$ be an abelian group generated, not necessarily freely, by the set $\Delta = \{\gamma_1, \ldots, \gamma_n\}$. Given volume-preserving Cartan C^r-actions (φ_0, Δ) and (φ_1, Δ) on an n-manifold X, for $r = 1, \infty$ or ω, suppose that*

(1) φ_0 is a trellised action;

(2) $H\colon X \to X$ is a homeomorphism conjugating φ_1 to φ_0;

(3) for all $1 \le i \le n$ and for each $x \in \Lambda(\varphi_0)$ the maximally contracting exponent of $D_x\varphi_0(\gamma_i)$ equals the maximally contracting exponent of $D_{H(x)}\varphi_1(\gamma_i)$.

Then H is a C^r-diffeomorphism. Moreover for $l = 1$ (or $l = \infty$ if $r = \infty$ or ω) suppose there are given

(4) a $C^{0,l}$-deformation $\{(\varphi_t, \Delta) | 0 \le t \le 1\}$ through volume-preserving Cartan C^r-actions, and

(5) a continuous family of homeomorphisms $\{H_t | 0 \le t \le 1\}$ conjugating φ_t to φ_0,

which satisfy (2.19.2) and (2.19.3) for all $0 \le t \le 1$. Then the C^r-diffeomorphism H_t varies C^0 with t in the C^l-topology on C^r-maps.

COROLLARY 2.20. *Let (φ_0, Δ) be a volume-preserving, trellised Cartan C^r-action on an n-manifold X, for $r = \infty$ or ω, with $\mathscr{A}$ the abelian group* *generated, not necessarily freely, by the set* $\Delta = \{\gamma_1, \dots, \gamma_n\}$. *Suppose that* $H\colon X \to X$ *is a* C^1*-conjugacy between an arbitrary* C^r*-action* $\varphi_1\colon \mathscr{A} \times X \to X$ *and the given action* φ_0. *Then* (φ_1, Δ) *is a volume-preserving Cartan* C^r*-action, and* H *is a* C^r*-diffeomorphism.*

Proof of corollary. Here H induces a continuous splitting of TX, corresponding to the stable and unstable foliations of φ_0, which is invariant under $D\varphi_1$. The Anosov condition given in (3) requires only a continuous decomposition of TX, so the C^1-diffeomorphisms $\varphi_1(\gamma_i)$ are Anosov. For the same reason, the C^r-diffeomorphisms $\varphi_1(\gamma_i)$ have one-dimensional, strongest stable foliations, which are transverse. Thus φ_1 is a Cartan C^r-action on X. The invariant volume form Ω for φ_0 is conjugated by H to a continuous volume form $H^*(\Omega)$ on X, which is $\varphi_1(\mathscr{A})$-invariant. The theorem of Livsic and Sinai [25] implies that $H^*(\Omega)$ is C^r. We then apply Theorem 2.19 to conclude that H is C^r. □

Our second result is a higher-dimensional generalization of Theorem 1 of [32]. For an oriented Cartan action φ let $x \in \Lambda$ be a periodic point and let $\mathscr{A}_x$ be the isotropy subgroup of x. The linear isotropy representation

$$D_x\varphi\colon \mathscr{A}_x \to \mathrm{GL}(T_xX)$$

has its image in a maximal diagonal subgroup. The choice of a trellis $\{\mathscr{F}_i\}$ for the action defines a basis in each tangent space T_xX for which the action is diagonal. Introduce the abelian (multiplicative) diagonal group $\mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$; then we can consider the isotropy representations as homomorphisms $D_x\varphi\colon \mathscr{A}_x \to \lambda^n$. A Cartan action is said to have *constant exponents* if there exist homomorphisms $\lambda_i\colon \mathscr{A} \to \mathbf{R}^+$ for $1 \le i \le n$ such that, for each $x \in \Lambda$ and $\gamma \in \mathscr{A}_x$,

$$D_x\varphi(\gamma) = \lambda_1(\gamma) \oplus \cdots \oplus \lambda_n(\gamma).$$

THEOREM 2.21. *Let* (φ, Δ) *be an oriented Cartan* C^r*-action on the n-torus* $\mathbf{T}^n$, *for* $r = 1, \infty$ *or* ω. *If* φ *has constant exponents, then there is a subgroup* $\tilde{\mathscr{A}} \subset \mathscr{A}$ *of finite index so that the restriction* $\varphi|\tilde{\mathscr{A}}$ *is* C^r*-conjugate to a standard linear action, and* φ *is* C^r*-conjugate to an affine action of* $\mathscr{A}$ *on* $\mathbf{T}^n$.

The proof of this result is given in Section 6. Theorem 2.21 has the following application [19]:

THEOREM 2.22 (Rigidity). *Let* $\varphi\colon \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ *be a Cartan* C^r*-action on the n-torus* $\mathbf{T}^n$, *for* $r = 1, \infty$ *or* ω. *Suppose that* Γ *is a higher rank lattice and the subgroup* $\mathscr{A} \subset \Gamma$ *generated by the* Δ *(cf. Definition 2.13) is a cocompact lattice in a maximal* **R***-split torus in* G. *If the action* φ *preserves an absolutely continuous probability measure on* $\mathbf{T}^n$, *then* φ *is* C^r*-conjugate to an affine action of* Γ.

Remark 2.23. Theorem 2.22 extends a previous, unpublished result of J. Lewis and R. Zimmer that applied to the case of the standard action of a subgroup $\Gamma \subset SL(n, \mathbf{Z})$ of finite index, for $n \geq 3$. There are three essential points to the proof of Theorem 2.22: an ε-C^1-perturbation of the action φ has an invariant, absolutely continuous probability measure, by the Kazhdan property T, and this measure must be smooth by the theorem of Livsic and Sinai [25] characterizing the invariant measures for a smooth Anosov diffeomorphism. Furthermore a sufficiently small C^1-perturbation of a Cartan action is a Cartan action by Proposition 2.18. Thus one need only consider the case of Cartan actions that preserve a C^r-volume form. From the cocycle super-rigidity theorem of Zimmer ([51], [48], [47]) and the measurable Livsic theorem ([23], [24]) one deduces that the Cartan subaction must have all exponents equal. It is then immediate that the action is algebraic for the coordinates on $\mathbf{T}^n$ that are provided by Theorem 2.21. The complete proof is given in [19].

Remark 2.24. The methods of this paper all require the existence of a dense set of periodic orbits for the action under study. The periodic points can be viewed as an "atomic" invariant measure for the group action whose closed support is all of X. On the other hand, the Zimmer super-rigidity theorem is applied (in geometric contexts) when there is given an absolutely continuous invariant measure whose closed support is all of X. Atomic measures and absolutely continuous measures are "dual" under Fourier transform, so one surmises that there is a common rigidity principle underlying Theorems 2.9, 2.19 and 2.22.

3. Topological rigidity for Anosov actions

As noted in the Introduction, studying rigidity of group actions naturally breaks into two parts: topological equivalence and smooth equivalence. We say that two actions have the same *topological dynamics* if there is a topological conjugacy between them. This is the natural notion of equivalence for dynamical systems generated by one endomorphism (cf. [39], [40]). Anosov showed that two Anosov diffeomorphisms, which are C^1-close, are topologically conjugate ([2]; cf. also Mather, Appendix to [40]) so that an Anosov diffeomorphism is always topologically stable. Anosov actions of groups with more than one generator need not be topologically rigid or deformation rigid; it then follows that topological conjugacy is a nontrivial notion of equivalence.

Let $\varphi\colon \Gamma \times X \to X$ be a smooth Anosov action that is strong-infinitesimally rigid at each periodic orbit, and with a dense set of periodic orbits. In this section we prove that there exists $\varepsilon > 0$ so that given an ε-C^1-deformation $\{\varphi_t | 0 \leq t \leq 1\}$ of φ, which varies C^k with the parameter t in the C^1-topology on maps, there exists a $C^{k,0}$-family of homeomorphisms $\{H_t | 0 \le t \le 1\}$ satisfying the conjugacy equation (2).

Fix $\gamma_0 \in \Gamma$, which is φ-hyperbolic. Choose $\varepsilon_0 > 0$ so that every diffeomorphism that is ε_0-C^1-close to $\varphi(\gamma_0)$ is necessarily Anosov. Then choose $\varepsilon > 0$ so that, for every ε-perturbation φ_1 of φ, the diffeomorphism $\varphi_1(\gamma_0)$ is ε_0-C^1-close to $\varphi_0(\gamma_0)$. (For example, if γ_0 is one of the generators of Γ used to define the C^1-norm on Γ-actions, then $\varepsilon = \varepsilon_0$.) It follows that for an ε-C^1-deformation $\{\varphi_t | 0 \le t \le 1\}$ of φ, each $\varphi_t(\gamma_0)$ is an Anosov diffeomorphism for $0 \le t \le 1$.

The C^1-topological stability of Anosov diffeomorphisms in the C^1-topology on maps (Appendix A, [28]) implies that we can find a $C^{k,0}$-family of homeomorphisms $\{H_t | 0 \le t \le 1\}$ such that H_0 is the identity and

$$H_t^{-1} \circ \varphi_t(\gamma_0) \circ H_t = \varphi(\gamma_0) \qquad \text{for all } 0 \le t \le 1. \tag{5}$$

We will show that this family of homeomorphisms satisfies (2) for all $\gamma \in \Gamma$. We first observe that the set of periodic points $\mathrm{Per}(\varphi(\gamma_0))$ of $\varphi(\gamma_0)$ contains the dense set Λ of periodic points for the full action of φ. Therefore the family $\{H_t\}$ is uniquely determined on the closure of the periodic points Λ. The strategy is to show that (2) holds on the set Λ for all $\gamma \in \Gamma$. Then we invoke the continuity of the actions to deduce that (2) holds on the closure of Λ, which is all of X.

A set $\Sigma_t \subset X$ is said to be *φ_t-saturated* if $x_t \in \Sigma_t$ implies that $\varphi_t(\gamma)(x_t) \in \Sigma_t$ for all $\gamma \in \Gamma$.

Definition 3.1. A *φ-filtration* of Λ is an ascending sequence of φ-saturated, finite sets $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_p \subset \cdots$ whose union is all of Λ.

LEMMA 3.2. *An Anosov action φ admits a natural φ-filtration.*

Proof. For each positive integer p let $\Lambda_p \subset \Lambda$ be the subset of points whose φ-orbit $\Gamma(x)$ contains at most p points. Clearly $\Lambda_p \subset \Lambda_{p+1}$ with the union over all p yielding Λ. We must check that each Λ_p is a finite set. Observe that each $x \in \Lambda_p$ is a periodic point for $\varphi(\gamma_0)$ and hence is a fixed point for some power $\varphi(\gamma_0)^q$ with $0 \le q \le p$. Thus Λ_p is contained in the set of fixed points for the Anosov diffeomorphism $\varphi(\gamma_0)^{p!}$. The fixed-point set is isolated in the compact manifold X; hence it is finite. □

We call the filtration produced in Lemma 3.2 the *length filtration* of Λ. For an Anosov action of an arithmetic lattice Λ there is another natural filtration, the *congruence filtration*, corresponding to the chain of congruence subgroups in Γ. A paradigmatic example of this is discussed in Example 7.3, as part of our analysis of the rigidity of $\mathrm{SL}(n, \mathbf{Z})$-actions.

Fix a φ-filtration $\{\Lambda_p\}$ of Λ. For each $p \geq 1$ define Γ_p to be the stabilizer subgroup of Λ_p. That is,

$$\gamma \in \Gamma_p \Leftrightarrow \varphi(\gamma)(x) = x \qquad \text{for all } x \in \Lambda_p.$$

Clearly $\Gamma_{p+1} \subset \Gamma_p$, and Λ dense in X implies that the intersection over all Γ_p is the set of $\gamma \in \Gamma$, which acts as the identity on X.

LEMMA 3.3. *Γ_p is a normal subgroup of Γ with finite index*:

$$[\Gamma_p: \Gamma] \leq \{\mathrm{Card}(\Lambda_p)\}!.$$

Proof. The action of Γ on Λ_p defines a representation $\Gamma \to \mathrm{Perm}(\Lambda_p)$ into the permutation group on the set Λ_p with kernel Λ_p; and the finite group of permutations has order $\{\mathrm{Card}(\Lambda_p)\}!$. □

For each $0 \leq t \leq 1$ set $\Lambda_p(t) = H_t(\Lambda_p)$. For $x \in \Lambda$ let $x_t = H_t(x)$.

The strategy for the proof of Theorem 2.9 is to prove that (2) holds on each subset Λ_p successively. We use a theorem of Stowe, whose hypotheses involve the isotropy action of Γ at each $x \in \Lambda_p$. Let us first give the necessary notation and some preliminary remarks. For each $x \in \Lambda$ let Γ_x denote the stabilizer subgroup of x. Thus $\Gamma_p \subset \Gamma_x$ if $x \in \Lambda_p$. The differential at x of the restricted action φ: $\Gamma_x \times X \to X$ yields the *isotropy linear representation* denoted by

$$D_x\varphi: \Gamma_x \to \mathrm{GL}(T_xX). \tag{6}$$

The subgroup Γ_p has finite index, so there is a positive integer m for which $\gamma_0^m \in \Gamma_p$. The spectrum of the linear automorphism $D_x\varphi(\gamma_0^m)$ is bounded away from 1 in modulus by the Anosov property of (3). Thus $0 \in T_xX$ is the unique fixed point for the linear action of $D_x\varphi(\gamma_0^m)$ on T_xX and so is also the unique fixed point for the linear action of the larger groups $D_x\varphi(\Gamma_x)$ and $D_x\varphi(\Gamma_p)$, when $x \in \Lambda_p$. Note that the corresponding remarks are true for the Anosov diffeomorphism $\varphi_t(\gamma_0^m)$ and all $x \in \Lambda_p(t)$.

We quote Stowe's result (Theorem A, [41] and Theorem 2.1, [42]) for the case of an isolated fixed point for a differentiable group action:

THEOREM 3.4 (Stowe). *Let α: $G \times X \to X$ be a C^r-action of a finitely generated group G, for $1 \leq r \leq \infty$. Let $x \in X$ be an infinitesimally rigid fixed point for the action. Then x is stable under perturbations of the action. In fact, for each C^r-action β near α in the C^r-topology, there exist C^r-embeddings Ψ_β: $T_xX \to X$ such that*

(1) $\Psi_\alpha(0) = x$, $D_0\Psi_\alpha = \mathrm{Id}|_{T_xX}$;

(2) $\Psi_\beta(0) = x_\beta$ *is the unique fixed point of the action β in the open set* $\Psi_\beta(T_xX)$;

(3) Ψ_β *varies continuously with β in the C^r-topology on embeddings.*

We deduce from Stowe's theorem the following application to our family of actions:

COROLLARY 3.5. *Fix* $0 \leq s \leq 1$ *and suppose there is given a subgroup* $G \subset \Gamma$ *such that*

(1) *there exists* $\gamma \in G$ *such that* $\varphi_s(\gamma)$ *is Anosov;*

(2) $y \in X$ *is a fixed point for* $\varphi_s(G)$ *with* $H^1(G; (T_y X)_{D_y\varphi_s}) = 0$.

Then there exists $\varepsilon = \varepsilon(G) > 0$ *and embeddings* $\Psi_t \colon T_y X \to X$ *for* $s - \varepsilon < t < s + \varepsilon$ *such that* $y_t = \Psi_t(0)$ *is the unique fixed point of the action* $\varphi_t(G)$ *in the open set* $\Psi_t(T_y X)$. *Moreover the embeddings* Ψ_t *depend continuously on* t.

We combine the stability under perturbation of the hypotheses of strong-infinitesimal rigidity with Corollary 3.5 to deduce the following key result for the proof of topological rigidity. It implies that each $x_t \in \Lambda_t$ is a periodic point of the action φ_t, for all t uniformly in the range $0 \leq t \leq 1$ independent of the choice of x.

PROPOSITION 3.6. *For each* $x \in \Lambda_p$, $x_t = H_t(x)$ *is an isolated fixed point of the action* $\varphi_t(\Gamma_p)$ *for all* $0 \leq t \leq 1$.

Proof. Fix $p > 0$ and choose $m = m(p) > 0$ so that $\gamma_0^{m(p)} \in \Gamma_p$. The representation $D_x\varphi \colon \Gamma_p \to \mathrm{GL}(T_x X)$ is not compact, as it contains in the image the hyperbolic linear automorphism $D_x\varphi(\gamma_0^m)$. Thus conditions (3.5.1) and (3.5.2) are satisfied for $s = 0$ and we obtain:

- $t_1 > 0$;
- a continuous path $\{y_t \mid 0 \leq t \leq t_1\}$ with $y_0 = x$;
- y_t, an isolated fixed point for $\varphi_t(\Gamma_p)$.

In particular y_t is a fixed point for the Anosov diffeomorphism $\varphi_t(\gamma_0^m)$. The path $\{x_t \mid 0 \leq t \leq t_1\}$ also consists of fixed points for the family of actions $\{\varphi_t(\gamma_0^m)\}$. The fixed points of an Anosov diffeomorphism are isolated so that $x_0 = x = y_0$ implies that these two continuous paths must coincide for all $0 \leq t \leq t_1$. We conclude, for some $t_1 > 0$, that x_t is an isolated fixed point for the action $\varphi_t(\Gamma_p)$ for all $0 \leq t \leq t_1$.

Let $s > 0$ be the supremum of values of $\varepsilon \leq 1$ such that x_t is an isolated fixed point for $\varphi_t(\Gamma_p)$ for all $0 \leq t < \varepsilon$. Suppose that $s < 1$. We show that this leads to a contradiction.

It is given that $\varphi_t(\Gamma_p)(x_t) = x_t$ for all $0 \leq t < s$. The continuity of φ_t in t implies that the limit point $y = x_s$ is a fixed point for the action $\varphi_s(\Gamma_p)$. Moreover it is a fixed point for the Anosov diffeomorphism $\varphi_s(\gamma_0^m)$ and hence is also isolated for the action of Γ_p. The image of the representation $D_y\varphi_s \colon \Gamma_p \to \mathrm{GL}(T_y X)$ contains the hyperbolic element $D_y\varphi_s(\gamma_0^m)$, so it is not compact. The conditions of Corollary 3.5 are therefore satisfied, and we obtain a continuous path of isolated fixed points $\{y_t | s - \varepsilon < t < s + \varepsilon\}$ for $\varphi_t(\Gamma_p)$, with $y_s = y = x_s$. By the local uniqueness of the isolated fixed points for $\varphi_t(\gamma_0^m)$, it must follow that $y_t = x_t$ for $s - \varepsilon < t \leq s$. The path $\{y_t\}$ thus extends the path $\{x_t\}$, contradicting the maximality of s.

We conclude that $s = 1$. The proof above also established that $\varphi_1(\gamma_0^m)$ is Anosov so that $x_1 = H_1(x)$ is an isolated fixed point for the action $\varphi_1(\Gamma_p)$. □

The proof of Theorem 2.9 is completed by the next proposition and its corollary.

PROPOSITION 3.7. *For each $x \in \Lambda$ and $\gamma \in \Gamma$,*

$$H_t^{-1} \circ \varphi_t(\gamma) \circ H_t(x) = \varphi(\gamma)(x) \qquad \text{for all } 0 \leq t \leq 1. \tag{7}$$

Proof. Let $x \in \Lambda$, $\gamma \in \Gamma$, and set $z = \varphi(\gamma)(x)$, $x_t = H_t(x)$ and $z_t = H_t(z)$. Then (7) is equivalent to showing that

$$\varphi_t(\gamma)(x_t) = z_t \quad \text{for } 0 \leq t \leq 1.$$

Choose $p > 0$ with $x \in \Lambda_p$. Then $z \in \Lambda_p$ also, and by Proposition 3.6 each z_t is an isolated fixed point for $\varphi_t(\Gamma_p)$. On the other hand, Γ_p is a normal subgroup of Γ so that $\varphi_t(\gamma)(x_t)$ is also an isolated fixed point for $\varphi_t(\Gamma_p)$. As both families of fixed points agree at $t = 0$, $\varphi_0(\gamma)(x_0) = z = z_0$, we therefore find that $\varphi_t(\gamma)(x_t) = z_t$ for all $0 \leq t \leq 1$. □

COROLLARY 3.8. *For each $\gamma \in \Gamma$ and $0 \leq t \leq 1$, H_t conjugates $\varphi_t(\gamma)$ to $\varphi(\gamma)$.*

Proof. All of the mappings in (7) are continuous; so for each $\gamma \in \Gamma$ we have (7) holding for x in the closure of the periodic set Λ, which is all of X. □

Remark 3.9. The constant t_1 appearing in the proof of Proposition 3.6, which exists by Stowe's theorem, depends upon the subgroup Γ_p and there is no a priori estimate of its size. It may, for example, tend to 0 as p tends to infinity. We overcome this limitation in the above proof by using a supremum-principle to obtain a path of fixed points defined for all $0 \leq t \leq 1$. Our method depends upon the connectivity of the interval $[0, 1]$, and therefore this approach does not directly apply to the study of topological stability under perturbations of the action. One approach toward proving topological stability in the generality of Theorem 2.9 would be to obtain a uniform estimate on the constant t_1, which bounds it away from 0 independent of p. Katok and Lewis' proof [21] of topological rigidity for $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ of finite index for $n \geq 4$ circumvents the above difficulties by using Stowe's theorem only once to obtain the stability of the "origin". The authors then rely on additional algebraic properties of the group Γ and on dynamical properties of the standard algebraic action to identify the other periodic points under a perturbation.

4. Rigidity of linear isotropy representations

In this section we give two results concerning the rigidity of the linear isotropy representations of an Anosov action. These results are used in the proof of the regularity of a homeomorphism $H: X \to X$, which conjugates two Anosov actions.

PROPOSITION 4.1. *Let $\varphi_0: \Gamma \times X \to X$ be a C^1-action of Γ on a closed n-manifold X such that*

(1) *Γ is a higher rank lattice subgroup of a connected semisimple Lie group G;*

(2) *φ_0 is trellised by a regular trellis $\mathscr{T}_0$ whose associated hyperbolic elements $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ determine an abelian Cartan subaction $\varphi_0|\mathscr{A}$;*

(3) *the set Δ extends to a commuting set $\tilde{\Delta}$ of semisimple elements, which span a lattice in a maximal* **R***-split torus in G;*

(4) *$\varphi_1: \Gamma \times X \to X$ is a C^1-action such that $\varphi_1|\mathscr{A}$ is an abelian Cartan subaction;*

(5) *φ_1 is conjugate to φ_0 by a homeomorphism $H: X \to X$.*

Then for each periodic point $x \in \Lambda(\varphi_0)$ there are a subgroup $\Gamma'_x \subset \Gamma$ of finite index and a linear isomorphism $\Phi_x: T_x X \to T_{H(x)}X$ so that

(1) *Φ_x maps the tangent space $T_x\mathscr{F}_{0,i}$ at x to the tangent space $T_{H(x)}\mathscr{F}_{1,i}$;*

(2) *Φ_x conjugates the linear isotropy representation $D_x\varphi_0: \Gamma'_x \to \mathrm{SL}(T_x X)$ to $D_{H(x)}\varphi_1: \Gamma'_x \to \mathrm{SL}(T_{H(x)}X)$.*

Proof. Let G be a semisimple Lie group with at most a finite number of connected components such that Γ is a lattice in G. Let $\tilde{G}^0$ denote the universal covering group of the connected component G^0 and $\Gamma' \subset \Gamma$ denote a finite-index subgroup that is the monomorphic image of a subgroup $\tilde{\Gamma} \subset \tilde{G}$.

For each point $x \in \Lambda(\varphi_0)$ choose a linear isomorphism $L_x: T_x X \to T_{H(x)}X$, which for $1 \leq i \leq n$ maps $T_x\mathscr{F}_{0,i}$ to $T_{H(x)}\mathscr{F}_{1,i}$ and is oriented with respect to the restriction of the homeomorphism H to the leaf of $\mathscr{F}_{0,i}$ through x. Let $\rho_0 = D_x\varphi_0$ and $\rho_1 = L_x^{-1} \circ D_{H(x)}\varphi_1 \circ L_x$ be the isotropy representations of the isotropy subgroup Γ_x on $T_x X$. We must show that there is a choice of L_x such that the representations ρ_0 and ρ_1 are equal when restricted to a subgroup of finite index.

There are two ideas used in the proof. First we conclude that the representations ρ_0 and ρ_1 extend to the universal covering group $\pi: \tilde{G} \to G$. We then use the classification of the finite-dimensional representations of a semisimple

Lie group by their weight spaces to show that the topological type of the action of the lattice Γ_x on $T_x X$ determines the representation of $\tilde{G}$ and hence the linear type of ρ_0 and ρ_1.

Let $G_{x,t}$ be the algebraic closure of $\rho_t(\Gamma_x) \subset \mathrm{SL}(T_x X)$ for $t = 0, 1$ and let $G_{x,t}^0$ be the connected component of the identity.

LEMMA 4.2. *$G_{x,t}^0$ is semisimple with finite center $\mathscr{Z}_{x,t}^0$ and without compact factors.*

Proof. Let $G_{x,t}^0 = L \ltimes U$, where L is reductive and U is the unipotent radical. Then U must be trivial, as it is contained in a conjugate of the unipotent, lower-triangular, matrix subgroup of $\mathrm{SL}(n, \mathbf{R})$ and is normalized by the hyperbolic elements $\rho_t(\gamma_i)$, which have one-dimensional, maximally contracting eigenspaces. The reductive factor L must have finite center, as every homomorphism $\Gamma_x \to \mathbf{R}$ is trivial by Theorem 2.8. The existence of a compact factor would imply that the eigenspaces of the matrices $\rho_t(\gamma_i)$ have dimension greater than 1, contrary to our assumption. $\square$

The Margulis super-rigidity theorem (Theorem 2, p. 2 in [33]) and Lemma 4.2 imply that the quotient homomorphisms $\tilde{\rho}_t$: $\tilde{\Gamma} \to PG_{x,t} = G_{x,t}^0/\mathscr{Z}_{x,t}^0$, onto the group modulo its center, extend to homomorphisms $\tilde{\rho}_t$: $\tilde{G}^0 \to PG_{x,t}$. The group $\tilde{G}^0$ is simply connected; so $\hat{\rho}_t$ lifts (possibly nonuniquely) to a homomorphism $\tilde{\rho}_t$: $\tilde{G}^0 \to G_{x,t}^0$, with the ambiguity determined by the elements of the finite group $\mathscr{Z}_{x,t}^0$. Therefore there is a subgroup $\Gamma_x' \subset \Gamma_x$ such that the restriction of $\hat{\rho}_t$ to Γ_x' is uniquely determined. The claim is that the restricted representations to Γ_x' are conjugate in $\mathrm{SL}(T_x X)$.

Consider first the case where Γ is a lattice in $G = \mathrm{SL}(n, \mathbf{R})$. There are two conjugacy classes of nontrivial representations of $\mathrm{SL}(n, \mathbf{R})$ into $\mathrm{SL}(T_x X) \cong \mathrm{SL}(n, \mathbf{R})$: the conjugates of the identity and the conjugates of the contra-gradient representation. Note that the contra-gradient representation reverses the signs of the weights of the representation and so replaces a contracting eigenvalue of a hyperbolic element γ_h with an expanding eigenvalue, and vice versa. The existence of a topological conjugacy between the actions of $\varphi_0(\gamma_h)$ and $\varphi_1(\gamma_h)$ in a neighborhood of x implies that the signs of the weights of the representations must agree on γ_h, and hence the representations $\hat{\rho}_0$ and $\hat{\rho}_1$ are both equivalent to lifts of the identity, or to the contra-gradient representation. Finally the hyperbolic element $\rho_t(\gamma_i)$ has a maximally contracting direction, which is tangent to $\mathscr{F}_{0,i}$ at x for $t = 0, 1$ by our choice of L_x. The intertwining operator must preserve this direction; so it will be a diagonal matrix with respect to the basis of $T_x X$ by vectors tangent to the leaves of $\mathscr{F}_{0,i}$ at x. We then adjust the choice of L_x by this intertwining operator to obtain our conclusion.

For the general case we use the classification of finite-dimensional representations by their weights. The Lie algebra $\mathbf{a}$ of the algebraic hull of $\mathscr{A}$ consists of semisimple elements, as they can be diagonalized in the representation onto $\mathrm{SL}(T_x X)$. Extend $\mathbf{a}$ to a maximal, $\mathbf{R}$-split, Cartan subalgebra $\tilde{\mathbf{a}}$ for the Lie algebra $\mathbf{g}$ of G, with $\tilde{\mathscr{A}}$ the abelian extension of $\mathscr{A}$ to a lattice in the connected abelian Lie subgroup of G defined by $\tilde{\mathbf{a}}$.

Next note that a topological conjugacy between Anosov C^1-actions must be Hölder continuous (this follows from H being defined globally on a compact manifold X). Restricting the homeomorphism H to a neighborhood of the periodic orbit x, we obtain a local Hölder conjugacy between the actions of $\varphi_0(\Gamma_x)$ and $\varphi_1(\Gamma_x)$ in a neighborhood of x. The Hölder topological type near x of the C^1-actions of Γ_x determines the dimensions of the expanding, contracting and invariant weight spaces for the representations ρ_0 and ρ_1 on $T_x X$ for the elements of $\tilde{\mathscr{A}}$. This suffices to determine two things: first the irreducible summands of $T_x X$ for the representations; and secondly the maximal weight vector in each irreducible summand, and from this, the maximal weight. It follows that the representations have isomorphic irreducible summands and hence are isomorphic. The fact that the intertwining operator preserves the strongest stable direction follows, as before. □

The proof of differentiable deformation rigidity requires the rigidity of representations under continuous deformation, which is a direct consequence of the well-known rigidity theory of Weil ([43], [44], [45]; cf. also Chapter VI of [38]). Let $\tilde{\Gamma}$ be a finitely generated group and G be a Lie group. Let $\mathscr{R}(\tilde{\Gamma}, G)$ be the set of all homomorphisms of $\tilde{\Gamma}$ in G with the topology of pointwise convergence on a fixed set of generators of $\tilde{\Gamma}$. A representation $\rho \in \mathscr{R}(\tilde{\Gamma}, G)$ is *locally rigid* if the orbit of ρ in $\mathscr{R}(\tilde{\Gamma}, G)$ under the conjugacy action of G is open in $\mathscr{R}(\tilde{\Gamma}, G)$. Let $\mathbf{g}$ denote the Lie algebra of G and Ad: $G \to \mathrm{SL}(\mathbf{g})$ denote the adjoint representation.

THEOREM 4.3 (Weil). *$\rho \in \mathscr{R}(\tilde{\Gamma}, G)$ is locally rigid if $H^1(\tilde{\Gamma}; \mathbf{g}_{\mathrm{Ad} \circ \rho}) = 0$.* □

COROLLARY 4.4. *Let Γ be a finitely generated group satisfying the vanishing cohomology condition* SVC(N). *Then for every subgroup of finite index $\tilde{\Gamma} \subset \Gamma$ and any Lie group G whose Lie algebra* $\mathbf{g}$ *has dimension at most N, every point $\rho \in \mathscr{R}(\tilde{\Gamma}, G)$ is locally rigid.* □

We now apply this corollary to the case of a group action.

COROLLARY 4.5. *Let Γ be a finitely generated subgroup satisfying condition* $\mathrm{SVC}(n^2 - 1)$ *and let X be a closed smooth manifold of dimension n. Given a continuous one-parameter family of C^1-actions, $\varphi_t\colon \Gamma \times X \to X$ for $0 \leq t \leq 1$, let*

$x_t \in \Lambda_t$ be a continuous path of fixed points for a finite-index subgroup Γ_x. Then the conjugacy class of the isotropy representation $D_{x_t}\varphi_t\colon \Gamma_x \to \mathrm{GL}(T_{x_t}X)$ is constant. □

5. Regularity theory for trellised actions

In this section we will prove the regularity of a homeomorphism $H\colon X \to X$, conjugating two trellised Anosov actions. The following is the main technical result from which Theorems 2.12, 2.15 and 2.19 are deduced.

THEOREM 5.1. *Let X be a compact Riemannian manifold of dimension n and let $r = 1, \infty$ or ω. Assume that the following data are given*:

(1) *Two C^r-actions $\varphi_0, \varphi_1\colon \Gamma \times X \to X$, such that the set of periodic orbits Λ_0 for φ_0 is dense, and $H^1(\tilde{\Gamma}; \mathbf{R}) = 0$ for each subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index*;

(2) *An oriented regular trellis $\mathscr{T}_0 = \{\mathscr{F}_{0,i} | 1 \le i \le n\}$ and φ_0-hyperbolic elements $\gamma_i \in \Gamma$ such that $\mathscr{F}_{0,i}$ is invariant under the orientation-preserving Anosov diffeomorphism $\varphi_0(\gamma_i)$*;

(3) *An oriented trellis $\mathscr{T}_1 = \{\mathscr{F}_{1,i} | 1 \le i \le n\}$ such that $\mathscr{F}_{1,i}$ is invariant under the orientation-preserving* Anosov *diffeomorphism $\varphi_1(\gamma_i)$*;

(4) *A homeomorphism $H\colon X \to X$ conjugating φ_1 to φ_0 and mapping the trellis $\mathscr{T}_1$ to $\mathscr{T}_0$*;

(5) *For each $x \in \Lambda$, a linear equivalence $\tilde{\Phi}_x\colon T_xX \to T_{H(x)}X$ between the isotropy representations $D_x\varphi_0$ and $D_{H(x)}\varphi_1$ so that $\tilde{\Phi}_x$ maps each one-dimensional tangent space $T_x\mathscr{F}_{0,i}$ to the corresponding space $T_{H(x)}\mathscr{F}_{1,i}$.*
Then the homeomorphism H is C^r.

Remark. Conditions (2) and (3) of Theorem 5.1 require that the same elements of Γ be hyperbolic and trellis preserving for both actions, while we only require regularity for the source trellis.

Proof. There are three steps in proving the regularity of the homeomorphism H:

(1) Construct a map of tangent bundles $\Phi\colon TX \to TX$, which covers the map H;

(2) Show that H restricted to each leaf of the trellis foliations is smooth, with the derivative given by a scalar multiple of the restriction of Φ;

(3) Invoke the web regularity theorem for smooth Anosov systems (cf. Lemma 2.3 of [28] for the case $n = 2$, Theorem 2.6 of [18] for the case $n > 2$, and [27] for the analytic case) to conclude that H has the same regularity as the actions.

In addition, if the homeomorphism $H = H_t$ is part of a one-parameter family, then we also will show in Proposition 5.11 that the C^r-map H_t varies C^0 continuously with the parameter in the C^l-topology on maps.

We first require a preliminary result for Anosov diffeomorphisms that arise from the restriction of group actions with vanishing first cohomology:

LEMMA 5.2. *Let X be a closed manifold of dimension n and let $\varphi\colon \Gamma \times X \to X$ be a C^r-action on X by orientation-preserving diffeomorphisms, for $r = 1$ or ∞. Assume that the periodic orbits of φ are dense and that $H^1(\tilde{\Gamma}, \mathbf{R}) = 0$ for each subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index. Then for each hyperbolic element $\gamma_h \in \Gamma$ there is a unique C^r-volume form Ω_{γ_h} on X, which has total mass 1 and is invariant under the action of $\varphi(\gamma_h)$.*

Proof. Fix a smooth volume form Ω_0 on X. For each $k \in \mathbf{Z}$ there is a smooth function $f_{\gamma,k}\colon X \to \mathbf{R}$ defined by the relation

$$\varphi(\gamma^k)^* \Omega|_x = \exp\{f_{\gamma,k}(x)\}\Omega|_x. \tag{8}$$

The functions $\{f_{\gamma,k}\}$ satisfy the cocycle equation over the action φ:

$$f_{\gamma,k+p}(x) = f_{\gamma,k}(\varphi(\gamma^p)(x)) + f_{\gamma,p}(x) \qquad \text{for all } k, p \in \mathbf{Z},\ x \in X. \tag{9}$$

At a periodic point $x \in \Lambda$ with period $p(x)$, the function $k \mapsto f_{\gamma,k\cdot p(x)}(x)$ is a homomorphism into the additive group $\mathbf{R}$. Let Γ_x be the isotropy subgroup of x and recall the divergence representation $\mathrm{Div}_x\colon \Gamma_x \to \mathbf{R}$ at x: For each $\delta \in \Gamma_x$, $\mathrm{Div}_x(\delta) = \log\{|\det(D_x\varphi(\delta))|\}$. Clearly the function $k \mapsto f_{\gamma,k\cdot p(x)}(x)$ is obtained by restricting the divergence representation of Γ_x to the subgroup generated by the powers of $\gamma^{p(x)}$.

The hypothesis that $H^1(\tilde{\Gamma}, \mathbf{R}) = 0$ implies Div_x is the trivial homomorphism and hence that $f_{\gamma,k\cdot p(x)}(x) = 0$ at each periodic orbit $x \in \Lambda$ and each $k \in \mathbf{Z}$.

Now fix a hyperbolic element γ_h. Then $f_{\gamma_h,p(x)}(x) = 0$ for a dense set of periodic points of the smooth Anosov diffeomorphism $\varphi(\gamma_h)$. By the Livsic theorem ([23], [24], [28]; cf. also [8], [18]) there is a C^r-function $F\colon X \to \mathbf{R}$ such that $f_{\gamma_h}(x) = F(\varphi(\gamma_h)(x)) - F(x)$ for all $x \in X$. We then define a volume form $\Omega_{\gamma_h} = \exp\{-F\}\Omega_0$, which is invariant for $\varphi(\gamma_h)$, by an elementary calculation.

By the theorem of Livsic and Sinai [25] there is a unique, absolutely continuous, invariant density for $\varphi(\gamma_h)$, up to constant scalar multiples. Rescale Ω_γ so that it has total mass one, and then it is unique. □

Let us return now to the proof of Theorem 5.1. We can assume without loss of generality that the foliations $\mathscr{F}_{t,i}$ are orientable. (There always exists a finite cover of X so that the lift of the foliations to the cover becomes orientable, and there is a subgroup of finite index in Γ whose action also lifts to the cover. The hypotheses of the theorem are again satisfied for this lifted action.) For each $1 \leq i \leq n$ and $t = 0, 1$ choose unit vector fields $X_{t,i}$ on X tangent to the leaves of $\mathcal{F}_{t,i}$. Each vector field $X_{t,i}$ is Hölder continuous by hypothesis (2.10.3).

Next construct the map Φ on tangent bundles. Fix $1 \leq i \leq n$. The diffeomorphism $\varphi_t(\gamma_i)$ preserves the foliation $\mathcal{F}_{t,i}$ so that the differential $D\varphi_t(\gamma_i)$ maps the vector field $X_{t,i}$ into a multiple of itself. Moreover, by replacing each element γ_i with its square γ_i^2, we can assure that $D\varphi_t(\gamma_i)(X_{t,i})$ is a positive multiple of $X_{t,i}$. Introduce the Hölder continuous cocycle $\mu_{t,i}$: $\mathbf{Z} \times X \to \mathbf{R}$ over the action of $\varphi_t(\gamma_i)$, defined by the relation

$$D\varphi_t(\gamma_i^k)(X_{t,i}(x)) = \exp\{\mu_{t,i}(k, x)\} \cdot X_{t,i}(\varphi_t(\gamma_i^k)(x)). \tag{10}$$

For notational convenience set $x_0 = x$ and $x_1 = H(x)$ for any $x \in X$. For $x \in \Lambda$ recall that $p(x)$ is the least positive integer l such that $\varphi_0(\gamma^l)(x) = x$ for all $\gamma \in \Gamma$.

LEMMA 5.3. *For each* $x \in \Lambda_0$, $\mu_{0,i}(p(x), x_0) = \mu_{1,i}(p(x), x_1)$.

Proof. We have $\gamma_i^{p(x)} \in \Gamma_x$ so that $D_{x_0}\varphi_0(\gamma_i^{p(x)})$ is linearly equivalent to $D_{x_1}\varphi_1(\gamma_i^{p(x)})$, and the similarity $\tilde{\Phi}_x$ sends the tangent space $T_{x_0}\mathcal{F}_{0,i}$ to $T_{x_1}\mathcal{F}_{1,i}$. By its definition, $\exp\{\mu_{t,i}(p(x), x)\}$ is the exponent of $D_{x_t}\varphi_t(\gamma_i^{p(x)})$ in the one-dimensional subspace $T_{x_t}\mathcal{F}_{t,i}$, and the lemma follows. □

LEMMA 5.4. *There exists a Hölder continuous, vector bundle map* Φ: $TX \to TX$ *covering* H *so that*

$$\Phi_{\varphi_0(\gamma)(x_0)} \circ D_{x_0}\varphi_0(\gamma) = D_{x_1}\varphi_1(\gamma) \circ \Phi_{x_0} \text{ for all } \gamma \in \Gamma. \tag{11}$$

In particular, for $x \in \Lambda$, Φ_{x_0} *conjugates the linear isotropy action of* $D_{x_0}\varphi_0(\gamma^{p(x)})$, *restricted to the tangent space to* $\mathcal{F}_{0,i}$ *at* x_0, *to the corresponding restricted linear isotropy action of* $D_{x_1}\varphi_1(\gamma_i^{p(x)})$, *restricted to the tangent space to* $\mathcal{F}_{1,i}$ *at* x_1.

Proof. Fix $1 \leq i \leq n$. Define a Hölder continuous 1-cocycle over the action $\varphi_0(\gamma_i)$:

$$M_i: \mathbf{Z} \times X \to \mathbf{R}, \tag{12}$$

$$M_i(k, x) = \mu_{1,i}(k, x_t) - \mu_{0,i}(k, x_0).$$

Our hypotheses and Lemma 5.3 imply that $M_i(p(x), x) = 0$ for all $x \in \Lambda$. By Lemma 5.2 there exists a C^r-volume form $\Omega_{0,i}$, which is $\varphi_0(\gamma_i)$-invariant. Therefore by the Livsic theorem ([23], [24]) there exists a Hölder continuous function F_i: $X \to \mathbf{R}$ such that $M_i(k, x) = F_i(\varphi_1(\gamma_i)(x)) - F_i(x)$ for all $x \in X$, and F_i is unique up to an additive constant.

Set $\tilde{X}_{1,i} = \exp\{-F_i\}X_{1,i}$ and define the map Φ by specifying it on the frame field $\{\tilde{X}_{0,1}, \ldots, \tilde{X}_{0,n}\}$,

$$\Phi(X_{0,i}(x)) = \tilde{X}_{1,i}(H(x)),$$

and extending it linearly on each of the fibers of TX. The choice of F_i ensures that equation (11) holds when Φ is restricted to any of the frame fields $X_{0,i}$ in TX; hence it holds on all of TX. □

For each $1 \le i \le n$ and each $x \in X$ let

$$\Psi_{t,i,x}: \mathbf{R} \to X$$

be the smooth immersion obtained by integrating the vector field $\tilde{X}_{t,i}$ with an initial condition $\Psi_{t,i,x}(0) = x_t$. The image of $\Psi_{t,i,x}$ is the stable manifold through x_t, denoted by $L_{t,i,x} \hookrightarrow X$. The homeomorphism H restricted to L_{0,i,x_0} maps into the stable manifold L_{1,i,x_1} so that, composing with these coordinates, we obtain a family of functions of one variable:

$$\begin{aligned} &H_{i,x}: \mathbf{R} \to \mathbf{R}, \\ &H_{i,x}(r) = \left(\Psi^i_{1,x_1}\right)^{-1} \circ H \circ \Psi^i_{0,x_0}(r). \end{aligned} \tag{13}$$

A key point in the proof of Theorem 5.1 is that it suffices to show that the restricted maps $H_{i,x}$ are smooth, with uniform estimates on all derivatives; the same will then hold for the restrictions of H composed with the inclusion $H_{i,x}: \mathbf{R} \to L_{1,i,x} \subset X$ (as a consequence of our assumption that the leaves of the trellis foliation $\mathscr{F}_{1,i}$ are smoothly immersed, with uniform estimates on the derivatives of the inclusion maps).

Lemma 5.5. *Let φ_0, φ_1 and H satisfy the hypotheses of Theorem* 5.1. *Then for all $1 \le i \le n$ and $x \in X$ the function $H_{i,x}$ is C^r, and the C^r-jet depends continuously on x. If the actions φ_0 and φ_1 are analytic, then $H_{i,x}$ is analytic and admits an analytic extension to a strip, with uniform width as a function of x.*

Proof. The proof is an adaptation with a few technical modifications of the proof of Theorem 2 in [29]. We indicate the steps and necessary modifications, and leave the remaining details to the reader. First observe the following:

Lemma 5.6. *There exists a nonzero integer k_i so that the restriction of $D\varphi_t(\gamma^{k_i})$ to the tangential distribution $T\mathscr{F}_{t,i}$ is uniformly contracting.*

Proof. For $t = 0$ or 1, $\varphi_t(\gamma_i)$ is Anosov, so there is a dichotomy: an invariant one-dimensional distribution $\mathbf{E} \subset TX$ for the diffeomorphism must be contained either in the expanding subbundle $\mathbf{E}^+_{t,i}$ of $\varphi_t(\gamma_i)$ or in the contracting subbundle $\mathbf{E}^-_{t,i}$. In particular $T\mathscr{F}_{t,i}$ must be a subbundle of either $\mathbf{E}^+_{t,i}$ or $\mathbf{E}^-_{t,i}$.

Choose $k_i > 0$ if $T\mathscr{F}_{0,i} \subset \mathbf{E}_{0,i}^-$, and $k_i < 0$ if $T\mathscr{F}_{0,i} \subset \mathbf{E}_{0,i}^+$, with $|k_i|$ sufficiently large so that, for both $t = 0$ and $t = 1$, $D\varphi_t(\gamma_i)$ is uniformly contracting on $\mathbf{E}_i^-$ and $D\varphi_t(\gamma_i)$ is uniformly expanding on $\mathbf{E}_i^+$. For $t = 0$ this implies the claim of the lemma.

For $t = 1$ note that H conjugates the leaves of the foliation $\mathscr{F}_{0,i}$ to those of $\mathscr{F}_{1,i}$ so that $\varphi_0(\gamma_i^{k_i})$ uniformly contracting on the leaves of $\mathscr{F}_{0,i}$ implies the same holds (topologically) for $\varphi_1(\gamma_i^{k_i})$ on $\mathscr{F}_{1,i}$. Hence the sign of k_i is also correct for $t = 1$. □

We next show that the function $H_{i,x}$ is C^r with uniform estimates in the C^r-norm.

The first step is to show that the maps $H_{i,x}$ are uniformly Lipshitz. Apply the lemma (p. 187, [29]) to the trellis foliation $\mathscr{F}_{0,i}$ to obtain a homeomorphism $\tilde{H}$, which is C^0-close to H and maps the foliation $\mathscr{F}_{0,i}$ to $\mathscr{F}_{1,i}$ and which is uniformly monotone increasing and C^r along the foliation $\mathscr{F}_{0,i}$. Introduce corresponding coordinate maps $\tilde{H}_{i,x}$. We show that the composition $H_{i,x}^{-1} \circ \tilde{H}_{i,x}$ is uniformly Lipshitz, and our claim for $H_{i,x}$ follows.

The argument at the top of page 188, [29] requires only that $\mathscr{F}_{0,i}$ be an invariant, uniformly contracted, one-dimensional foliation for the Anosov map $\varphi(\gamma_i)$; so it also proves the following lemma:

LEMMA 5.7.

$$H^{-1} \circ \tilde{H} = \lim_{t \to \infty} \left\{ \varphi_{0,t} \circ \tilde{H}^{-1} \circ \varphi_{1,-t} \circ \tilde{H} \right\}. \tag{14}$$ □

The existence of a Hölder continuous, tangent-bundle map Φ, intertwining the (uniformly contracting) linear actions of $\varphi_t(\gamma_i^{k_i})$ on the leaves of $\mathscr{F}_{t,i}$ for $t = 0, 1$, allows us to make uniform estimates on the derivatives of the restrictions of the compositions $\varphi_{0,t} \circ \tilde{H}^{-1} \circ \varphi_{1,-t} \circ \tilde{H}$ to the leaves of $\mathscr{F}_{0,i}$. These uniform estimates and the uniform convergence in (14) above imply that $H_{i,x}^{-1} \circ \tilde{H}_{i,x}$ is a Lipshitz map (cf. pp. 188–9, [29]).

A Lipshitz continuous map has a derivative almost everywhere, and the derivatives of the maps $H_{i,x}$ form a 1-coboundary for the differential 1-cocycle M defined in equation (12) over the Anosov map $\varphi_0(\gamma_i^{k_i})$. The absolute continuity of the foliation $\mathscr{F}_{0,i}$ implies that the coboundary is defined almost everywhere for the standard Lebesgue measure on X. Then by the measurable Livsic theorem ([23], [24]) the derivatives of the Lipshitz maps $H_{i,x}$ exist everywhere (cf. Theorem 3, [26]; and also the proof of Proposition 6.3, [18]) with uniform estimates on their norms.

When $r > 1$, we next apply the standard "bootstrap" technique (cf. Lemma 2.2, [28] and Theorem 1, [26]) to the derivative of $H_{i,x}$ along $\mathscr{F}_{0,i}$ to deduce that the function $H_{i,x}$ must be C^r. This step in the argument works for any $1 < r \leq \infty$.

For the case $r = \omega$ we are given that the actions φ_t are analytic and that the associated trellises are real analytic. Then the coordinate maps $\Psi_{t,i,x}$ are uniformly analytic. We modify the bootstrap method to include radius-of-convergence estimates. (This approach is worked out in detail in the proof of Theorem 2, [27].) This yields uniform estimates on the rate of decay of the Fourier coefficients for the smooth functions $H_{i,x}$, with the conclusion that each function $H_{i,x}$ extends to an analytic function in a uniform strip (cf. §3, [27]). □

PROPOSITION 5.8. *Let φ_t and H satisfy the hypotheses of Theorem* 5.1. *Then H is a C^∞-diffeomorphism. If the actions φ_t are analytic, then H is a real analytic diffeomorphism.*

Proof. Theorem 2.3 of [28] characterizes the smooth functions on an open set by their restrictions to a pair of transverse foliations satisfying a regularity hypothesis: The smooth functions are precisely those functions whose restrictions to individual leaves are uniformly smooth. The web regularity theorem, Theorem 2.6 of [18], reproves this theorem by elementary methods of Fourier series and also extends the characterization of smooth functions to include restrictions to multiple transverse foliations. The regular trellis $\mathscr{T}_0$, associated to the action φ_0, exactly satisfies the necessary foliation regularity hypotheses to apply Theorem 2.6, [18].

Lemma 5.5 establishes that H restricts to uniformly smooth functions on leaves of the trellis foliations $\mathscr{F}_{0,i}$. Thus H is locally smooth on X and hence is smooth.

We have used so far only that the trellis $\mathscr{T}_0$ is regular to obtain that H is smooth. We prove that H^{-1} is smooth without a regularity assumption of the trellis $\mathscr{T}_1$ via a technical observation:

LEMMA 5.9. *The tangent bundle maps Φ and DH agree up to fiberwise-scale factors $\{C_1, \ldots, C_n\}$, which are constant on X.*

Proof. With the notation of Lemma 5.4 let $\hat{X}_{1,i} = DH(X_{0,i})$. Then there exist functions $\hat{F}_i$ on X so that $\hat{X}_{1,i} = \exp\{\hat{F}_i\} \cdot X_{1,i}$, and hence $\hat{F}_i$ is also a coboundary for the cocycle M_i. By uniqueness of solutions there exist constants c_i so that $\hat{F}_i = F_i + c_i$. Introduce the tangent bundle map $\mathbf{C}$: $TX \to TX$, which on the typical fiber T_xX acts on $X_{1,i}(x)$ by multiplication by $C_i = \exp\{c_i\}$. We then have $DH = \mathbf{C} \circ \Phi$. □

Lemma 5.9 implies that DH is uniformly injective, as Φ is injective. It follows that H^{-1} is also a C^1-diffeomorphism; hence H is C^r implies that H is a C^r-diffeomorphism.

The analytic case of Proposition 5.8 follows from the analytic extension of the web regularity theorem. The Fourier series method of [18] is used by R. de la Llavé in [27] to characterize the real analytic functions as those functions which are (locally) uniformly real analytic when restricted to the individual leaves of a foliation whose leaves are analytically immersed submanifolds, with appropriate transversal hypotheses. The proof in [27] is only for a pair of foliations, but the method extends directly to the case of multiple transverse foliations. We then apply the analytic conclusion of Lemma 5.5 to obtain that H is a real analytic diffeomorphism (cf. §3, [27]). □

Proof of Theorem 2.12. We apply Theorem 5.1 to the topological conjugacies H_t, which are given between the trellised action φ_0 and the actions φ_t, for $0 \le t \le \varepsilon$. The condition $\mathrm{SVC}(n^2 - 1)$ on Γ implies the vanishing of the cohomology groups $H^1(\tilde{\Gamma}; \mathbf{R})$. The Weil rigidity theorem, as applied in Corollary 4.5, implies that the isotropy representations at $x \in \Lambda$ are independent of t up to a linear isomorphism of tangent spaces, which yields hypothesis (5) of Theorem 5.1. Thus by this theorem we have each H_t is C^r. The C^0-dependence of H_t on t in the C^l-topology follows from Proposition 5.11 below.

Proof of Theorem 2.15. We apply Theorem 5.1 to the topological conjugacy H between the trellised action φ_0 and the action φ_1, and we need to verify hypotheses (1) and (5). First observe:

LEMMA 5.10. *Assume that Γ is topologically determined in dimension $N \ge 1$. Then for every subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index*, $H^1(\tilde{\Gamma}; \mathbf{R}) = 0$.

Proof. If $H^1(\tilde{\Gamma}; \mathbf{R}) \neq 0$, then there exists a nontrivial representation $\tilde{\Gamma} \to \mathbf{R}$ with discrete image, which induces a hyperbolic action on the line. The topological type of a discrete hyperbolic action on $\mathbf{R}$ does not determine the exponent of the action, which yields a contradiction. □

Next observe that, at each periodic point $x \in \Lambda$ for the action $\varphi_t = \varphi$, the representations of the isotropy group Γ_x on $\mathbf{R}^n$ are topologically determined. Moreover we are given that H conjugates the action $\varphi_0(\Gamma_x)$ to the action $\varphi_1(\Gamma_x)$ in a neighborhood of x. The trellis structure on each action implies these are hyperbolic representations; therefore the isotropy representations $D_x\varphi_t\colon \Gamma_x \to \mathrm{GL}(n, \mathbf{R})$ are linearly conjugate, which establishes (5). Theorem 2.15 now follows.

The last result of this section addresses the regularity of an ε-$C^{k,l}$-deformation $\{\varphi_t | 0 \le t \le 1\}$ of a trellised action, which is assumed to be topologically trivial by a continuous family of homeomorphisms $\{H_t | 0 \le t \le 1\}$. We have established in Theorem 5.1 that the conjugating map H_t is C^r for each

$0 \le t \le \varepsilon$. Let $\mathrm{Maps}^l(X)$ denote the set of C^r-maps of X to itself, endowed with the C^l-Frechet topology.

PROPOSITION 5.11. *Let $\{\varphi_t | 0 \le t \le 1\}$ be a $C^{0,l}$-family of Anosov actions, with φ_0 trellised. Suppose that $\{H_t | 0 \le t \le 1\}$ satisfies the conjugating equation (5) with each map H_t a C^r-diffeomorphism for $r \ge 1$. Suppose also that the image trellis $\mathscr{T}_t$, whose foliations $\{\mathscr{F}_{t,i}\}$ are the push-forward of the foliations $\{\mathscr{F}_{0,i}\}$ of the trellis $\mathscr{T}$, has leaves that vary C^0 continuously with the parameter t in the C^l-topology on immersions. Then the curve*

$$\hat{H}: [0,1] \to \mathrm{Maps}^l(X); \qquad t \mapsto H_t$$

is continuous.

Proof. Fix a C^r-Riemannian metric on TX and define the vector fields $X_{t,i}$ for all $0 \le t \le \varepsilon$ as the unit, positively oriented, tangent field to the trellis foliation $\mathscr{F}_{t,i}$. The hypothesis on the trellises implies that these vector fields depend continuously on t in the C^l-topology. Define vector fields $\hat{X}_{t,i} = DH_t(X_i)$ and introduce the corresponding scale functions $\hat{F}_{t,i}$ such that $\hat{X}_{t,i} = \exp\{-\hat{F}_{t,i}\}X_{t,i}$, as in the proof of Lemma 5.4.

The differential DH_t is completely determined by the functions $\hat{F}_{t,i}$ and the vector fields $X_{t,i}$; so the claim for $l = 1$ follows by showing that the functions $\hat{F}_{t,i}$ depend continuously on t in the C^1-topology. Each function $\hat{F}_{t,i}$ solves the t-dependent cocycle equation (12), for $M_{t,i}(k,x) = \mu_{t,i}(k,x_t) - \mu_{0,i}(k,x_0)$. The family of diffeomorphisms $\varphi_t(\gamma_i)$ varies C^0 continuously with t in the C^r-topology by hypothesis; hence in the direction of $X_{t,i}$, the expansion function $\exp(\mu_{t,i}(k,x_t))$ and the difference of the exponents $M_{t,i}$ depend continuously on t in the C^l-topology. By Livsic theory, the solution function $\hat{F}_{t,i}$ will vary C^0 continuously with t in the C^l-topology (cf. Theorem 2.2 of [28]) when $l = 1$ or $l = \infty$.

The proof that H_t is $C^{0,l}$ is more delicate. First note that the functions of one variable $H_{t,i,x}$ defined in (13) have derivatives determined by the restrictions of $\hat{F}_{t,i}$ and therefore vary C^0 continuously with the parameter t in the C^l-topology. For $l = 1$ we are then done. For $l = \infty$ note that the proofs of the regularity results (Theorem 2.3, [28] and Theorem 2.6, [18]) give explicit control over the local Fourier transforms of H_t in terms of the Fourier transforms of the functions $H_{t,i,x}$ on the line. As the latter data depend continuously on the parameter, the local Fourier transforms of H_t vary continuously with t in the Schwartz topology. We obtain that H_t varies $C^{0,\infty}$ with t by the Fourier inversion formula.

The cases $1 < l < \infty$ are excluded above, but the reader can now see the nature of the extension to these intermediate cases. The point is that we obtain continuous control in t on the local Fourier transforms of H_t in the l-Schwartz norms. Applying the inverse Fourier transform yields only that H_t varies $C^{0,l-\nu}$ with t for some $0 < \nu < n$. □

6. Differential rigidity for Cartan actions

The classification of Cartan C^r-actions on a closed manifold X of dimension $n > 2$ is a special case of the problem of classifying the centralizers of Anosov C^r-diffeomorphisms (cf. [35], [36]). We give the proofs of Theorems 2.19 and 2.21 in this section, which show that volume-preserving Cartan actions can be classified by their linear isotropy data at periodic orbits. Cartan actions thus admit a complete generalization of the smooth stability and rigidity theory for a single Anosov diffeomorphism of $\mathbf{T}^2$, developed in a series of papers by R. de la Llavé, J. M. Marco and R. Moriyon ([26], [27], [29], [28], [31], [32]).

The results of this section do not give the definitive classification of Cartan C^r-actions (except possibly in the case of constant exponents), and it remains an interesting problem to develop moduli for them. For example, the classification of a single, volume-preserving, Anosov diffeomorphism of $\mathbf{T}^2$ was investigated by S. Hurder and A. Katok [18], where the *Anosov class* of a codimension-one Anosov diffeomorphism of $\mathbf{T}^2$ was introduced. This is a cohomology invariant of the action, constructed as an obstacle to the differentiability of the stable and unstable foliations of the given Anosov diffeomorphism. It provides a very effective parameter on the space of volume-preserving Anosov diffeomorphisms on $\mathbf{T}^2$. The study of the regularity of the invariant foliations for a Cartan action on $\mathbf{T}^n$ for $n > 2$ is expected similarly to yield cohomology invariants, which are effective for the C^r-classification of Cartan actions.

We first establish that a Cartan C^∞-action, which is "volume preserving" at periodic points, always preserves a smooth volume form. Hence by Moser's theorem [34] such an action can be assumed to preserve (up to smooth conjugacy) the "standard" smooth volume form on X.

Theorem 2.16 is a key technical result and is proven next, establishing that a Cartan action always preserves a (not necessarily regular) trellis.

Theorem 2.19 states that a smooth Cartan action on X is determined by the exponents of the group action at periodic orbits. Equivalently by the Livsic theorem ([23], [24]) the cohomology class of the diagonal exponent 1-cocycle $D\varphi\colon \mathscr{A} \times X \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$ also parametrizes the smooth Cartan actions. This is proven next. Moreover a $C^{0,1}$-deformation of a Cartan action, whose exponents at periodic orbits are independent of t, is implemented by conjugating with a $C^{0,1}$-path of diffeomorphisms. A similar result holds for $C^{0,\infty}$-deformations.

We finally consider Cartan actions with constant exponents and show that they must be algebraic. This is a critical ingredient for establishing perturbation rigidity of Cartan lattice actions.

The first result extends Lemma 5.2 to Cartan actions:

LEMMA 6.1. *Let $\varphi\colon \mathscr{A}\times X \to X$ be a Cartan C^r-action with dense periodic orbits, for $r = 1, \infty$. Assume that for each periodic orbit $x \in \Lambda$ the divergence homomorphism*

$$\mathrm{Div}_x\colon \Gamma_x \to \mathbf{R},$$
$$\mathrm{Div}_x(\delta) = \log\{|\det(D_x\varphi(\delta))|\},$$

is trivial. Then there exists a unique C^r-volume form Ω on X of total mass 1, which is invariant for the action φ.

Proof. Fix a hyperbolic element $\gamma \in \mathscr{A}$. By the proof of Lemma 5.2 there is a unique invariant C^∞-volume form Ω_γ of total mass 1, which is invariant under $\varphi(\gamma)$. For each $\delta \in \mathscr{A}$ the volume form $\varphi(\delta)^*\Omega_\gamma$ is again $\varphi(\gamma)$-invariant and so must equal Ω_γ. □

Proof of Theorem 2.16. Assume that we are given a Cartan C^r-action $\varphi\colon \mathscr{A}\times X \to X$, with hyperbolic generating set $\Delta = \{\delta_1, \dots, \delta_n\}$. The stable manifold theory of Hirsch and Pugh ([12]; see also [39]) implies that each leaf of the one-dimensional, strongest stable foliation $\mathscr{F}_i^{ss}$ is a C^r-immersed submanifold of X, and the immersion depends continuously on the point in X. We have assumed that the strongest stable foliations are pairwise transverse; so this shows that the collection $\mathscr{T} = \{\mathscr{F}_1^{ss}, \dots, \mathscr{F}_n^{ss}\}$ is a C^r-trellis on X.

Let $\varphi\colon \mathscr{A}\times \mathbf{T}^n \to \mathbf{T}^n$ be a maximal Cartan C^r-action. Anosov proved that the stable foliation of an Anosov diffeomorphism is absolutely continuous ([2], [1]; a more detailed proof following Anosov's ideas is given in Lemma 2.5, [28]). Thus the collection $\mathscr{T}$ of stable foliations of the Anosov maps $\{\varphi(\gamma_i)\}$ is a regular C^r-trellis on X.

The third part of Theorem 2.16 follows from the next lemma:

LEMMA 6.2. *Let (φ, Δ) be a smooth Cartan C^r-action for $r \geq 3$, which leaves invariant a smooth volume form Ω on $\mathbf{T}^n$. Then for each $1 \leq i \leq n$ the stable foliation $\mathscr{F}_i$ is $C^{1,\alpha}$ for some $\alpha > 0$.*

Proof. By Moser's theorem [34] we can assume without loss of generality that Ω is the standard volume form for the euclidean metric on $\mathbf{T}^n$. Let $\mathscr{F}_1$ be the stable foliation of the Anosov diffeomorphism $\varphi(\delta_1)$. Choose a (continuous) Riemannian metric on $T\mathbf{T}^n$ so that the foliations $\mathscr{F}_i$ are pairwise orthogonal, the volume form of the new metric is the standard volume form on $\mathbf{T}^n$ and the

Anosov condition (3) holds for $c = 1$ and $\lambda = 1/\mu$. Let X^i be the corresponding oriented, unit, tangent vector field to $\mathcal{F}_i$. For any point $x \in \mathbf{T}^n$ let $\mu_i(x) = \mu(\delta_i, x)$ denote the exponent of $\varphi(\delta_i)$ at x for the action on $\mathcal{F}_i$. Then

$$\mu_1(x) < 0 < \mu_i(x), \qquad 2 \le i \le n,$$

$$0 = \mu_1(x) + \cdots + \mu_n(x),$$

which together imply that $-\mu_1(x) > \mu_i(x) > 0$ for all $2 \le i \le n$, uniformly in x. Thus there exists some $\alpha_1 > 0$ so that the flow is *α_1-spread* in the sense of Definition 1.1 (Hasselblatt, [11]). The stable foliation regularity theory of Theorem I ([11]) implies that the foliation is C^{1,α_1}. The same will hold for all $1 < i \le n$ by the same method, and we take α to be the infimum of the α_i. □

Proof of Theorem 2.19. Recall that, for this, we are given two volume-preserving Cartan C^r-actions (φ_0, Δ) and (φ_1, Δ) on a closed manifold X, for $r = 1$ or ∞, with φ_0 a trellised action preserving the regular trellis $\mathcal{T}_0$. For each $\gamma_i \in \Delta$ the Anosov diffeomorphism $\varphi_0(\gamma_i)$ is volume preserving and hence is transitive. We deduce that the set of periodic orbits for $\varphi_0(\gamma_i)$ is dense, and as $\mathcal{A}$ is abelian, the set Λ of periodic orbits for the full action $\varphi(\mathcal{A})$ must be dense.

It is also given that

(1) $H\colon X \to X$ is a homeomorphism conjugating φ_1 to φ_0;

(2) $\varphi_1(\delta_i)$ is Anosov for each $1 \le i \le n$;

(3) for all $1 \le i \le n$ and for each $x \in \Lambda(\varphi_0)$ the maximally contracting exponent of $D_x\varphi_0(\delta_i)$ equals the maximally contracting exponent of $D_{H(x)}\varphi_1(\delta_i)$.

We apply the method of proof of Theorem 5.1 to deduce that H is C^r. First we note that an invariant volume form is given for the action φ_0 so that we obtain the conclusion of Lemma 5.2 automatically.

The action of φ_1 is Cartan; so by Theorem 2.16 there is a C^r-trellis $\mathcal{T}_1$ on X, which is invariant under the action of $\varphi(\mathcal{A})$. By passing to covers, we can assume without loss of generality that the trellises $\mathcal{T}_0$ and $\mathcal{T}_1$ are oriented.

Hypothesis (5) of Theorem 5.1 on the linear isotropy representations was used to identify the exponents along the strong stable foliations at fixed points for the action of the elements of Δ. This conclusion is part of the given data (see 2.19.3).

The regularity of H now follows by the same proof as that given for Theorem 5.1. It remains to note that we can apply the method of proof of Proposition 5.11 to obtain the C^k-dependence of H_t on the parameter t. □

Proof of Theorem 2.21. We formulate the proof for a Cartan action on an infra-nilmanifold X. It is only at the last step that we require $X = \mathbf{T}^n$.

Let (φ, Δ) be a Cartan C^r-action on an infra-nilmanifold X. Manning [30] proved that an Anosov C^1-diffeomorphism of an infra-nilmanifold has a dense nonwandering set and hence, by the Anosov closing lemma, has dense periodic orbits. It follows that the abelian group $\varphi(\mathscr{A})$ also acts with dense periodic orbits. We then make an elementary observation.

LEMMA 6.3. *A Cartan action on an infra-nilmanifold with constant exponents preserves a smooth volume form.*

Proof. The sum of the exponents for each $\delta \in \mathscr{A}$ is constant after scaling by the reciprocal of the length of an orbit. The proof of Lemma 5.2 (which uses Livsic's theorem and thus requires that the periodic orbits be dense) shows that there is a volume form Ω on X, which is uniformly expanded by the exponential of this average sum of exponents. As a diffeomorphism must preserve the total volume, the sum of the exponents must be zero. We then apply Lemma 5.2 to obtain the invariant volume form Ω. □

Proposition 2.18 implies that there is a subgroup $\tilde{\mathscr{A}} \subset \mathscr{A}$ of finite index, generated by the set Δ^p, so that the restricted action $\varphi | \tilde{\mathscr{A}}$ is topologically conjugate, via a homeomorphism H, to a linear action φ_* on X, which is determined by the action $\varphi_\#\colon \tilde{\mathscr{A}} \times \pi_1(X) \to \pi_1(X)$ induced from φ.

The Cartan action φ has an invariant trellis (which need not be regular); via the topological conjugacy H, we obtain a "topological trellis" on X, which is invariant under the linear action $\varphi_*(\tilde{\mathscr{A}})$. As (φ_*, Δ^p) is an abelian linear action with constant exponents, the existence of a topological trellis, which is invariant under $\varphi_*(\tilde{\mathscr{A}})$, implies there exists a linear (hence regular!) trellis, which is $\varphi_*(\tilde{\mathscr{A}})$-invariant. Thus the action φ_* is trellised.

It remains to show that the exponents of the action φ at periodic orbits $x_0 \in \Lambda$ equal the exponents at the periodic orbits $\Lambda_* = H(\Lambda)$ for the algebraic action φ_*. We can then invoke Theorem 2.19 to conclude that H is a C^r-conjugacy.

We restrict ourselves now to the case where $X = \mathbf{T}^n$. Corresponding to each one-dimensional foliation $\mathscr{F}_i$ of the trellis $\mathscr{T}_0$ for φ is an asymptotic, nonzero, one-dimensional homology class $[C_i] \in H_1(\mathbf{T}^n; \mathbf{R})$. The induced action of φ on homology defines a "homology expansion rate" for each $\alpha \in \mathscr{A}$, given by $\varphi_*(\alpha)[C_i] = \tilde{\lambda}_i(\alpha)[C_i]$. The homology expansion rates form a homomorphism $\tilde{\lambda}\colon \mathscr{A} \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$. This homomorphism is clearly a homeomorphism invariant of the action φ so that the linear action φ_* has the same exponent function (this is actually a tautological fact).

LEMMA 6.4. *Let $x_* \in \Lambda_*$ and let $\mathscr{A}_{x_*}$ be the isotropy subgroup for φ_* at x_*. Then the restriction of the linear isotropy representation*

$D_{x_*}\varphi_*: \mathscr{A}_{x_*} \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$ equals the restriction of the homology homomorphism $\tilde{\lambda}: \mathscr{A}_{x_*} \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$.

Proof. The action φ_* is linear, so its exponents at a periodic orbit equal its expansion rate along an integral curve through the orbit and hence its homology expansion rates. □

LEMMA 6.5. *Let $x \in \Lambda$ and let $\mathscr{A}_x$ be the isotropy subgroup for φ at x. Then the restriction of the linear isotropy representation*

$$D_x\varphi: \mathscr{A}_x \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$$

equals the restriction of the homology homomorphism

$$\tilde{\lambda}: \mathscr{A}_x \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+.$$

Proof. It is given that the restriction $D_x\varphi: \mathscr{A}_x \to \mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$ is independent of the point $x \in \Lambda$ (up to congruence equivalence of subgroups of $\mathscr{A}$); so we must establish the seemingly obvious, except that an application of the Livsic theorem is required to prove it.

Use the technique in the proof of Proposition 5.1 (hence the Livsic theorem) to continuously parametrize the line fields $T\mathscr{F}_i$ so that the differential $D\varphi$ acts via a constant scale multiplier in each strong-stable manifold. Let $\tilde{X}_i$ be the oriented unit vector fields for this parametrization, with $\tilde{X}_i$ tangent to $\mathscr{F}_i$. Each integral curve for $\tilde{X}_t$ defines an asymptotic cycle in the class of $[C_i]$; so we can use this cycle to determine the homology expansion rate of an element $\alpha \in \mathscr{A}_x$. Since we have linearized the action of $\varphi(\alpha)$ along $\mathscr{F}_i$, the exponents at a periodic point x determine the expansion rates of the integral curves through this point and thus the expansion rate on homology. □

We have now shown that H conjugates the action $\varphi|\mathscr{A}$ to a linear action. The full action of $\mathscr{A}$ commutes with the subgroup action and so must be affine in the coordinates provided by H.

7. Applications and examples

The purpose of this section is to discuss some of the examples of algebraic lattice group actions, which are rigid by the theorems of the previous sections. The list is not exhaustive, but it is sufficient to give the reader an idea of the available constructions. We discuss in each case the applications of the theorems of Section 2, as to whether the actions are topologically deformation rigid, C^r-deformation rigid or C^r-rigid for $r = 1, \infty, \omega$. Theorems 1.1, 1.2 and 1.3 of the Introduction follow from the lemmas and discussion of this section.

We start with the central example of integer matrices acting on the torus. The constructions of the congruence filtration of the periodic orbits are completely natural in this context, and there are several elementary observations that apply in full generality to all of the arithmetic examples discussed afterwards.

The second class of examples presented is the subgroups of the integer symplectic matrices. These lattices always contain Cartan subgroups whose standard action is Cartan.

Weyl's restriction-of-scalars technique gives a third construction of basic examples. The issue with these examples is to obtain the Anosov condition; if the standard action of such a lattice is Anosov, then it will also be Cartan.

The three basic classes of examples can be combined via the constructions of geometric sums, products, diagonal actions and what we will call arithmetic products (see Example 7.20) to obtain many examples that are C^r-deformation rigid by Theorems 2.9 and 2.12 (e.g., the tensor product examples and the diagonal actions); the remainder are C^r-rigid by Theorem 2.22.

The last example of this section gives a construction of an analytic deformation of the standard action of $\mathrm{SL}(2, \mathbf{Z})$ on the 2-torus, which is not topologically trivial. Thus the standard action of $\mathrm{SL}(2, \mathbf{Z})$ is not topologically rigid for $n = 2$.

We begin by recalling some standard facts regarding lattices. The fundamental result on the existence of lattices is due to Borel and Harish–Chandra ([4]; cf. also Ch. XIV, [38]).

THEOREM 7.1 (Borel–Harish-Chandra). *Let $G \subset \mathrm{SL}(N, \mathbf{C})$ be a semisimple algebraic group defined over $\mathbf{Q}$. Then the group of integer points $G_{\mathbf{Z}}$ is a lattice in the group of real points $G_{\mathbf{R}}$.* □

The group $\Gamma = G_{\mathbf{Z}}$ preserves the integer lattice in $\mathbf{R}^N$ and so descends to a standard action on $\mathbf{T}^N$.

The Margulis vanishing theorem (Theorem 2.8) discussed in Section 2 implies that SVC(N) holds for all $N \geq 1$ for every subgroup $\Gamma \subset G_{\mathbf{Z}}$ of finite index, where G is as in Theorem 7.1, $(G_{\mathbf{R}})_0$ has no compact factors and $G_{\mathbf{R}}$ has $\mathbf{R}$-split rank at least 2. Applying Theorem 2.9 to a standard action requires only this and the existence of at least one hyperbolic element in Γ.

Theorems 2.15 and 2.22 require the additional data of a Cartan subaction, which for a standard action require a commuting subset $\Delta \subset \Gamma$ of hyperbolic elements with one-dimensional maximal eigenspaces. The existence of these subgroups for any $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ of finite index can be shown by number-theoretic methods. However there is a much more powerful existence theorem, which applies to every subgroup $\Gamma \subset G_{\mathbf{Z}}$ of finite index, due to G. Prasad and M. S. Raghunathan (Theorem 2.8 and Corollary 2.9, [37]):

THEOREM 7.2 (Prasad–Raghunathan). *Let G be a semisimple analytic Lie group and Γ be a lattice in G. Let H be a Cartan subgroup of G; then there exists $g \in G$ such that $\Gamma_H = \Gamma \cap g^{-1}Hg$ is a uniform lattice in $g^{-1}Hg$.*

If G is a semisimple linear group with no compact factors, then we can apply Theorem 7.2 for H a maximal $\mathbf{R}$-split torus to conclude that a lattice Γ in G always contains a free abelian subgroup $\mathscr{A}$ of rank equal to the rank of G such that the generators of $\mathscr{A}$ are represented by commuting hyperbolic elements that can be made diagonal. We call the resulting subgroup Γ_H a *Cartan* subgroup for Γ. The standard action of Γ_H on $\mathbf{T}^N$ will be a Cartan action if the Lie group H can simultaneously be made diagonal by a basis $\{v_1, \dots, v_N\}$ of $\mathbf{R}^N$ so that each v_i is the (unique up to scalar multiples) maximal eigenvector for some $g_i \in H$. This is a Lie algebraic question, which can be easily determined in all examples.

Example 7.3 ($\mathrm{SL}(n, \mathbf{Z})$). Let $\mathbf{Z}/\mathbf{pZ}$ denote the finite cyclic group of order p. Reducing the entries $\mathrm{mod}(p)$ defines a natural quotient homomorphism of groups $\mathrm{SL}(n, \mathbf{Z}) \to \mathrm{SL}(n, \mathbf{Z}/\mathbf{pZ})$ whose kernel $\tilde{\Gamma}_p$ is called the p-congruence subgroup. It is clearly of finite index. For any subgroup $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ we call the sequence of normal subgroups formed by the intersections $\Gamma_p = (\Gamma \cap \tilde{\Gamma}_p)$ the *congruence filtration* of Γ. Technically these do not form a filtration of Γ, as they are not successively included into each other. To obtain a filtration, as defined in Section 3, we can take the subsequence $\{\Gamma_{p!} | p = 1, 2, \dots\}$.

Let $\mathbf{Z}^n \subset \mathbf{R}^n$ be the standard embedding, which is an $\mathrm{SL}(n, \mathbf{Z})$-invariant lattice. For each integer $p > 0$ introduce the lattice $((1/p)\mathbf{Z})^n$, characterized by the property that

$$\vec{v} \in \left(\frac{1}{p}\mathbf{Z}\right)^n \Leftrightarrow p\vec{v} \in \mathbf{Z}^n.$$

Each lattice descends to a finite subgroup of the torus $((1/p)\mathbf{Z})^n/(\mathbf{Z})^n \subset \mathbf{T}^n$. Let $(\mathbf{Q}/\mathbf{Z})^n$ denote the rational torus in $\mathbf{T}^n$. The following is an easy exercise, left to the reader:

LEMMA 7.4. *Let $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ contain a hyperbolic element for the standard action of Γ on $\mathbf{T}^n$.*

(1) *The periodic points of the standard action of Γ are $\Lambda = (\mathbf{Q}/\mathbf{Z})^n$ and hence are dense;*

(2) *the fixed-point set Λ_p for the congruence filtration subgroup Γ_p contains the p-torus $((1/p)\mathbf{Z})^n/(\mathbf{Z})^n \subset \mathbf{T}^n$; for the full congruence subgroup $\tilde{\Gamma}_p$ it is exactly the p-torus $((1/p)\mathbf{Z})^n/(\mathbf{Z})^n \subset \mathbf{T}^n$.* □

For a subgroup $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ of finite index for $n \geq 3$, the Margulis vanishing theorem (Theorem 2.8) implies that SVC(N) holds for all $N \geq 1$. Theorem 2.9 therefore applies to the standard action of Γ to show that it is topologically deformation rigid. For C^r-rigidity we require the additional data of an invariant regular trellis:

LEMMA 7.5. *Let $\Gamma \subset \mathrm{SL}(n, \mathbf{Z})$ be a subgroup of finite index. Then there is a subset of commuting hyperbolic elements $\Delta \subset \Gamma$ so that each $\gamma_i \in \Delta$ has a one-dimensional contracting eigenspace $E_i \subset \mathbf{R}^n$ with internal direct sum $E_1 \oplus \cdots \oplus E_n \cong \mathbf{R}^n$. Consequently the standard action of Γ on $\mathbf{T}^n$ is maximal Cartan.*

Proof. Let $\mathscr{A} \subset \Gamma$ be a Cartan subgroup obtained from the Prasad–Raghunathan theorem. The Zariski closure of $\mathscr{A}$ in $\mathrm{SL}(n, \mathbf{R})$ is isomorphic to a rank $(n-1)$ subgroup of a "diagonal" subgroup $\mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+ \subset \mathrm{SL}(n, \mathbf{R})$ of rank $(n-1)$. Via the logarithm map applied to the diagonal entries we identify $\mathscr{A}$ with an additive lattice in the codimension-1 subspace of $\mathbf{R}^n$ consisting of vectors whose coordinate sum is zero. For each $1 \leq i \leq n$ this hyperplane intersects the sector of $\mathbf{R}^n$, where $x_i < 0$ and the other coordinates of $\mathbf{R}^n$ are positive. By the Zariski dense condition there is an element $\gamma_i \in \mathscr{A}$ whose image is in this sector. From the definitions we see that γ_i is a matrix that can be made diagonal with exactly one contracting eigendirection, and all other eigendirections expanding. The collection of these elements forms the set Δ. $\square$

Combining the above remarks with Lemmas 7.4 and 7.5, we conclude that the standard action of a subgroup of finite index of $\mathrm{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n$, for $n \geq 3$, is C^r-rigid for $r = 1, \infty$ and ω.

Example 7.6 ($\mathrm{Sp}(n, \mathbf{Z})$). The previous example of $\mathrm{SL}(n, \mathbf{Z})$ corresponds to the "A" series of simple Lie groups. There are corresponding Anosov actions for the symplectic groups, or the "C" series. The rigidity of the standard action of one Anosov element in $\mathrm{Sp}(2, \mathbf{Z})$ was studied in [6].

LEMMA 7.7. *Let $n \geq 1$ and let $\Gamma \subset \mathrm{Sp}(n, \mathbf{Z}) \subset \mathrm{SL}(2n, \mathbf{Z})$ be a subgroup of finite index. Then the standard action of Γ on $\mathbf{T}^{2n}$ is Anosov.*

Proof. The real Lie group $\mathrm{Sp}(n, \mathbf{R})$ contains a Cartan subgroup H with a noncompact factor whose Lie algebra has a basis of n semisimple hyperbolic elements (as matrices in $\mathrm{SL}(2n, \mathbf{R})$). Theorem 7.2 states that Γ intersects a translate of H in a cocompact subgroup; so Γ must contain a hyperbolic element for the standard action on $\mathbf{R}^{2n}$ and hence on $\mathbf{T}^{2n}$. $\square$

This lemma suffices to establish the topological deformation rigidity of the standard action for $n \geq 2$. Differential rigidity will follow by showing that the action is also Cartan with invariant linear foliations, and hence that the action is C^{ω}-trellised.

LEMMA 7.8. *Let $n \geq 1$ and let $\Gamma \subset \mathrm{Sp}(n, \mathbf{Z})$ be a subgroup of finite index. Then there exist commuting matrices $\delta_1, \ldots, \delta_n \in \Gamma$ such that the set $\Delta = \{\delta_1, \delta_1^{-1}, \ldots, \delta_n, \delta_n^{-1}\}$ generates an abelian subgroup $\mathscr{A}$ whose standard action on $\mathbf{T}^{2n}$ is trellised.*

Proof. There exists a Cartan subgroup $\mathscr{A} \subset \Gamma$ by the Prasad–Raghunathan theorem, and we can assume without loss of generality that the eigenvalues of the elements of $\mathscr{A}$ are all positive. The algebraic hull of $\mathscr{A}$ will be an n-dimensional, "diagonal," $\mathbf{R}$-split, Cartan, connected subgroup $\mathscr{C} \subset \mathrm{Sp}(n, \mathbf{R})$. The subgroup $\mathscr{C}$ is identified (via the logarithm map followed by a symplectomorphism) with the maximal Lagrangian subspace

$$\mathscr{L} = \{(x_1, -x_1, \ldots, x_n, -x_n) | (x_1, \ldots, x_n) \in \mathbf{R}^n\} \subset \mathbf{R}^{2n}.$$

The pairing of the coordinates corresponds to the fact that the eigenvectors for a symplectic semisimple hyperbolic matrix are naturally paired by the invariant symplectic form.

The algebra $\mathscr{A}$ maps to a lattice $\tilde{\mathscr{A}}$ in $\mathscr{L}$. Therefore we can choose $\delta_i \in \mathscr{A}$ whose image in $\tilde{\mathscr{A}}$ has coordinates satisfying $x_i(\delta_i) < x_j(\delta_i) < 0$ for all $j \neq i$. The matrix δ_i has a one-dimensional, maximal contracting eigenspace, and the same holds for its inverse. The set of maximal eigendirections for the collection Δ is the same as the basis making the algebra $\mathscr{A}$ diagonal; so we obtain a linear trellis, which is invariant for the standard action of $\mathscr{A}$. □

Example 7.9 ($\mathrm{SL}(n, \mathscr{O}(\mathbf{k}))$). Let $\mathbf{k} \subset \mathbf{R}$ be an algebraic number field of degree d over $\mathbf{Q}$, let $\mathscr{O}(\mathbf{k})$ be the ring of integers for the field and let $\mathrm{SL}(n, \mathscr{O}(\mathbf{k}))$ be the subgroup of $\mathrm{SL}(n, \mathbf{k})$ with entries from $\mathscr{O}(\mathbf{k})$. The restriction-of-scalars technique of Weil yields a wide range of lattice actions.

PROPOSITION 7.10. *For $n \geq 2$ and $\Gamma \subset \mathrm{SL}(n, \mathscr{O}(\mathbf{k}))$ a subgroup of finite index,*

(1) *there exists an analytic "standard" action of Γ on $\mathbf{T}^{dn}$, and*

(2) *if the group $G_{\mathbf{R}} = R_{\mathbf{k}/\mathbf{Q}}(\mathrm{SL}(n, \mathbf{R}))_{\mathbf{R}}$ of real points (for the group G obtained by the restriction of scalars) has no compact factors, then the standard action of Γ is Anosov.*

Proof (cf. pp. 115–6, [48]). Let $\{\sigma_1, \ldots, \sigma_d\}$ be distinct field embeddings of $\mathbf{k}$ into $\mathbf{R}$ with σ_1 the identity inclusion. Each embedding σ_i defines a map $\sigma_i^n \colon \mathbf{k}^n \to \mathbf{R}^n$, and so we get a $\mathbf{Q}$-linear map

$$\sigma^n \colon \mathbf{k}^n \to \mathbf{R}^{dn},$$
$$\sigma^n(w) = (\sigma_1(w), \ldots, \sigma_d(w)),$$

whose extension to $\mathbf{R}$ over $\mathbf{Q}$ is an isomorphism. This induces an isomorphism of

$\mathrm{SL}(n, \mathbf{k})$ with an algebraic subgroup $G \subset \mathrm{SL}(dn, \mathbf{R})$, which is defined over $\mathbf{Q}$. The image of the group $\mathrm{SL}(n, \mathscr{O}(\mathbf{k}))$ is then seen to equal the integral points $G_{\mathbf{Z}}$ of G. We define the standard action of $\mathrm{SL}(n, \mathscr{O}(\mathbf{k}))$ on $\mathbf{T}^{dn}$ via this embedding.

The group G defined over $\mathbf{k}$ is equal to the product of the embeddings $G^{\sigma_i} = \sigma_i(\mathrm{SL}(n, \mathbf{k}))$, and the set of real points has a similar product structure

$$G_{\mathbf{R}} \cong \prod_{i=1}^{d} (G^{\sigma_i})_{\mathbf{R}}. \tag{15}$$

The image of $\sigma^n(\Gamma) \subset G_{\mathbf{R}}$ is a lattice by Weil's theory of the restriction of scalars so that if no factor $(G^{\sigma_i})_{\mathbf{R}}$ is compact, then we can find a Cartan subgroup for $G_{\mathbf{R}}$ containing a hyperbolic element for the standard action. Then by Theorem 7.2 of Prasad and Raghunathan the image of Γ will contain a hyperbolic element. □

The usual application of Weil's restriction-of-scalars theory is to produce cocompact lattices in an arithmetic Lie group (cf. Ex. 6.1.5, [48], or p. 216, [38]). In these constructions the field extension has degree 2, with $(G^{\sigma_1})_{\mathbf{R}}$ isomorphic to $(\mathrm{SL}(n, \mathbf{k}))_{\mathbf{R}}$ and $(G^{\sigma_2})_{\mathbf{R}}$ isomorphic to a compact Lie group. These examples do *not* give Anosov standard actions.

COROLLARY 7.11. *Let $n \geq 3$ and let k be an algebraic number field of degree d over $\mathbf{Q}$ such that the group $R_{\mathbf{k}/\mathbf{Q}}(\mathrm{SL}(n, \mathbf{k}))_{\mathbf{R}}$ has no compact factor. For any subgroup $\Gamma \subset \mathrm{SL}(n, \mathscr{O}(\mathbf{k}))$ of finite index the standard action of Γ on $\mathbf{T}^{dn}$ is C^r-rigid for $r = 1, \infty, \omega$.*

Proof. The action has dense periodic orbits, by Lemma 7.4, and is Anosov by Proposition 7.10. The product $\prod_{i=1}^{d}(G^{\sigma_i})_{\mathbf{R}}$ of (15) has $\mathbf{R}$-rank $d(n-1)$, as no factor is compact; and by the Prasad–Raghunathan theorem there is a Cartan subalgebra $\mathscr{A} \subset \Gamma$. The standard action of $\mathscr{A}$ on $\mathbf{T}^{dn}$ is Cartan with linear trellising, as this is true for each of the real Cartan Lie algebras in the factorization (15) (cf. the next example). We can thus apply all of the results of Section 2 in this case; in particular, by Theorem 2.22, the standard action is C^r-rigid. □

Example 7.12 (Geometric sums and products). Let $\{\varphi_i\colon \Gamma_i \times X_i \to X_i \mid 1 \leq i \leq d\}$ be given C^r-actions. Then the *direct product action* of $\Gamma = \Gamma_1 \times \cdots \times \Gamma_d$ on $X = X_1 \times \cdots \times X_d$ is obtained by letting the subgroup Γ_i act on the factor X_i via φ_i and via the identity on X_j for $j \neq i$, and then extending to all of Γ via products.

LEMMA 7.13. *Suppose that each action φ_i for $1 \leq i \leq d$ is Anosov (respectively, trellised and Cartan). Then the direct product action $\varphi\colon \Gamma \times X \to X$ is Anosov (respectively, trellised and Cartan).*

Proof. For each $1 \leq i \leq d$ choose a φ_i-hyperbolic element $\gamma_i \in \Gamma_i$. Then $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_d)$ is hyperbolic for φ, as the product of Anosov diffeomorphisms is Anosov.

If the action φ_i is trellised, then for each foliation $\mathscr{F}_{i,j}$ of the trellis $\mathscr{T}_i$ there is a hyperbolic element $\gamma_{i,j} \in \Gamma_i$ whose action leaves $\mathscr{F}_{i,j}$ invariant. We define a foliation $\tilde{\mathscr{F}}_{i,j}$ on the product space X whose leaves are one-dimensional and obtained by taking the product of those of $\mathscr{F}_{i,j}$ with the point foliations on the other factors. Then $\tilde{\mathscr{F}}_{i,j}$ is invariant for any extension $\tilde{\gamma}_{i,j}$ of $\gamma_{i,j}$ to an Anosov diffeomorphism of X via products, as before. Note that if $\mathscr{F}_{i,j}$ is the strongest stable foliation of $\gamma_{i,j}$, then it remains so for positive powers $\gamma_{i,j}^p$. By choosing a sufficiently large power $p = p_{i,j}$, we can ensure that the extended foliation $\tilde{\mathscr{F}}_{i,j}$ is the strongest stable foliation for the hyperbolic element $\tilde{\gamma}_{i,j}$. The collection of all such foliations $\tilde{\mathscr{T}} = \{\tilde{\mathscr{F}}_{i,j}\}$ forms a trellis for the product space X, which is invariant for the set of Anosov extensions $\{\tilde{\gamma}_{i,j}\}$.

Finally the Anosov elements Δ_i in a Cartan action for φ commute, and their extensions $\tilde{\Delta}$ can be chosen to commute, yielding a collection of commuting elements $\Delta = \tilde{\Delta}_1 \cup \cdots \cup \tilde{\Delta}_d$, which preserves the trellis $\tilde{\mathscr{T}}$. By the previous remark we can chose each $\tilde{\gamma}_{i,j}$ so that $\tilde{\mathscr{F}}_{i,j}$ is its strongest stable foliation; hence the product action φ is Cartan. □

COROLLARY 7.14. *For $1 \leq i \leq d$ let $\Gamma_i \subset \mathrm{SL}(n_i, \mathbf{Z})$ be isomorphic to a higher rank lattice and so that the standard action $\varphi_i\colon \Gamma_i \times \mathbf{T}^{n_i} \to \mathbf{T}^{n_i}$ is Cartan with linear trellising. Then the product action $\varphi = \varphi_1 \times \cdots \times \varphi_d$ on $\mathbf{T}^n = \mathbf{T}^{n_1} \times \cdots \times \mathbf{T}^{n_d}$ is C^r-rigid for $r = 1, \infty, \omega$.*

Proof. The product action is Cartan by Lemma 7.13, and an Anosov action of a product of higher rank lattices satisfies Zimmer's super-rigidity theorem (Theorem 2.2, [47]). We can then apply our Theorem 2.22 to obtain the conclusion. □

Theorem 1.3 of the Introduction is deduced by applying Corollary 7.14 to Examples 7.3 and 7.6. Note that an explicit construction of Anosov arithmetic examples, as discussed in Example 7.9, would greatly extend the list in Theorem 1.3.

Suppose that each space $X_i = \mathbf{T}^{n_i}$ for integers $n_i > 2$, and $\Gamma_i \subset \mathrm{SL}(n_i, \mathbf{Z})$. The *geometric tensor product* of the standard actions $\{\varphi_i\colon \Gamma_i \times \mathbf{T}^{n_i} \to \mathbf{T}^{n_i} | 1 \leq i \leq d\}$ is obtained by taking the induced action of the lattices Γ_i on the tensor product $\mathbf{R}^{n_1} \otimes \cdots \otimes \mathbf{R}^{n_d}$ and observing that this preserves the tensor product lattice $\mathbf{Z}^{n_1} \otimes \cdots \otimes \mathbf{Z}^{n_d}$. We obtain the tensor product action φ of $\Gamma = \Gamma_1 \times \cdots \times \Gamma_d$ on $\mathbf{T}^N$, where $N = n_1 \ldots n_d$.

LEMMA 7.15. *Suppose that each action φ_i for $1 \le i \le d$ is Anosov (respectively, trellised). Then the tensor product action $\varphi\colon \Gamma \times \mathbf{T}^N \to \mathbf{T}^N$ is Anosov (respectively, trellised).*

Proof. The proof is virtually identical to that of Lemma 7.13. We need only note that the exponent spectrum of a product element $\gamma_{i_1}^{p_1} \times \cdots \times \gamma_{i_d}^{p_d}$ is the sum of the exponent spectrum of the individual factors (in contrast to the direct product case, where the spectrum is the union). Thus by varying the choices of the factors γ_i and the powers p_i, we can obtain one-dimensional, contracting eigenspaces, which span $\mathbf{R}^N$. □

Remark. Note that a tensor product action will never be a Cartan action, as it is impossible to have a basis of maximally contracting eigenspaces.

COROLLARY 7.16. *For $1 \le i \le d$ let $\Gamma_i \subset \mathrm{SL}(n_i, \mathbf{Z})$ be isomorphic to a higher rank lattice and so that the standard action $\varphi_i\colon \Gamma_i \times \mathbf{T}^{n_i} \to \mathbf{T}^{n_i}$ is linearly trellised. Then the tensor product action $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_d$ on $\mathbf{T}^N$, for $N = n_1 \cdots n_d$, is $C^{0,r}$-deformation rigid for $r = 1, \infty, \omega$.* □

Example 7.17 (Diagonal actions). The *d-fold diagonal action* of an action $\varphi_1\colon \Gamma_1 \times X_1 \to X_1$ is obtained by restricting the product action of d-copies of φ_1 to the d-fold diagonal. There is a slightly more general construction available. Let actions $\{\varphi_i\colon \Gamma \times X_i \to X_i | 1 \le i \le d\}$ be given; then we obtain an action of Γ on $X = X_1 \times \cdots \times X_d$ by setting

$$\varphi(\gamma)(x_1, \ldots, x_d) = (\varphi_1(\gamma)(x_1), \ldots, \varphi_d(\gamma)(x_d)).$$

LEMMA 7.18. *Let φ be the generalized diagonal action obtained from the Anosov actions $\{\varphi_i | 1 \le i \le d\}$. If there exists $\gamma \in \Gamma$ such that γ is φ_i-hyperbolic for all $1 \le i \le d$, then φ is an Anosov action.* □

COROLLARY 7.19. *Let $\{\varphi_i\colon \Gamma \times X_i \to X_i | 1 \le i \le d\}$ be Anosov actions with dense periodic orbits and with a common hyperbolic element γ. If Γ satisfies the cohomology condition* SVC(n) *for $n = n_1 + \cdots + n_d$, then φ is C^k-topologically deformation rigid.* □

A diagonal action with $\rho_i = \rho$ the same for all i cannot be Cartan for $d \ge 2$, as the dimensions of the eigenspaces for the hyperbolic elements are always at least d; hence the strongest stable direction is always of dimension at least d.

Diagonal actions provide a large collection of examples where topological deformation rigidity is the best result known. It seems difficult, at the present state of research, to decide whether these actions are differentiably rigid. A natural test case is to show they are C^1-deformation rigid; for example, by studying the properties of cocycles over product actions.

Example 7.20 (Arithmetic products). Let $\{\varphi_i\colon \Gamma_i \times \mathbf{T}^{n_i} \to \mathbf{T}^{n_i} | 1 \le i \le d\}$ be Anosov standard actions of arithmetic subgroups $\Gamma_i = (G_i)_{\mathbf{Z}}$, where $G_i \subset \mathrm{SL}(n_i, \mathbf{R})$ is a connected, semisimple, algebraic group defined over $\mathbf{Q}$, with real rank at least 2. There is an alternate construction of a standard action of a group Γ on a torus constructed from this data, which we call the *arithmetic product*.

The product group $G = G_1 \times \cdots \times G_d \subset \mathrm{SL}(n, \mathbf{R})$ is defined over $\mathbf{Q}$, for $n = n_1 + \cdots + n_d$. The group of real points $G_{\mathbf{R}}$ admits an arithmetic *irreducible* lattice subgroup $\Gamma \subset G_{\mathbf{R}}$. That is, for some $N \ge n$ there exists a group $\tilde{G} \subset \mathrm{SL}(N, \mathbf{R})$ containing a lattice $\tilde{\Gamma} = \tilde{G}_{\mathbf{Z}}$, and there is a natural homomorphism $\pi\colon \tilde{G} \to G$ whose restriction to $\tilde{\Gamma}$ is an isomorphism.

The arithmetic product of the actions $\{\varphi_i\}$ is the action of Γ on $\mathbf{T}^N$ via the inverse map

$$(\pi|_{\tilde{\Gamma}})^{-1}\colon \Gamma \to \mathrm{SL}(N, \mathbf{Z}).$$

This construction is similar to that of Example 7.9. To determine whether such an action is Anosov or Cartan, we first must determine whether $\tilde{G}$ contains a nontrivial compact factor. This entails a more extensive discussion of cases, which we omit.

Example 7.21 (A deformation of the standard action of $\mathrm{SL}(2, \mathbf{Z})$). We construct an example showing that the standard action of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{T}^2$ is not topologically deformation rigid, even though the actions are real analytic. Thus the Anosov hypothesis is not sufficient for the topological rigidity of a group action with more than one generator, and additional hypotheses are necessary to obtain rigidity; for example, on the cohomology of the group, as used in this paper.

THEOREM 7.22. *There exists an analytic family $\{\varphi_t | 0 \le t \le 1\}$ of volume-preserving, real analytic actions of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{T}^2$, with $\varphi_0 = \varphi$ the standard action, such that φ_t is not topologically conjugate to φ for all $0 < t \le 1$.*

Proof. Let us first note some standard facts about the algebraic structure of $\mathrm{SL}(2, \mathbf{Z})$:

LEMMA 7.23. (1) *The pair of matrices $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ generates* $\mathrm{SL}(2, \mathbf{Z})$.

(2) *A has order 4, B has order 6, and $A^2 = B^3 = -I$.*

(3) $\mathrm{SL}(2, \mathbf{Z})$ *is isomorphic to the amalgamated product* $(\mathbf{Z}/4\mathbf{Z}) \times_{\mathbf{Z}/2\mathbf{Z}} (\mathbf{Z}/6\mathbf{Z})$ *generated by* $\{A, B\}$. □

Let $\vec{Z}_1 = x(\partial/\partial y) - y(\partial/\partial x)$ be the rotational vector field about the origin. Then for any smooth function $\psi(s)$, the vector field $\vec{Z}_\psi = \psi(x^2 + y^2) \cdot \vec{Z}_1$ is divergence free.

We first form a nontrivial family of C^∞-deformations, then indicate the modifications necessary for the real analytic case. Choose a smooth function ψ such that $\psi(0) = 1$, $\psi(s) \geq 0$ for all s, and $\psi(s) = 0$ for $s \geq 10^{-4}$. Form the translate of the vector field $\vec{Z}_\psi$, centered at the point $[1/2, 0] \in \mathbf{R}^2$:

$$Z_+ = DT_{[1/2,0]}(Z_\psi). \tag{16}$$

Introduce the companion vector field $Z_- = D(A^2)(Z_+) = D(-I)(Z_+)$ and form the sum $Z = Z_+ + Z_-$. Note that $D(A^2)(Z) = Z$.

We want the vector field Z to be invariant under the translation action of the lattice $\mathbf{Z}^2$; so we form the infinite sum

$$\tilde{Z} = \sum_{[m,n] \in \mathbf{Z}} DT_{[m,n]}(Z), \tag{17}$$

which is well defined since the supports of the translates are disjoint.

Let $F(t)$: $\mathbf{R}^2 \to \mathbf{R}^2$ be the flow of the vector field $\tilde{Z}$ and observe that

$$F(t) \circ A^2 = A^2 \circ F(t), \tag{18}$$

$$T_{[m,n]} \circ F(t) = F(t) \circ T_{[m,n]}. \tag{19}$$

From equation (19) the maps $F(t)$ descend to a family of diffeomorphisms of $\mathbf{T}^2$ denoted by $\tilde{F}(t)$. Moreover from the identity (18) we have

$$\left\{\tilde{F}^{-1}(t) \circ \varphi(A) \circ \tilde{F}(t)\right\}^2 = -I, \tag{20}$$

which by Lemma 7.23 implies there is a well-defined C^∞-deformation of the standard action φ of $\mathrm{SL}(2, \mathbf{Z})$, by the declaration

$$\varphi_t(A) = \tilde{F}(t)^{-1} \circ \varphi(A) \circ \tilde{F}(t), \tag{21}$$

$$\varphi_t(B) = \varphi(B). \tag{22}$$

LEMMA 7.24. *If there exists a homeomorphism H: $\mathbf{T}^2 \to \mathbf{T}^2$ conjugating φ_t to φ_0, then $t = 0$.*

Proof. The standard action of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{T}^2$ has a unique fixed point $x_0 \in \mathbf{T}^2$ corresponding to the coset of the origin $[0, 0] \in \mathbf{R}^2$, which also remains fixed for the perturbed actions φ_t. The homeomorphism H must preserve x_0; so H admits a unique lift $\hat{H}$: $\mathbf{R}^2 \to \mathbf{R}^2$, which fixes the origin $[0, 0]$ and conjugates the given action $\hat{\varphi}_t$: $\mathrm{SL}(2, \mathbf{Z}) \times \mathbf{R}^2 \to \mathbf{R}^2$ to the linear action of $\mathrm{SL}(2, \mathbf{Z})$. We then have a pair of identities for the actions on $\mathbf{R}^2$:

$$\hat{H} \circ A = \hat{\varphi}_t(A) \circ \hat{H}, \tag{23}$$

$$\hat{H} \circ B = B \circ \hat{H}. \tag{24}$$

Observe that (23) implies

$$G \circ A \circ G^{-1} = A, \tag{25}$$

where $G = \tilde{F}(t) \circ \hat{H}$ so that G fixes the origin $[0, 0]$ and commutes with the period-4 rotation.

The action of the element B on $\mathbf{R}^2$ has period 6. A fundamental domain for the action is given by the cone

$$S_1 = \{[r\cos(\theta), r\sin(\theta)] | 0 \le r, 0 \le \theta \le \pi/3\}.$$

Label the translates of this domain by $S_j = B^{j-1}S_1$. For example, S_3 is the second quadrant of the plane. Label the restrictions of $\hat{H}$ to these sectors by $\hat{H}_j = \hat{H}|S_j$ so that (24) becomes the identity

$$B^{-1} \circ \hat{H}_{j+1} \circ B = \hat{H}_j. \tag{26}$$

We then need two more observations: the restriction of $F(t)$ to the sectors S_2 and S_3 has support either outside the ball of radius $\sqrt{2}$ or on the x-axis. And secondly the element $A^{-1}B^2$ fixes the x-axis, and the element B^2A^{-1} fixes the y-axis so that the homeomorphism $\hat{H}$ must map each sector S_j to itself, excluding the "spill-over" in a neighborhood of $B^{j-1}[0, 1/2]$ contained in the ball of radius $1/50$, and its $\mathbf{Z}^2$-translates.

The identity (23), the two observations above and then (26) imply that

$$A \circ F(t) \circ \hat{H}_1 \circ A^{-1} = \hat{H}_3, \tag{27}$$

$$C^{-1} \circ F(t) \circ \hat{H}_1 \circ C = \hat{H}_1, \tag{28}$$

where $C = A^{-1}B^2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. The action of C on $\mathbf{R}^2$ is the identity along the x-axis and preserves the lines $y = c$. So restricting to the line $y = 0$ in (28) yields

$$F(t) \circ \hat{H}_1[x, 0] = C \circ \hat{H}_1[x, 0]. \tag{29}$$

The identity (29) is impossible for $t \neq 0$, however: C preserves lines $y = c$, while the rotational flow $F(t)$ does not preserve any of the lines $y = c$ for c sufficiently small. The curve $x \mapsto H(t)[x, 0]$ lies in the x-axis for x away from $1/2$; so there must exist x' such that $H(t)[x', 0] = [x'', c]$ with c sufficiently small, and (29) cannot hold for this $[x', 0]$.

Analytic deformations are easily obtained by using the cut-off function $\psi(s) = \exp\{-(100s)^2\}$. The support of the exponential function is no longer compact, but the sum (17) will still yield an analytic vector field, for the index set grows linearly with the weight $|n| + |m|$, and the function $\exp\{-10000(n^2 + m^2)\}$ decays super-exponentially fast in this weight. The remainder of the proof is essentially the same as for a compactly supported cut-off function. □

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(Received July 6, 1990)