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# MANIFOLDS WHICH CANNOT BE LEAVES OF FOLIATIONS

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The question of when an open manifold is the leaf of a foliation of a compact manifold has been studied for the last 20 years. For open surfaces, this was first addressed by Sondow [28], and solved by Cantwell and Conlon [5] who proved that every complete two manifold can be realized as leaf of a foliation of a compact 3-manifold. In contrast, Ghys [10] and independently Inaba *et al.* [20] constructed open manifolds of dimension 3 which cannot be homotopy equivalent to a leaf of any codimension one topological foliation of a compact manifold, on account of "non-recurrence" properties of their fundamental groups. It is very difficult to obtain non-realization results for this generality, and no such results are known for codimension greater than one.

The realization problem can also be formulated to include metric information: given an open manifold $X$ and a complete Riemannian metric $g_X$ of bounded geometry on $TX$, the question is whether there is a leaf $L$ of foliation $\mathscr{F}$ on a compact Riemannian manifold $(V, g_V)$ so that $(X, g_X)$ is quasi-isometric to $(L, g_L)$ for the induced Riemannian metric $g_L = g_V \mid T\mathscr{F}$? In 1978, Phillips and Sullivan [24] introduced the average Euler characteristic of a complete open manifold $(X, g_X)$ with subexponential volume growth, and the non-vanishing of this invariant is a quasi-isometry invariant. A leaf of a $C^0$ codimension-$q$ foliation with non-zero average Euler characteristic determines a non-zero class in $H^q(V; \mathbb{R})$. Hence, if $H^q(V; \mathbb{R})$ is trivial then there is no such leaf for $\mathscr{F}$. Januszkiewicz [21] generalized the Phillips–Sullivan result to include the average Pontrjagin numbers as obstructions to realizing the quasi-isometry class of $(X, g_X)$ as a leaf.

In this paper we study three approaches to the realization problem. The first yields new classes of simply connected manifolds which cannot be realized (up to homeomorphism) as leaves of a codimension-one foliation, using a combination of techniques of bounded geometric surgery with straightforward extensions of known foliation techniques. The second approach yields new classes of simply connected manifolds of subexponential growth type which cannot be realized as leaves (up to quasi-isometry) of a codimension-one foliation.

Our third class of results is the most novel, and is based on a new invariant for a complete open manifold $(X, g_X)$ – its *entropy* $h(X, g_X)$. It seems to be a new observation in foliation theory is that $h(X, g_X)$ must vanish when $(X, g_X)$ is quasi-isometric to a leaf of either a codimension one foliation, or of a transversally $C^1$-foliation. We give constructions of complete manifolds of bounded geometry $(X, g_X)$ with exponential growth and positive entropy, providing a new class examples that cannot be quasi-isometric to leaves. This notion of entropy for complete metric spaces also has further generalizations and applications [17].

Our first result is an extension of Ghys' results [10], giving a new class of manifolds which are not homeomorphic to leaves of foliations.

THEOREM 1. *There exists an uncountable set of homeomorphism types of simply connected Riemannian manifolds of bounded geometry, each homotopy equivalent to the infinite connected sum* $\#_{-\infty}^{\infty} S^4 \times S^2$, *yet none is homeomorphic to a leaf of a codimension-one,* $C^0$*-foliation of a compact manifold.*

The idea behind the proof of Theorem 1 is to replace the role of the leafwise fundamental groups in Ghys's paper with the leafwise Pontrjagin classes. Theorem 1 follows directly from Theorem 7, Section 5.

The Cantwell–Conlon results mentioned above, combined with the obstructions produced by Phillips–Sullivan, yields Riemann surfaces of bounded geometry which are diffeomorphic to leaves, but whose quasi-isometry class cannot be realized as a leaf of codimension-$q$ foliation of a manifold with $H^q(V; \mathbb{R}) = 0$. The next result gives examples of complete manifolds of bounded geometry whose quasi-isometry classes cannot be realized as a leaf in codimension one, without restrictions on the ambient cohomology, yet the manifolds are diffeomorphic to leaves:

THEOREM 2. *There exists an uncountable set of quasi-isometry types of Riemanian manifolds of bounded geometry with linear volume growth, none of which is quasi-isometric to a leaf of a codimension-one,* $C^0$*-foliation of a compact manifold. Yet all of these manifolds are diffeomorphic to* $S^3 \times S^2 \times \mathbb{R}$, *which is trivially a leaf of a smooth codimension one foliation.*

Theorem 2 follows directly from Theorem 6, Section 4. One novelty of its conclusion is that all of the example manifolds $M(\mathbf{a})$ constructed are all diffeomorphic to leaves, yet the obstructions to being a leaf result from local Pontrjagin classes. Moreover, the manifolds $M(\mathbf{a})$ are *bg homotopy equivalent* to open manifolds which are embeddable as leaves of foliations. S. Weinberger has pointed out to us that these examples thus show that for embeddings of non-compact manifolds as leaves, the Casson–Haefliger–Sullivan–Wall theorem is false: that is, the existence of an embedding is not determined by the homotopy type or even the *bg* homotopy type of the manifold.

The non-realization results for the quasi-isometry classes of manifolds mentioned above are all based on an averaging procedure, which requires an asymptotic cycle of subexponential growth in the manifold – or more generally, it must contain an asymptotic cycle of some degree [21]. These methods generally fail for complete manifolds of exponential volume growth. This leaves open a basic problem.

QUESTION. *What restrictions are imposed on a complete open manifold of bounded geometry with exponential growth type, which is quasi-isometric to a leaf of a foliation of a compact manifold?*

This problem can be reformulated as demanding a refined understanding of what recurrence properties are forced on an open complete manifold which is a leaf of a foliation. The *entropy* $h(X, g_X)$ defined in Section 6 is a very crude measure of the lack of uniform recurrence in the coarse metric structure of $(X, g_X)$. For a leaf of a foliation, there are *a priori* estimates on this entropy – we prove that it must be zero for codimension-one foliations, and also zero for non-linear growth leaves in $C^1$-foliations. Then in Section 7 we give a construction to show:

THEOREM 3. *There exists an uncountable set of quasi-isometry types of Riemannian manifolds of bounded geometry and exponential volume growth, with positive entropy. Hence, none of these is quasi-isometric to a leaf of a codimension-one $C^0$-foliation, nor to a leaf of a $C^1$-foliation of any codimension.*

## 1. STRUCTURE THEORY OF TOPOLOGICAL FOLIATIONS

A *$C^0$-foliation* $\mathcal{F}$ of a paracompact smooth manifold $V^m$ is a continuous partition of $V$ into tamely embedded $C^2$-submanifolds (the leaves) of constant dimension $p$ and codimension $q$. We require that these leaves be locally given as the level sets (plaques) of local foliation coordinate charts which satisfy four conditions..

(1.1) There is given a uniformly locally finite covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ of $V$; that is, there exists $m(\mathcal{A}) > 0$ so that for any $\alpha \in \mathcal{A}$ the set $\{\beta \in \mathcal{A} \mid U_\alpha \cap U_\beta \neq \emptyset\}$ has cardinality at most $m(\mathcal{A})$.

(1.2) There are local coordinate charts $\phi_\alpha : U_\alpha \to (-1, 1)^m$, so that each map $\phi_\alpha$ admits an extension to a homeomorphism $\tilde{\phi}_\alpha : \tilde{U}_\alpha \to (-2, 2)^m$ where $\tilde{U}_\alpha$ contains the closure of the open set $U_\alpha$.

(1.3) For each $z \in (-2, 2)^q$, the preimage $\tilde{\phi}_\alpha^{-1}((-2, 2)^p \times \{z\}) \subset \tilde{U}_\alpha$ is the connected component containing $\tilde{\phi}_\alpha^{-1}(\{0\} \times \{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\} \times \{z\})$ with the set $\tilde{U}_\alpha$.

The extensibility condition in (1.2) is made to guarantee that the topological structure on the leaves remains tame out to the boundary of the chart $\phi_\alpha$. The collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ is called a *regular foliation atlas* for $\mathcal{F}$.

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1, 1)^p \times \{z\}) \subset U_\alpha$$

are topological discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated with this atlas. *We will assume that the covering is chosen so that all plaques have diameter at most* 1. One thinks of the plaques as "tiling stones" which cover the leaves in a regular fashion. The plaques are indexed by the *complete transversal* $\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$ associated with the given covering, where $\mathcal{T}_\alpha = (-1, 1)^q$. The charts $\phi_\alpha$ define tame embeddings

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : \mathcal{T}_\alpha \to U_\alpha \subset V.$$

We will implicitly identify the set $\mathcal{T}$ with its image in $V$ under the maps $t_\alpha$, though the union of these maps may not be not injective, but is at most finite-to-one.

Finally, the fourth condition ensures that the leaves are $C^2$-manifolds.

(1.4) For each $z \in (-2, 2)^q$, and $\beta$ so that $\phi_\alpha((-1, 1)^p \times \{z\}) \cap U_\beta \neq \emptyset$ the transition function $\tilde{\phi}_{\alpha\beta, z}$ is $C^2$ uniformly in the parameter $z$, where

$$\tilde{\phi}_{\alpha\beta, z} = \tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} : (-2, 2)^p \times \{z\} \cap \tilde{\phi}^{-1}(U_\alpha \cap U_\beta) \to (-2, 2)^p.$$

The foliation $\mathcal{F}$ is said to be $C^r$ if the foliation charts $\{\phi_\alpha \mid \alpha \in \mathcal{A}\}$ can be chosen to be $C^r$-diffeomorphisms.

The *Product Neighborhood Theorem* is a key property of foliated manifolds, which is a direct generalization of the foliated neighborhood theorem for a compact leaf with finite holonomy (cf. [13]). For $K \subset V$ and $\varepsilon > 0$, let $\mathcal{N}(K, \varepsilon)$ be the open neighborhood consisting of points which lie within $\varepsilon$ of $K$.

PROPOSITION. *Let $L$ be a leaf in a foliated space $(V, \mathscr{F})$ with holonomy covering $\tilde{L}$. Given a compact subset $K \subset \tilde{L}$ and $\varepsilon > 0$, there exists a foliated immersion $\Pi: K \times (-1, 1)^q \to V$ so that the restriction $\Pi: K \times \{0\} \to L \subset V$ coincides with the restriction to $K$ of the covering map $\pi: \tilde{L} \to L$, and $\Pi(K \times (-1, 1)^q) \subset \mathscr{N}(\pi(K), \varepsilon)$.*

*Remark.* The details of the proof of the proposition can be found in [16].

An *exhaustion sequence* for a leaf $L$ is an increasing sequence of connected compact sets

$$K_1 \subset K_2 \subset \cdots K_n \subset \cdots \subset L$$

whose union is all of $L$. The $\omega$-limit set of a leaf $L$ is the intersection $\omega(L) = \bigcap_{n=1}^{\infty} \overline{L - K_n}$ where the closures are formed with respect to the topology on $V$.

PROPOSITION. *$\omega(L)$ is a compact, saturated (i.e. if a leaf $L' \cap \omega(L) \neq \emptyset$ then $L' \subset \omega(L)$) set, independent of the choice of exhaustion sequence. Moreover, if $L - K_n$ is connected for all $n$ then $\omega(L)$ is also connected.*

A leaf $L$ is *proper* if the inclusion $L \hookrightarrow V$ induces from $V$ the metric topology on $L$. It is an easy exercise that a leaf is proper exactly when $L \cap \omega(L) = \emptyset$.

An *end* $\varepsilon$ of a non-compact manifold $L$ is determined by a choice of an open neighborhood system of $\varepsilon$, which is a collection $\{U_\alpha\}_{\alpha \in A}$ such that

- each $U_\alpha$ is an unbounded open subset of $L$,
- each finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}$ is connected and non-empty,
- the infinite intersection $\bigcap_1^\infty U_{\alpha_i} = \emptyset$.

Given an open neighborhood system $\{U_\alpha\}_{\alpha \in A}$ of $\varepsilon$, the $\varepsilon$-limit set $\lim_\varepsilon(L) = \bigcap_{\alpha \in A} \bar{U}_\alpha$. Clearly, for each end $\varepsilon$, we have $\lim_\varepsilon(L) \subset \omega(L)$. But $\omega(L)$ may include more points than just the union of the $\varepsilon$-limit sets of $L$. An end $\varepsilon$ of $L$ is *proper* if $L$ is not contained in $\lim_\varepsilon(L)$, and $\varepsilon$ is *totally proper* if $\lim_\varepsilon(L)$ is a union of proper leaves.

A leaf $L'$ is said to be the *asymptote* of a leaf $L$ if $\omega(L) = L'$. Note this implies that $\omega(L') = \emptyset$. $L'$ is compact.

A compact, non-empty, $\mathscr{F}$-saturated set $X$ is *minimal* for $\mathscr{F}$ if each leaf of $X$ is dense in $X$. Equivalently, $X$ is minimal with respect to the properties that it be closed, non-empty and $\mathscr{F}$-saturated. Zorn's Lemma implies that for each end $\varepsilon$ of $L$, there is a minimal set contained in $\lim_\varepsilon(L)$.

A key to our study of the entropy of leaves, are the notions of *expansion rate* and *geometric entropy* of a foliation. Let $D: V \times V \to [0, 1]$ be the path-length metric associated with a Riemannian metric on $TV$ of diameter 1, and $D_L: L \times L \to [0, \infty]$ on each leaf $L$.

A *leafwise path* $\gamma$ is a $C^1$-map $\gamma: [0, 1] \to V$ whose image is contained in a single leaf of $\mathscr{F}$. The length of $\gamma$ will denoted by $|\gamma|$. Suppose that a leafwise path $\gamma$ has initial point $\gamma(0) = t_\alpha(z_0)$ and final point $\gamma(1) = t_\beta(z_1)$ on transversals to $\mathscr{F}$, then $\gamma$ determines a *local holonomy map*, $h_\gamma$, which is a local homeomorphism from a neighborhood of $z_0$ in $\mathscr{T}_\alpha$ to a neighborhood of $z_1$ in $\mathscr{T}_\beta$. Note that the holonomy of a concatenation of two paths is the composition of their local holonomy maps (possibly restricted to a smaller domain).

For this work, when we speak of the holonomy of $h_\gamma$ of a path $\gamma$, it is implicitly assumed that there are points $z_0 \in \mathscr{T}_\alpha$ and $z_1 \in \mathscr{T}_\beta$ for some $\alpha, \beta$ so that the endpoints of $\gamma$ are $\gamma(0) = t_\alpha(z_0)$ and $\gamma(1) = t_\beta(z_1)$.

Let $\lambda(U_\alpha) > 0$ be the Lebesgue number of the covering $\{U_\alpha \mid \alpha \in \mathscr{A}\}$. For each $R > 0$ we define a metric on $\mathscr{T}$ by setting, for $x, y \in \mathscr{T}_\alpha$,

$$d_R(x, y) = \inf\left\{ \max_{|\gamma| \leq R} D(h_\gamma(x), h_\gamma(y)), \lambda(U_\alpha) \right\}$$

and for $x$ and $y$ on distinct transversals we set $D(x, y) = 1$. The metrics $d_R$ strongly depend upon the choice of the foliation covering.

For $0 < \varepsilon < 1$ and $R > 0$, we say that a finite subset $\{x_1, \ldots, x_\ell\} \subset \mathscr{T}$ is $(\varepsilon, R)$-*spanning* if for any $x \in \mathscr{T}$ there exists $x_i$ such that $d_R(x, x_i) < \varepsilon$. Let $H(\mathscr{F}, \varepsilon, R)$ denote the minimum cardinality of an $(\varepsilon, R)$-spanning subset of $\mathscr{T}$. The $\varepsilon$-expansion growth of $\mathscr{F}$ is the growth class of the function $R \mapsto H(\mathscr{F}, \varepsilon, R)$. This function is one of the basic measures of the "transverse dynamics" of a foliation (cf. [11, Section 3]). We will need the following

PROPOSITION (Egashira [9]). *The growth rate of $H(\mathscr{F}, \varepsilon, R)$ at most $[a^{b^R}]$ for $a, b > 1$, and is at most $[a^R]$ if the foliation is transversally $C^1$.*

Let $Z \subset V$ be an $\mathscr{F}$-saturated set. The restricted spanning function $H(Z \mid \mathscr{F}, \varepsilon, R)$ equals the minimum cardinality of an $(\varepsilon, R)$-spanning subset of $\mathscr{T} \cap Z$. Clearly, $H(Z \mid \mathscr{F}, \varepsilon, R) \leq H(\mathscr{F}, \varepsilon, R)$.

Note the two properties: $\varepsilon' < \varepsilon$ implies $H(\mathscr{F}, \varepsilon', R) \geq H(F, \varepsilon, R)$ for all $R > 0$ and $R' > R$ implies $H(\mathscr{F}, \varepsilon, R') \geq H(\mathscr{F}, \varepsilon, R)$ for all $\varepsilon > 0$. Introduce the quantity

$$h_g(\mathscr{F}, \varepsilon) = \limsup_{R \to \infty} \frac{\log H(\mathscr{F}, \varepsilon, R)}{R}.$$

The *geometric entropy* [11] of $\mathscr{F}$ is the limit $h_g(\mathscr{F}) = \lim_{\varepsilon \to 0} h(\mathscr{F}, \varepsilon)$. The limit is finite for a transversally $C^1$-foliation, but may be infinite for topological foliations.

Finally, we recall part of the structure theory for codimension-one $C^0$-foliations (cf. [15].) We assume that $\mathscr{F}$ is transversally orientable, and fix a topological foliation $\mathscr{N}$ of dimension 1 transverse to $\mathscr{F}$. Let $U$ be an open set in $V$ saturated by $\mathscr{F}$. The *completion* $\hat{U}$ of $U$ is a manifold with boundary equipped with

- a codimension 1 $C^0$-foliation $\hat{\mathscr{F}}$ tangent to the boundary,
- a continuous map $i : \hat{U} \to V$ which restricts to a homeomorphism from the interior of $\hat{U}$ onto $U$, so that
- the restriction of $\hat{\mathscr{F}}$ to the interior of $\hat{U}$ agrees with $i^*\mathscr{F}$.

THEOREM (Dippolito [7]). *Under the preceding conditions, there is a compact submanifold with boundary and corners $K$ of $\hat{U}$ so that $\partial K = \partial^{tg} \cup \partial^{tr}$ with*

(i) $\partial^{tg} \subset \partial\hat{U}$.

(ii) *$\partial^{tr}$ is saturated by the foliation $i^*\mathscr{N}$.*

(iii) *The complement of the interior of $K$ in $\hat{U}$ is the finite union of non-compact submanifolds $B_i$ with boundary and corners homeomorphic to $S_i \times [0, 1]$ by a homeomorphism $\phi_i : S_i \times [0, 1] \to B_i$ so that $\phi_i(\{*\} \times [0, 1])$ is a leaf of $i^*\mathscr{N}$.*

The foliation restricted to $B_i$ is defined by suspension of a representation of the fundamental group of $S_i$ into the group of homeomorphisms of the interval [0, 1].

TRIVIALIZATION LEMMA (Hector [14]; cf. Ghys [10, Lemma 3.2]. *Let $J$ be an arc contained in a leaf of $\mathscr{N}$. Suppose that each pair of distinct points of $J$ belong to distinct leaves of $\mathscr{F}$. Then the saturation of $J$ by $\mathscr{F}$ is homeomorphic to $L \times J$ by a homeomorphism taking $L \times \{*\}$ to a leaf of $\mathscr{F}$, and $\{*\} \times J$ to a leaf of $\mathscr{N}$.*

## 2. UNIFORMLY FINITE HOMOLOGY

Let $M$ and $M'$ denote complete Riemannian manifolds of bounded geometry with length metrics $d$ and $d'$, respectively. Recall that

*Definition.* A homeomorphism $f: M \to M'$ is a *quasi-isometry* if there exist constants $\lambda(f) > 1$ and $D(f) > 0$ for which

$$\lambda(f)^{-1} d_M(x, y) - D(f) \leqslant d_{M'}(f(x), f(y)) \leqslant \lambda(f) d_M(x, y) + D(f)$$

for all $x, y \in M$. We say that $f$ is a quasi-isometry with *dilation* at most $\lambda(f)$. For a quasi-isometry with *dilation* $\lambda(f) = 1$, the constant $D(f)$ is the deviation from an isometry, and we say $f$ has *translational distortion* at most $D(f)$.

The space of real-valued $p$-forms which are bounded with respect to the norm $\|\alpha\| = \sup|\alpha(x)| + |d\alpha(x)|$ form a Banach space denoted by $\Omega^p_\beta(M)$. This gives rise to a complex $d_i: \Omega^i_\beta(M) \to \Omega^{i+1}_\beta(M)$. The *bounded de Rham groups* are defined by

$$H^p_\beta(M; \mathbb{R}) = [\operatorname{Ker} d_p]/[\operatorname{Im} d_{p-1}].$$

(In taking the quotient by the image of $d$ one does not take the closure.) Standard de Rham theory techniques show that a $C^2$-quasi-isometry $f: M \to M'$ induces an isomorphism

$$f^*: H^*_\beta(M'; \mathbb{R}) \to H^*_\beta(M; \mathbb{R}).$$

Januszkiewicz [21] has defined characteristic classes in $H^*_\beta(M; \mathbb{R})$:

THEOREM. *Any $C^2$ bounded metric on $M$ defines a Chern–Weil homomorphism from the ring of polynomials on the dual Lie algebra of the group $O(n)$ invariant under the Ad-action into $H^*_\beta(M)$. The bounded Pontrjagin classes*

$$\{p_1(M), \ldots, p_d(M)\} \subset H^{4*}_\beta(M; \mathbb{R})$$

*are invariants of the $C^2$-quasi-isometry class of $M$.*

To analyze the bounded Pontrjagin classes geometrically we introduce a Poincaré dual homology theory to $H^*_\beta(M; \mathbb{R})$.

*Definition.* Let $X$ be a simplificial complex of bounded geometry. Define the *uniformly bounded chains* $C^{\text{uff}}_q(X; \mathbb{Z})$ to be the group of formal sums of simplices in $X$, $c = \sum a_\sigma \sigma$ so that there exists $K > 0$ depending on $c$ so that $|a_\sigma| \leqslant K$ and the number of simplices $\sigma$ lying in a ball of given size is uniformly bounded. The boundary is defined to be the linear extension of the usual singular boundary. The homology group $H^{\text{uff}}_i(X; \mathbb{Z})$ is defined to be the homology of the complex $C^{\text{uff}}_q(X; \mathbb{Z})$.

We recall the following result from [2].

THEOREM. *Let $M$ be an $n$-dimensional, oriented complete manifold of bounded geometry. Then there is a Poincaré duality isomorphism*

$$H^{\text{uff}}_i(M, \mathbb{Z}) \otimes \mathbb{R} \simeq H^{n-i}_\beta(M; \mathbb{R}).$$

This allows us to re-express Januszkiewicz' characteristic classes in terms of degeneracy sets.

*Definition.* Let $E \to M$ be a rank $k$ vector bundle of bounded geometry and $\sigma = (\sigma_1, \ldots, \sigma_k)$ be $k$ global $C^\infty$ sections of $E$. Then the degeneracy set $D_i(\sigma)$ is the set of points $x \in M$ where the $\sigma_1, \ldots, \sigma_i$ are linearly independent, i.e.,

$$D_i(\sigma) = \{x \colon \sigma_1(x) \wedge \cdots \wedge \sigma_i(x) = 0\}.$$

The collection of sections $\sigma$ is generic if for each $i$, $\sigma_{i+1}$ intersects the subspace of $E$ spanned by $\sigma_1, \ldots, \sigma_i$ transversally and if integration over $D_{i+1}(\sigma) - D_i(\sigma)$ in the sense of averaging over a regular exhaustion, is a closed current. Then $D_i(\sigma)$ is a uniformly bounded cycle, and $[D_i(E)] \in H_i^{\mathrm{uff}}(M, \mathbb{Z})$ is called the $i$th *degeneracy class* for $E$.

THEOREM. *Let $E \to M$ be a real vector bundle with bounded geometry. Then the degeneracy class $[D_{n-4l}(E)] \in H_{4\ell}^{\mathrm{uff}}(M; \mathbb{Z})$ is Poincaré dual to the $\ell$th Pontrjagin class $p_\ell(M) \in H_\beta^{4\ell}(M; \mathbb{R})$.*

## 3. MANIFOLDS OF BOUNDED GEOMETRY

We consider related notions of bounded geometry (*bg*) for simplicial complexes in this section; then formulate the PL surgery groups and recall their calculations in certain cases. Details of the *bg* theory of simplicial complexes are given in [1].

*Definition.* A simplicial complex $X$ has *bounded geometry* if there is a uniform bound on the number of simplices in the link of each vertex of $X$. A simplicial map $f: X \to Y$ of simplicial complexes of bounded geometry is said to have bounded geometry if the inverse image of each simplex $\Delta$ of $Y$ under the map $f$ contains a uniformly bounded number of simplices in $X$.

*Definition.* A subdivision of a simplicial complex of bounded geometry is said to be *uniform* if

(i) Each simplex is subdivided a uniformly bounded number of times on its $n$-skeleton, where the $n$-skeleton is the union of $n$-dimensional subsimplices of the simplex.

(ii) The distortion $\mathrm{Max}(\mathrm{length}(e), \mathrm{length}(e)^{-1})$ of each edge $e$ of the subdivided complex is uniformly bounded in the metric given by barycentric coordinates of the original complex. (This is the PL version of the dilation.)

*Definition.* A metric space $P$ is a *bg* polyhedron if:

(i) It is topologically a subset $P \subset \mathbb{R}^n$.

(ii) Each point $a \in P$ has a cone neighborhood $N = a * L$ of $P$ in the given Euclidean space, where $L$ is compact and there is a uniform upper bound for the number of simplices needed to triangulate $L$, independent of $a \in P$.

*Definition.* A map $f: P \to Q$ between *bg* polyhedra is *bg* PL if it is piecewise linear and has bounded distortion.

*Definition.* A PL manifold of bounded geometry is a *bg* polyhedron so that each point $x \in M$ has a neighborhood in $M$ which is PL homeomorphic to an open set in $\mathbb{R}^n$, with a uniform bound on the distortion of the PL homeomorphism over $M$.

*Definition.* A homotopy of bounded geometry between two maps $f_0$ and $f_1$ of bounded geometry is a map of bounded geometry $F: X \times I \to Y$ so that $F | X \times 0 = f_0$ and $F | X \times 1 = f_1$. (We write this $f_0 \sim_{bg} f_1$.) A homotopy equivalence of bounded geometry is a map $f$ of bounded geometry so that there is a map $g$ of bounded geometry with $f \circ g$ and $g \circ f$ *bg* homotopic to the identity.

*Definition.* A CW-complex of bounded geometry is defined to be a CW-complex with a uniformly bounded number of cells attached to each cell and a finite number of homeomorphism types of attaching maps. A *bg* $n$-cell is a discrete collection of $n$-cells $\Sigma \times I^n$, equipped with an attaching map $\psi: \Sigma \times I^n \to X$. Two attaching maps $\psi_1, \psi_2: \Sigma \times I^n \to X$ are of the same homeomorphism type if there is a cellular homeomorphism $h: X \to X$ so that $h\psi_1 h^{-1} = \psi_2$.

*Definition.* Let $X$ be a CW complex of bounded geometry. An expansion of bounded geometry is a *bg* CW complex $Y$ so that

(i) $(Y, X)$ is a *bg* CW pair.

(ii) $Y = X \cup_f (\Sigma \times I^r) \cup_g (\Sigma \times I^{r+1})$ for *bg* $(r+i)$-cells $\Sigma \times I^{r+i}$, $i = 0, 1$ and attaching maps $f$, $g$.

(iii) There is a characteristic map $\psi_{r+1} : \Sigma \times I^{r+1} \to Y$ for the *bg* $(r+1)$-cell so that $\psi_{r+1} | \Sigma \times I^r : \Sigma \times I^r \to Y$ is characteristic.

If $Y$ is an expansion of $X$, then $Y$ is said to collapse to $X$. A *bg* homotopy equivalence is said to be simple if it can be obtained by a series of expansions and collapses.

*Definition.* The *bg* simple structure set of a manifold of bounded geometry $X$, denoted $\mathscr{S}^{bg,s}(X)$ is the set of *bg* simple homotopy equivalences $\phi : N \to X$ modulo the equivalence relation $\phi \sim \phi' : N' \to X$, if there is a PL quasi-isometry $h : N \to N'$ so that $\phi' \circ h = \phi$.

*Definition.* Let $M$ be a compact manifold with boundary. The *PL structure set* $\mathscr{S}(M, \partial)$ of piecewise linear structures on $M$ relative to the boundary of $M$, is defined to be the equivalence classes of maps $h : X \to M$ of PL manifolds $X$ with boundary, which are homotopy equivalences and PL homeomorphisms when restricted to the boundary. Two maps $h : X \to M$ and $h' : X' \to M$ are said to be equivalent if there is a homeomorphism $\phi' : X \to X'$ so that $h = h' \circ \phi$.

The set $\mathscr{S}(M, \partial)$ is a group via "characteristic variety" addition [30].

Here is a key observation from differential topology.

THEOREM 4. *The PL structure set $\mathscr{S}(S^3 \times S^2 \times I, \partial)$ contains a summand isomorphic to $\mathbb{Z}$ and detected by $p_1 \in H^4(S^3 \times S^2 \times I, \partial; \mathbb{Z})) \cong H^4(S^3 \times S^2 \times S^1; \mathbb{Z})$.*

*Proof.* Standard Browder–Novikov surgery theory [3, 22, 23, 30, 32, 33] yields an exact sequence

$$0 \to \mathscr{S}(S^3 \times S^2 \times I, \partial) \to [\Sigma(S^3 \times S^2); G/PL] \to \mathbb{Z} \to 0.$$

Observe that

$$[\Sigma(S^3 \times S^2); G/PL] \otimes \mathbb{R} = H^4(\Sigma(S^3 \times S^2); \mathbb{R}) \oplus H^8(\Sigma(S^3 \times S^2); \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$$

and note that after tensoring with $\mathbb{R}$, the kernel of the above map to $\mathbb{R} \cong \mathbb{Z} \oplus \mathbb{R}$ is detected by $p_1$. This yields the result.

For each $a \in \mathbb{Z}$ let $(M(a), \partial)$ denote the smooth 6-manifold (with boundary) homotopy equivalent to $(S^3 \times S^2 \times I, \partial)$ with relative Pontrjagin class $p_1(M(a)) = a$. Given a bi-infinite sequence $\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ of integers, define a complete open manifold of bounded geometry

$$M(\mathbf{a}) = \bigcup_{n=-\infty}^{\infty} M(a_n)$$

to be the infinite-union manifold whose $n$th factor is $M(a_n)$, and the manifolds are "glued" along their boundaries.

A bi-infinite sequence $\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ is *uniformly bounded* if there is a bound $A*$ on the terms $a_n$ of the sequence: $|a_n| \leqslant A*$ for all $n$.

PROPOSITION. *If $\mathbf{a}$ is uniformly bounded, then the manifold $M(\mathbf{a})$ is diffeomorphic to $S^3 \times S^2 \times \mathbb{R}$.*

This follows from a standard application of the infinite process trick (see [29]). The following is a special case of the main result of [1].

THEOREM 5. *The set of strict quasi-isometry classes of the set of manifolds* $\{M(\mathbf{a}) \mid \mathbf{a}$ is uniformly bounded$\}$ *is in one to one correspondence with the set of sequences in* $\mathbb{Z}$ *of linear growth, modulo the set of bounded sequences. Furthermore, the correspondence is given by the first Pontrjagin class in* $H_\beta^*(S^3 \times S^2 \times \mathbb{R}; \mathbb{Z})$.

*Proof.* The correspondence is given as follows. Given a sequence of integers, take the sequence given by adding up the first $n$ terms. This has linear growth. If we apply the result of [2] which states that the degeneracy set of complexified tangent bundle is Poincaré dual to the Pontrjagin class, we see that gluing together representatives in the structure set with Pontrjagin classes $\{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$ corresponds to taking the cycle in $H_0^{\mathrm{uff}}(\mathbb{R}, \mathbb{Z})$ given by assigning this sequence of numbers the corresponding sequence of manifolds in $\mathcal{S}(S^3 \times S^2 \times I, \partial)$. Thus, the manifolds are distinguished by their first Pontrjagin classes in $H_\beta^4(S^3 \times S^2 \times \mathbb{R}; \mathbb{R}) \cong H_0^{\mathrm{uff}}(\mathbb{R}; \mathbb{Z}) \otimes \mathbb{R}$. The converse follows from the main result of [1].

A sequence $\mathbf{a}^+ = \{a_1, a_2, a_3, \ldots\}$ is *eventually periodic* in $H_0^{\mathrm{uff}}([0, \infty); \mathbb{Z})$ if it is uff-homologous to a periodic sequence.

Let $\{U_\alpha\}_{\alpha \in A}$ define an end $\varepsilon$ of $M$. We say that $\varepsilon$ is *periodic* if there exist $\alpha \in A$ and a homeomorphism into $\Pi: U_\alpha \to U_\alpha$, so that all $\beta, \delta \in A$ there exists $k = k(\beta, \delta) > 0$ with $\Pi^k(U_\beta \cap U_\alpha) \subset U_\delta$. We say that $\varepsilon$ is *bg*-periodic if the homeomorphism $\Pi$ can be chosen to be a *quasi-isometry with dilation* 1.

The Classification Theorem above has the following application.

COROLLARY 1. *Suppose that* $M(\mathbf{a})$ *has a bg-periodic positive end. Then the subseqence* $\mathbf{a}^+ = \{a_1, a_2, a_3, \ldots\}$ *is eventually periodic in* $H_0^{\mathrm{uff}}([0, \infty); \mathbb{Z})$.

*Proof.* Let $U_\alpha = \bigcup_{n=\alpha}^{\infty} M(a_n)$ define the positive end, for $\alpha \geqslant 0$. By considering a suitable power of the end shift map, we can suppose there exists a quasi-isometric homeomorphism into $\Pi: U_1 \to U_3$ of dilation 1.

LEMMA. *For each* $x \in M_1$ *there exists an open neighborhood* $x \in V_x \in M_1$ *so that* $V_x \cap \Pi^\ell V_x = \emptyset$ *for all* $\ell > 0$.

*Proof.* Let $k = k(x) \geqslant 0$ be the greatest integer so that $x \in \Pi^k U_1$ but $x \notin \Pi^{k+1} U_1$. There is a unique $y \in U_1$ with $\Pi^k(y) = x$. Choose an open neighborhood $y \in W_y \subset U_1 - \Pi U_1$; hence $W_y \cap \Pi^\nu W_y = \emptyset$ for all $\nu > 0$. Set $V_x = \Pi^k W_y$. If $V_y \cap \Pi^\ell V_y \neq \emptyset$, then $\Pi^k W_y \cap \Pi^{k+\ell} W_y \neq \emptyset$ and as $\Pi$ is 1–1, $W_y \cap \Pi^\ell W_y \neq \emptyset$ a contradiction.

It follows that the quotient space $\mathcal{U} = U_1/\Pi$ has the structure of a compact manifold without boundary. The first Pontrjagin class $p_1(\mathcal{U}) \in H^4(\mathcal{U}; \mathbb{R})$ lifts to a periodic class $\tilde{p}_1 \in H_\beta^4(U_1; \mathbb{R})$, which determines a periodic dual class

$$(b_1, b_2, \ldots, b_p, b_1, b_2, \ldots) \in H_0^{\mathrm{uff}}(U_1; \mathbb{Z}).$$

It remains to observe that the given PL structure on $U_1$ is *bg*-equivalent to the periodic structure induced by the local covering map $U_1 \to \mathcal{U}$; hence $(a_1, a_2, \ldots)$ is uff-homologous to $(b_1, b_2, \ldots, b_p, b_1, b_2, \ldots)$.

## 4. EXAMPLES OF NON-LEAVES IN CODIMENSION ONE

Given a bi-infinite bounded sequence $\mathbf{b} = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$ where $b_i \in \{1, 2\}$, form the bounded sequence

$$\mathbf{a} = (\ldots, b_{-3}, 0, 0, 0, 0, b_{-2}, 0, 0, b_{-1}, 0, b_0, 0, b_1, 0, 0, b_2, 0, 0, 0, 0, b_3, \ldots).$$

That is, $a_0 = b_0$, $a_{\pm 1} = 0$, $a_{\pm 2} = b_{\pm 1}$ and in general the sequence $\mathbf{a}$ is obtained from $\pm n$th entries "$b_{\pm n}$" of $\mathbf{b}$ then followed by $2^n$ zeros, followed by the $\pm(n+1)$th entries "$b_{\pm(n+1)}$" and so forth.

The complete manifold $M(\mathbf{a})$, constructed as in Section 3, has linear volume growth and is diffeomorphic to $S^3 \times S^2 \times \mathbb{R}$. In particular, $M(\mathbf{a})$ is diffeomorphic to a leaf of a codimension-one foliation of $S^3 \times S^2 \times S^1 \times S^1$. Yet none of the manifolds $M(\mathbf{a})$ is quasi-isometric to a leaf by the next result.

THEOREM 6. *Let* $\mathbf{a}$ *be a uniformly bounded sequence with non-negative entries such that neither endstrings* $(a_1, a_2, \ldots)$ *nor* $(a_{-1}, a_{-2}, \ldots)$ *are eventually periodic. Then* $M(\mathbf{a})$ *is not quasi-isometric to any leaf of a* $C^0$*-foliation of codimension one of a compact manifold.*

*Proof.* Let $M(\mathbf{a})$ be as in the theorem. We assume $M(\mathbf{a})$ is quasi-isometric to a leaf $L$ of a $C^0$-foliation $\mathscr{F}$ of codimension one of a compact manifold $V$, which we will then show is impossible.

Note that by Corollary 1 of the last section, $L$ cannot be *bg*-end-periodic.

Suppose first that $L$ is a non-proper leaf, then there exists a closed transversal $T: S^1 \hookrightarrow V$ to $\mathscr{F}$ whose intersection $L \cap T(S^1)$ is an infinite set. The restricted holonomy pseudogroup on $S^1$ carries a diffuse invariant probability measure $\mu$, defined by an averaging sequence derived from $L$. It follows that the holonomy pseudogroup on $S^1$ is actually a group, and semi-conjugate to a group of rotations of $S^1$ with dense orbits. (In fact, $L$ has precisely linear growth implies that the group of rotations is generated by a single transformation with irrational rotation number.)

LEMMA. *There exists a transformation of bounded geometry,* $S: L \to L$, *so that for any compact set* $K \subset L$ *there exists some* $m > 0$ *with* $S^m(K) \cap K = \emptyset$.

*Proof.* Let $\mathscr{N}$ be a one-dimensional foliation transverse to $\mathscr{F}$ for which $T(S^1)$ is a closed orbit (cf. [15, Theorem 1.1.2]). We can assume that $\mathscr{F}$ is transversally oriented, as $L$ lifts homeomorphically to a leaf of the covering foliation on any covering of $V$. Also, assume the foliations charts in the covering of $V$ are sufficiently small so that each plaque intersects $T(S^1)$ at most once. Choose a basepoint $* \in L$ and a positively oriented arc $i_*: [0, 1] \to \mathscr{I} \subset V$ of $\mathscr{N}$ starting at $*$ and ending on $L$. This has transverse measure $0 < a = \mu(\mathscr{I}) < \infty$. Now define $S$ as follows: For each $x \in L$, there is a positively oriented arc $i_x: [0, 1] \to \mathscr{I}_x \subset V$ contained in a leaf of $\mathscr{N}$ such that $i_x(0) = x$, $i_x(1) \in L$ and $\mu(\mathscr{I}_x) = a$. Set $S(x) = i_x(1)$. It is clear that $S$ is a quasi-isometric homeomorphism.

Choose a leafwise path $\gamma$ from $*$ to $S(*)$. Then the induced local holonomy transformation along $\gamma$ extends to a global holonomy transformation $h_\gamma: S^1 \to S^1$ which commutes with $S$: for $z = T(\theta) \in L \cap T(S^1)$ then $S(z) = T(h_\gamma(\theta))$. Now, given a compact set $K \subset L$, let $\hat{K}$ denote the union of the plaques in $L$ which intersect $K$ non-trivially. The plaque-saturation $\hat{K}$ is again compact, so there is a finite intersection $\hat{K} \cap T(S^1) = \{z_1, \ldots, z_p\}$. Let $\mathscr{P}_i$ denote the union of the plaques through $z_i$. Set $\theta_i \in S^1$ with $T(\theta_i) = z_i$. Then for $m$ sufficiently large, $S^m(\{\theta_1, \ldots, \theta_p\}) \cap \{\theta_1, \ldots, \theta_p\} = \emptyset$. Now if $T^m(K) \cap K \neq \emptyset$ then there exists some $i, j$ so that $T^m(\mathscr{P}_i) \cap \mathscr{P}_j \neq \emptyset$. Each plaque intersects $T(S^1)$ in a unique point (if at all) hence $T^m(z_i) = z_j$ which is a contradiction. This establishes the second claim of the lemma.

Remark now that the transformation $S: L \to L$ induces a *bg* periodic structure on each end of $L$, which is impossible by the previous remarks. So this rules out the possibility that $M(\mathbf{a})$ is quasi-isometric to a non-proper leaf $L$.

If $L$ is a proper leaf, we next follow the outline of the proof in Ghys [10] to arrive at a contradiction. (The use of simply connected open manifolds simplifies several of the steps in [10].)

LEMMA (Ghys [10, Lemma 4.4]). *$L$ possesses a saturated open neighborhood which is uniformly quasi-isometric to $L \times (-1, +1)$ by a homeomorphism $\phi$ taking $L \times \{t\}$ to a leaf of $\mathscr{F}$ for all $-1 < t < 1$, and for $t = 0$ is the isometric inclusion of $L$.*

*Proof.* Note that the flow along the transverse foliation $\mathscr{N}$ induces quasi-isometric homeomorphisms of the leaves.

Following [10], let $\Omega \subset M$ be the union of leaves of $\mathscr{F}$ quasi-isometric to $L$. Then $\Omega$ is open and the restriction of $\mathscr{F}$ to $\Omega$ is defined by a locally trivial fibration with base a manifold of dimension 1. Let $\Omega_1$ be a connected component of $\Omega$.

We claim that the Dippolito completion of $\Omega_1$ cannot be compact. For suppose $\hat{\Omega}_1$ is a compact manifold with boundary, and $\mathscr{F}$ restricted to the interior of $\hat{\Omega}_1$ has no holonomy, all its leaves are proper and quasi-isometric to $L$. If $F$ is a compact leaf of $\partial\hat{\Omega}_1$, then the holonomy of $F$ is without fixed points and has proper orbits. Thus, the holonomy is infinite cyclic. We deduce that the neighborhood of the end of $L$ is quasi-isometric with dilation 1 to a neighborhood of the end of the infinite cyclic holonomy cover of $F$. Hence, $L$ has a *bg* periodic end, which is impossible.

Lemma 4.6 of [10] also applies to the leaf $L$ above, hence the space of leaves of the restriction of $\mathscr{F}$ to $\hat{\Omega}_1$ can not be $\mathbb{R}$.

It remains to rule out the case where $\Omega_1$ is non-compact and the quotient space by the leaves is a circle. Then, there is a fibration $\Omega_1 \to S^1$ with fiberwise monodromy map $S: L \to L$. This map can be defined exactly as before, as there is a transverse circle $T: S^1 \to \Omega_1$ whose induced holonomy pseudo-group is trivial. Thus, there is an invariant diffuse probability measure on $T(S^1)$ which is used to define the quasi-isometric homeomorphism $S$. Lemma 5.1 of [10] shows that there is a compact subset $K \subset L$ for which $S$ induces on each component of $L \backslash K$ a *bg* homeomorphism, so that each end of $L$ is *bg* periodic which is a contradiction. Hence, $M(\mathbf{a})$ cannot be quasi-isometric to a proper leaf $L$ either, which completes the proof of Theorem 6.

## 5. NON-LEAVES WHICH ARE HOMOTOPY EQUIVALENT TO LEAVES

In this section we give a general procedure for constructing simply connected, six-dimensional complete open manifolds, none of which is homeomorphic to a leaf of any codimension-one, $C^0$-foliation of a compact manifold. Yet each of these manifolds is homotopy equivalent to a manifold which is a leaf of a smooth foliation of codimension one.

Say that a bi-infinite sequence $\mathbf{a} = \{a_n\}$ is *odd* if $a_n$ is an odd integer for all $n$.

For each integer $s$, let $N(s)$ denote the closed manifold homotopy equivalent to $S^4 \times S^2$ whose first Pontrjagin class evaluates to $s$ in $H^4(S^4 \times S^2; \mathbb{Z}) \cong \mathbb{Z}$. Given a bi-infinite sequence, $\mathbf{a}$, form the infinite connected sum

$$N(\mathbf{a}) = \#_{n=-\infty}^{\infty} N(2^{|n|} \cdot a_n)).$$

Note that $N(\mathbf{a})$ has a metric of bounded geometry, where the diameter of the summand $N(2^{|n|} \cdot a_n)$ tends to infinity. Note also that each manifold $N(\mathbf{a})$ is homotopy equivalent to the standard connected sum $\#_{n=-\infty}^{\infty} S^4 \times S^4$.

THEOREM 7. *Let $\mathbf{a}$ be an odd sequence. Then $N(\mathbf{a})$ is not homeomorphic to a leaf of any $C^0$-foliation of codimension one of a compact manifold.*

*Proof.* $N(\mathbf{a})$ cannot be homeomorphic to a non-proper leaf $L$ of $\mathcal{F}$ on $V$. Otherwise, for each point $x \in L$, there is a sequence of points $x_n \in L$ such that the leafwise distance from $x$ to $x_n$ tends to infinity with $n$, yet $x_n \to x$ in the topology on $V$. Now fix $K = N(a_0) \subset L$. As the fundamental group of $L$ is trivial, by the Product Neighborhood Theorem there is a (leafwise smooth) embedding $j: K \times [-1, +1] \hookrightarrow V$ so that $j(K \times t)$ is contained in a leaf of the foliation $\mathcal{F}$ for all $-1 \leqslant t \leqslant 1$, and $j(K \times 0)$ is the inclusion into $L$. Each point of $K$ must be a limit of points in $L$ which tend to infinity, hence $L$ must intersect $j(K \times [-1, 1]) - j(K \times 0)$.

If the positive end $N(\mathbf{a}^+) = \#_1^\infty N(a_n 2^{|n|})$ intersects $j(K \times [-1, 1])$, then we obtain a diffeomorphism into $N(a_0) \subsetneq N(\mathbf{a}^+)$. The first Pontrjagin class $p_1(N(a_0)) = a_0 \in H^4(S^4 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$ is odd. By naturality of the first Pontrjagin class, the existence of an embedding $N(a_0) \hookrightarrow N(\mathbf{a}^+)$ implies that $p_1(N(a_0))$ is the pull-back of $p_1(N(\mathbf{a}^+))$. But this latter class must be divisible by 2 by construction, which is a contradiction.

The proof that $N(\mathbf{a})$ cannot be homeomorphic to a proper leaf $L$ uses the same techniques of proof as in the last section. If the Dippolito completion of $\Omega_1$ is compact, then $L$ has a (positive) periodic end, asymptotic to a compact leaf $F$. This is impossible, as for $r \gg 0$ the class

$$p_1(\#_r^\infty N(a_n 2^{|n|})) \in H^4(\#_r^\infty N(a_n 2^{|n|}); \mathbb{Z})$$

is the lift of $p_1(F) \in H^4(F; \mathbb{Z})$. The first Pontrjagin class on summand $N(a_n 2^{|n|})$ is divisible by $2^n$ which contradicts that it is a lift of a fixed class on $F$.

Lemma 4.6 of [10] assumes only that the leaf $L$ has no holonomy along its ends, hence the space of leaves of the restriction of $\mathcal{F}$ to $\hat{\Omega}_1$ cannot be $\mathbb{R}$.

Finally, when $\Omega_1$ is non-compact and fibers over a circle, Lemma 5.1 of [10] proves that there is a monodromy map $S: L \to L$ with the property that for an appropriate choice of kernel $K_0 \subset L$, given any compact set $K \subset L$ disjoint from $K_0$ then there exists $n > 0$ so that $S^n(K) \cap (K_0 \cup K) = \emptyset$. In particular, for $r \gg 0$ we can take $K_0 = \#_{1-r}^{r-1} N(a_n 2^{|n|})$ and $K = N(a_r 2^r)$. Then

$$S^n: N(a_r 2^r) \to \#_{r+1}^\infty N(a_n 2^{|n|}) \cup \#_{-\infty}^{-r-1} N(a_n 2^{|n|})$$

and we again get a contradiction by considering the divisibility of $p_1(N(a_r 2^r))$.

This establishes that $N(\mathbf{a})$ cannot be realized as a proper leaf either, which completes the proof of Theorem 7.

Finally, let us show that $\#_{n=-\infty}^\infty S^4 \times S^2$ is a leaf of a foliation. Observe this manifold is diffeomorphic to the universal covering $\tilde{W}$ of the manifold $W$ obtained from $S^4 \times S^2$ by attaching a 1-handle. Choose an irrational number $0 < \alpha < 1$ and let $R_\alpha: S^1 \to S^1$ be the rotation map by angle $\alpha\pi$. Then the diagonal action of $\mathbb{Z}$ on $\tilde{W} \times S^1$, acting via the deck translations on the first factor and via $R_\alpha$ on the second, yields a quotient manifold $V_\alpha = \mathbb{Z} \backslash (\tilde{W} \times S^1)$, which has a codimension-one smooth foliation $\mathcal{F}_\alpha$ whose leaves are the cosets of $\{\tilde{W} \times \{\theta\} \mid \theta \in S^1\}$. Thus, each leaf of $\mathcal{F}_\alpha$ is *bg* diffeomorphic to $\#_{n=-\infty}^\infty S^4 \times S^2$.

## 6. ENTROPY OF OPEN MANIFOLDS

The last two sections have used the recurrence properties of the ends of a leaf in codimension one to obtain non-embedding results. In higher codimensions, recurrence is much more difficult to categorize and quantize. Recall that every non-singular flow on a $(q + 1)$-manifold yields a codimension $q$-foliation, so the recurrence properties for a general codimension $q$ foliation are at least as complicated as that of flows, and this is just for the class of leaves with linear growth.

In this section, we introduce a new invariant for an open complete manifold $(X, g)$, its *entropy*. When $(X, g)$ is quasi-isometric to a leaf of a foliation of a compact manifold, then there is a relation between the entropy of $(X, g)$ and the geometry of the foliation. We use this to obtain non-embedding results. For example, there is a fundamental difference between the dynamics of a $C^0$ and a $C^1$-foliation, which we use to produce non-embedding results for leaves of exponential volume growth in $C^1$-foliations.

*Definition.* For $\varepsilon, R > 0$, and $(\varepsilon, R)$ *quasi-tiling* of a complete Riemannian manifold $(X, g)$ is a collection $\{K_1, \ldots, K_d\}$ of a compact metric spaces with diameters at most $R$, and a countable set of *homeomorphisms* into $\{f_i: K_{\alpha_i} \to M \mid i \in \mathscr{I}\}$ so that:

- Each $f_i$ is a quasi-isometry onto its image with $\lambda(f_i) \leqslant (1 + \varepsilon)$ and $D(f_i) \leqslant \varepsilon$.
- For any set $K$ of diameter at most $R/2$, there exist $i \in \mathscr{I}$ so that $K \subset f_i(K_{\alpha_i})$.

The integer $d$ is called the cardinality of the quasi-tiling. Note that the images $f_i(K_{\alpha_i})$ have diameter at most $\varepsilon(R + 1)$.

*Definition.* For $\varepsilon > 0$, the $\varepsilon$-*growth complexity function* of $(X, g)$ is

$$H(X, g, \varepsilon, R) = \min\{d \mid \text{there exists an } (\varepsilon, R) \text{ quasi-tiling of } (X, g) \text{ of cardinality } d\}$$

If no $(\varepsilon, R)$ quasi-tiling exists, then set $H(X, g, \varepsilon, R) = \infty$.

*Remark.* A few observations help to clarify what the function $H(X, g, \varepsilon, R)$ measures:

- When $(X, g)$ is a covering of a compact Riemannian manifold $B$, the isometric action of the deck translation group $\Gamma$ and $X$ provides a "geometric periodicity" for $X$ based on the translates of a fundamental domain. This implies that $H(X, g, \varepsilon, R) = 1$ for all $\varepsilon > 0$ and $R > 3\,(\operatorname{diam}(B) + \varepsilon)$.
- For a manifold with a periodic end, for $R$ sufficiently large and fixed $\varepsilon > 0$ the end can be covered by just one tile, so contributes marginally to the complexity.
- For the general complete, non-compact manifold of bounded geometry, the complexity function is a measure of the capacity of the image of $M$ in an appropriate space of metric spaces (cf. [17].)

The following is obvious from the definition.

PROPOSITION. *Given a quasi-isometry $f: M \to M'$, for $\varepsilon \gg 0$ there exists $\varepsilon' > 0$ so that for all $R > \varepsilon$*

$$H(X', g', \varepsilon', R - \varepsilon) \leqslant H(X, g, \varepsilon, R) \leqslant H(X', g', \varepsilon', R + \varepsilon).$$

This implies that a suitable equivalence class of the growth complexity function is a quasi-isometry invariant of $M$. We will extract two numerical invariants of the complexity – the *geometric* and *topological* entropies.

For a complete open manifold $(X, g)$, let $B(x, R) \subset X$ denote the ball of radius $R$ centered at $x$, and define $\mathscr{V}(X, g, R) = \sup\{\operatorname{vol}(B(x, R)) \mid x \in X\}$.

*Definition.* The *entropy* of $(X, g)$ is

$$h(X, g) = \lim_{\varepsilon \to \infty} \limsup_{R \to \infty} \frac{\ln\{\ln\{H(X, g, \varepsilon, R)\}\}}{\ln\{\mathscr{V}(X, g, R)\}}$$

and the *geometric entropy* of $(X, g)$ to defined to be

$$h_g(X, g) = \lim_{\varepsilon \to \infty} \limsup_{R \to \infty} \frac{\ln\{H(X, g, \varepsilon, R)\}}{R}.$$

When $(L, g_L)$ is a leaf of a foliation of a compact manifold $M$ endowed with the metric $g_L$ restricted from $g$ on $TM$, then we call $h(L, g_L)$ the *leaf entropy* of $(L, g_L)$ and $h_g(L, g_L)$ the *geometric leaf entropy*.

We have an immediate consequence of the above proposition.

COROLLARY. *Let $(X, g)$ be a complete Riemannian manifold of bounded geometry. Then the properties*

- $h(X, g) = 0$,
- $0 < h(X, g) < \infty$,
- $h(X, g) = \infty$,

*depend only on the quasi-isometry class of $(X, g)$. A corresponding conclusion also is true for the geometric entropy $h_g(X, g)$.*

We next develop relations between the growth complexity function for a leaf and the geometric entropy of the foliation. The most precise statement relates $H(X, g, \varepsilon, R)$ with the expansion growth function $H(\mathscr{F}, \varepsilon, R)$ of a topological foliation discussed in Section 1, which measures the rate of transverse mixing of the holonomy pseudogroup.

Here is the key technical observation. Let $\mathscr{F}$ be a $C^0$ codimension-$q$ foliation of a compact manifold $V$. Fix the path-length metric on $V$ associated to a Riemannian metric on $TV$. Choose local foliation coordinate charts $\phi_\alpha : U_\alpha \to (-1, 1)^m$ as in Section 1, for which we can then define the function $H(\mathscr{F}, \varepsilon, R)$. Recall that $H(L \mid \mathscr{F}, \varepsilon, R)$ denotes the maximal cardinality of an $(\varepsilon, R)$-spanning subset of the intersection of the leaf $\mathscr{T} \cap L$, so that $H(L \mid \mathscr{F}, \varepsilon, R) \leqslant H(\mathscr{F}, \varepsilon, R)$.

PROPOSITION. *Let $L \subset V$ be a simply connected leaf of a $C^0$-foliation. For each $R > 0$ there exists an open covering $\{\mathscr{V}_\beta \mid \beta \in \mathscr{B}\}$ of $V$ so that*

(1) *the cardinality $|\mathscr{B}| \leqslant H(L \mid \mathscr{F}, \frac{1}{2}, R)$;*

(2) *each $\mathscr{V}_\beta$ is a foliated product;*

(3) *for each leaf $L' \subset \bar{L}$ the restriction of the covering $\{\mathscr{V}_\beta\}$ to $L'$ has Lebesgue number at least $R - 3$.*

*In particular, there is a uniform estimate $|\mathscr{B}| \leqslant H(\mathscr{F}, \frac{1}{2}, R)$ independent of $L$.*

*Proof.* Choose a $(\frac{1}{2}, R)$-spanning subset $\{x_1, \ldots, x_{d(R)}\} \subset \mathscr{T} \cap L$ of cardinality $d(R) = H(L \mid \mathscr{F}, \frac{1}{2}, R)$. Let $\alpha_i$ be the index for which $x_i \in \mathscr{T}_{\alpha_i}$. Let $K_i \subset L$ denote the union of the plaques in $L$ which can be reached from $x_i$ by a leafwise path of length at most $R - 1$. Then each point in the intersection $K_i \cap \mathscr{T}$ can be joined to $x_i$ by a leafwise path of length at most $R$.

Let $B_R(x_i, \frac{3}{4}) \subset \mathscr{T}_{\alpha_i}$ be the ball centered at $x_i$ of radius $\frac{3}{4}$ in the metric $d_R$ restricted to the transversal. Define $\mathscr{V}_i$ to be the union of all plaques of $\mathscr{F}$ which can be joined to $t_{\alpha_i}(B_R(x_i, \frac{3}{4})$ by a leafwise path of length at most $R - 1$.

We show that the collection $\{\mathscr{V}_\beta \mid 1 \leqslant \beta \leqslant d(R)\}$ is a covering of $\bar{L}$. Let $x \in \bar{L}$, then $x \in U_\alpha$ for some $\alpha$ and so lies on a plaque $\mathscr{P}_\alpha(z_x)$ for some $z_x \in \mathscr{T}_\alpha$. The metric $d_R$ is quasi-isometric to the Riemannian metric on $V$, so there exists $z_x^* \in L \cap \mathscr{T}_\alpha$ so that $d_R(z_x, z_x^*) < \frac{1}{4}$. The $\frac{1}{2}$-spanning property then implies there exists $x_i \in \mathscr{T}_\alpha$ with $d_R(z_x^*, x_i) < \frac{1}{2}$, hence $x \in \mathscr{P}_\alpha(z) \subset \mathscr{V}_i$.

The proof of the product neighborhood theorem (cf. [16]) yields exactly that we can choose homeomorphisms onto, $\Pi_i : K_i \times (-1, 1)^q \to \mathscr{V}_i$ which satisfy

- the leafwise restriction $\Pi_i : K_i \times \{0\} \to L \subset V$ is the inclusion; and
- the transverse restriction $\Pi_i : \{x_i\} \times (-1, 1)^q \to t_{\alpha_i}(B_R(x_i, \frac{1}{2}))$ is a homeomorphism onto.

Finally, let $Z \subset L' \subset \bar{L}$ be a connected compact subset of diameter at most $R - 3$ in a leaf $L'$. Let $\hat{Z}$ be the union of the plaques with non-empty intersection with $Z$; then $\hat{Z}$ has diameter at most $R - 1$. Choose a point $z \in \hat{Z} \cap \mathscr{T}$ and a point $x_i \in \mathscr{T}_\alpha$ so that $d_R(z, x_i) < \frac{1}{2}$. Then clearly $\hat{Z} \subset \mathscr{V}_i$. This completes the proof of the proposition.

THEOREM 8. *Let $(L, g)$ be a simply-connected leaf of a $C^0$-foliation $\mathscr{F}$ of a compact manifold $V$. Then $H(L, g, 1, R - 3) \leqslant H(L \mid \mathscr{F}, \frac{1}{2}, R)$ for all $R > 3$.*

*Proof.* Let $\{\mathscr{V}_\beta \mid 1 \leqslant \beta \leqslant d(R)\}$ be a covering associated with a $(\frac{1}{2}, R)$-spanning set as above, with local homeomorphisms onto $\Pi_i : K_i \times (-1, 1)^q \to \mathscr{V}_i$. For each $z \in L \cap \mathscr{T}_{\alpha_i}$ define a homeomorphism $f_{i,z} = \Pi_i(\cdot, z) : K_i \to L$. As all plaques have diameter at most 1, each $f_{i,z}$ is a quasi-isometry onto its image with distortion $D(f_{i,z}) \leqslant 1$. So by (3) of the technical proposition above, the collection $\{K_1, \ldots, K_{d(R)}\}$ with maps $\{f_{i,z}\}$ forms a $(1, R - 3)$-quasi-tiling of $L$ of cardinality at most $H(L \mid \mathscr{F}, \frac{1}{2}, R)$.

COROLLARY. *Let $(X, g)$ be quasi-isometric (with dilation $\lambda = 1$) to a leaf of a $C^0$-foliation $\mathscr{F}$. Then the geometric entropy of $(X, g)$ is dominated by the geometric entropy of $F$*

$$h_g(X, g) \leqslant h_g(\mathscr{F}).$$

*In particular, if either $\mathscr{F}$ has codimension-one or is a $C^1$-foliation, then $(X, g)$ quasi-isometric to a leaf of $\mathscr{F}$ implies that $h_g(X, g) < \infty$.*

This corollary gives an effective restriction on when a complete open manifold $(X, g)$ is quasi-isometric to a leaf of a foliation $\mathscr{F}$. First, we leave it as a non-obvious challenge to the reader to construct an example of an open manifold $(X, g)$ with $h_g(X, g) \neq 0$ yet $h_g(L \mid \mathscr{F})$ must vanish for any leaf $L$ quasi-isometric to $(X, g)$. (This will be discussed in detail in a subsequent paper [17]). The second example is more complicated – we construct in the next section an open complete manifold $(X, g)$ with $h(X, g) = \infty$, hence the geometric entropy $h_g(X, g) \geqslant h(X, g)$ must also be infinity so $(X, g)$ cannot be a leaf of either a codimension one foliation, nor of a $C^1$-foliation of arbitrary codimension.

## 7. NON-LEAVES OF EXPONENTIAL GROWTH IN HIGHER CODIMENSIONS

In this section, we exhibit complete open manifolds $(X, g)$ of bounded geometry and exponential volume growth whose Pontrjagin classes in $H^*_\beta(X)$ are sufficiently "random" so that $H(X, g, \varepsilon, R)$ has $\varepsilon$-growth type $[a^{b^R}]$ for some constants $a, b > 1$ and for all $\varepsilon > 0$.

The general construction of the manifolds $(X, g)$ will connect-sum an infinite number of copies of $S^4 \times S^2$ onto the hyperbolic $n$-space $\mathbb{H}^6$, chosen so that we force every quasi-tiling to have maximum growth rate. The role of hyperbolic space can be replaced by the universal cover $\tilde{B}$ of any compact 6-manifold $B$ whose fundamental group $\Gamma$ has exponential growth, but we leave the details of the generalization to this reader.

Let $B(x, R)$ denote the ball of radius $R$ centered at $x \in \mathbb{H}^6$. Our construction is based on the following property of manifolds of uniformly exponential growth.

PROPOSITION. *There exists a constant $c > 1$ so that each $x \in \mathbb{H}^6$ and $R > r > 0$, the ball $B(x, R)$ contains at least $\lfloor c^{R-r} \rfloor$ pairwise disjoint balls of radius $r$.*

Given $x \in \mathbb{H}^6$ and $r > 0$, choose $d = \lfloor c^r \rfloor$ points $\{x_1, \ldots, x_d\} \subset B(x, r)$ such that the balls $\{B(x_i, 1) \mid 1 \leqslant i \leqslant d\}$ and contained in $B(x, r)$ and are pairwise disjoint.

Next, fix model manifolds $N_\ell$ for $0 \leqslant \ell \leqslant 2$, each homotopy equivalent to $S^4 \times S^2$, with $p_1(N_\ell) = \ell \in H^4(S^4 \times S^2; \mathbb{Z}) \cong \mathbb{Z}$. Fix a Riemannian metric on $N_\ell$ with injectivity radius at least $\frac{1}{2}$, and choose a disk of radius $\frac{1}{2}$ in $N_\ell$ which will be the center for a connected sum operation.

For each integer $1 \leqslant k < d$ construct a manifold $W^+(x, r, k)$ with boundary the sphere $S(x, r)$ of radius $r$: for $i \leqslant k$, connect sum $N_2$ to the ball $B(x_i, \frac{1}{2})$; and for $k < i < d$, connect sum $N_0$ to the ball $B(x_i, \frac{1}{2})$. Note that $W^+(x, r, d)$ has a standard collar neighborhood of radius $\frac{1}{2}$ about its boundary.

Modify this construction to define $W^-(x, r, d)$, where we now attach $N_1$ to the ball $B(x_d, \frac{1}{2})$ in $W^+(x, r, d)$.

We repeat this procedure a second time, where for $y \in \mathbb{R}^6$ and $R > s$ we choose points $\{y_1, \ldots, y_D\} \subset B(y, R)$ where $D = \lfloor c^{R-s} \rfloor$ so that the balls $B(y_i, s)$ are contained in $B(y, R)$ and are pairwise disjoint. Assume that $s \geqslant r$ and set $R = r + s$ so that $D \geqslant d = \lfloor c^{r-1} \rfloor$ and choose a sequence $\boldsymbol{k} = \{k_1, \ldots, k_d\}$ with each $k_i \in \{\pm\}$. For each $1 \leqslant i \leqslant d$, surger in a copy of $W^{k_i}(y_i, r, i)$ in place of the ball $B(y_i, r)$. Label the resulting manifold $N(y, r, s, \boldsymbol{k})$. Again, note that the boundary of $N(y, r, s, \boldsymbol{k})$ is a sphere of radius $R$ about $y$ and admits a product neighborhood (see Fig. 1).

The purpose of this complicated construction of the modified disks $N(y, r, s, \boldsymbol{k})$ of radius $R$ in $\mathbb{H}^6$ is to create a set of standard "models" which have distinct quasi-isometry types. There are $2^d$ choices of the sequences $\boldsymbol{k} = \{k_1, \ldots, k_d\}$, hence an equivalent number of manifolds $N(y, r, s, \boldsymbol{k})$. Let $\hat{N}(y, r, s, \boldsymbol{k})$ be the result of attaching $N(y, r, s, \mathbf{k})$ to $\mathbb{H}^6$ in place of the ball $B(y, r + s)$.

PROPOSITION. *Let $h: N(y, r, s, \mathbf{k}) \to \hat{N}(z, r, s, \mathbf{l})$ be a quasi-isometric homeomorphism with $\lambda(h) \leqslant \varepsilon$ and $D(h) \leqslant \varepsilon$. If $s > 2\varepsilon(2r + 1)$, then $\mathbf{k} = \mathbf{l}$.*

*Proof.* Let us show that $k_i = \ell_i$. Let $x_1 \in W^{k_i}(y_i, r, i)$ be the first point in the construction of this set. Then the image of the set $W^{k_i}(y_i, r, i)$ under the map $h$ must be contained in the

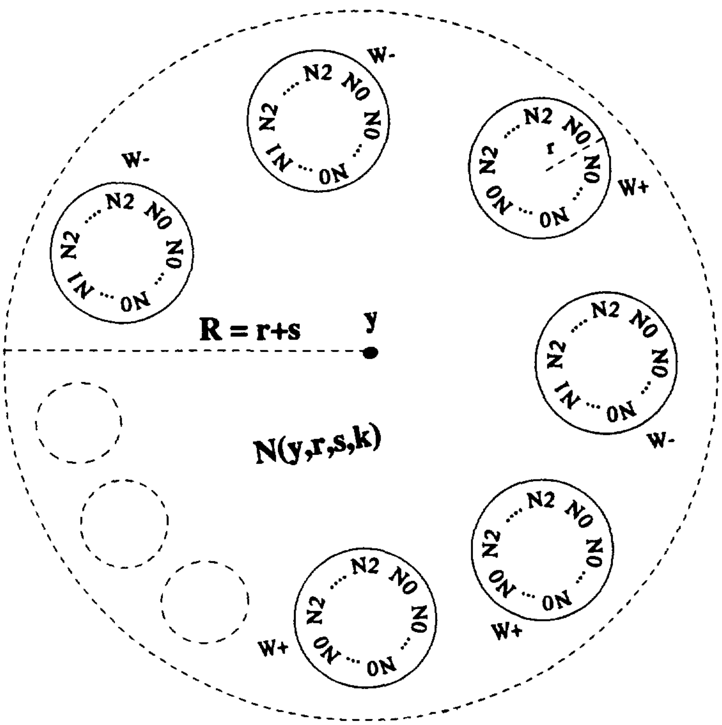


Fig. 1.

ball $B(h(x_1), \varepsilon(2r+1))$. The point $h(x_1)$ must lie in one of the sets $W^{\ell_a}(z_a, r, a)$ used to construct $\hat{N}(z, r, s, \mathbf{l})$. By the choice of s, the intersection $B(h(x_1), \varepsilon(2r+1)) \cap W^{\ell_b}(z_b, r, b)$ is empty unless $a = b$. It follows that $W^{k_i}(y_i, r, i)$ must be mapped quasi-isometrically onto $W^{\ell_a}(z_a, r, a)$.

We can now count the total number of summands of $S^4 \times S^2$ in $W^{\ell_a}(z_a, r, a)$ with positive even Pontrjagin class to obtain that $i = a$. Finally, if $k_i =$ " $-$ " then there must also be a summand of $S^4$ in $W^{\ell_i}(z_i, r, i)$ with positive odd Pontrjagin class, hence $\ell_i =$ " $-$ ". Otherwise, $\ell_i =$ " $+$ ". This proves the proposition.

Choose a geodesic curve $g: (-\infty, \infty) \to \mathbb{H}^6$. We observe that $g$ is a "straight" curve in the sense of Gromov; that is, the distance $d_{\mathbb{H}^6}(g(r), g(s)) = |r - s|$. For each integer $i > 0$, set $w_i = g(i!)$.

We are now in a position to define inductively the manifold $M$ which is not a leaf. Set $M(0) = \mathbb{H}^6$. Fix $n > 0$ and assume that $M(n-1)$ has been defined. There are $2^d$ choices of the manifolds $N(y, n, \mu n, \mathbf{k})$, where $d = \lfloor c^n \rfloor$ and $\mu$ is a positive integer. For each $1 \leqslant \mu \leqslant n^2$, attach these $2^d$ choices onto a subset of the points $\{w_i \mid i > n\}$ which have not been modified in a previous step. This produces $M(n)$. (That is, we are essentially implementing a diagonalization procedure in order to list all of the choices of these manifolds, spaced out along the increasingly distant points $\{w_i\}$.) Let $M$ be the direct limit manifold obtained by this inductive procedure.

Proposition. *There exists $b > 0$ so that for all $\varepsilon > 0$, $H(X, g, \varepsilon, R) \geqslant 2^{b^R}$ for $R \gg 0$.*

*Proof.* Fix $\varepsilon > 1$ and an integer $R = n > 10\varepsilon^2$. Let $\{K_1, \ldots, K_v\}$ be an $(\varepsilon, R)$ quasi-tiling of $M$ with countable set of homeomorphisms into $\{f_i : K_{\alpha_i} \to M\}$ so that:

- Each $f_i$ is a quasi-isometry onto its image with $\lambda(f_i)$, $D(f_i) \leqslant \varepsilon$.
- $\{f_i(K_{\alpha_i})\}$ is an open covering of $L$ with Lebesgue number at least $R$.

Set $\xi = 4(n+1)\varepsilon^2$. Distinct submanifolds $N(y, n, \xi, \mathbf{k})$ and $N(z, n, \xi, \mathbf{l})$ of $M$, each of diameter $\xi + n$, are separated by a distance at least $(n-1)! - 2(\xi + n) > \varepsilon(n+1)$. The diameter of each set $f_i(K\alpha_i)$ is at most $\varepsilon(n+1)$, so the image of the quasi-isometry $f_i$ which contains a set $N(y, n, \xi, \mathbf{k})$ will intersect no other set of this type.

Assume there are two such maps defined on a common $K_{\alpha_i}$, with $N(y, n, \xi, \mathbf{k}) \subset f_i(K_{\alpha_i})$ and $N(z, n, \xi, \mathbf{l}) \subset f_j(K_{\alpha_i})$. Then $f_j \circ f_i^{-1}$ restricts to a quasi-isometry from $N(y, n, \xi, \mathbf{k})$ to $N(z, n, \xi, \mathbf{l})$ with $\lambda(f_j \circ f_i^{-1}) \leqslant 2\varepsilon$ and $D(f_j \circ f_i^{-1}) \leqslant \varepsilon^2$. Apply the above proposition to conclude that $\mathbf{k} = \mathbf{l}$. In particular, $v \geqslant 2^d$ where $d = \lfloor c^{n-1} \rfloor$. Take $1 < b < c$ and the proposition follows.

The claim of Example 3 of the introduction now follows by the quasi-isometry invariance property of the growth complexity function.

## 8. SOME OPEN QUESTIONS

We conclude with a few questions about the embedding problem.

Problem 1. *Find a complete open manifold which cannot be embedded as a leaf of a $C^0$-foliation in any codimension.*

*Remark.* The theorems which imply the existence of periodic ends for leaves are unique to codimension one, so do not yield obstructions in higher codimensions. One approach might be to use to the quasi-tiling function for a complete open manifold to introduce finer numerical invariants which are obstructions to realization as a leaf in a $C^0$-foliation. For example, there might be such obstructions associated with the minimal sets in the ends of a leaf, which would play the role in higher codimensions of periodic end structure in codimension one.

PROBLEM 2. *Find examples of complete open manifolds in dimensions* 3, 4 *and* 5 *which are homotopy equivalent to leaves, but cannot be realized as leaves of a codimension-one, $C^0$-foliation.*

*Remark.* Our constructions use Pontrjagin classes and simply connected surgery, so are limited to manifolds of dimension at least 6. It should be a straightforward technical task to extend our results to leaves of dimension 5. Similarly, dimension 4 may also follow from the same approach used here, but using 4-manifold surgery techniques. The case of 3-manifolds seems to require a completely different approach, possibly using local torsion invariants in place of the Pontrjagin classes.

In another direction, Zeghib [34] observed that our construction in Section 7 can be modified to yield manifolds of dimension 2 which are diffeomorphic to the plane, but have positive entropy.

PROBLEM 3. *Is there a general obstruction theory to embedding a complete open manifold of bounded geometry as a leaf of a foliation? Is this part of some broader surgery classification scheme for leaves?*

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