Dynamics and matchbox manifolds

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BYU Dynamical Systems Seminar - November 15, 2011
Report on works with Alex Clark and Olga Lukina

Two simple questions

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Solutions in terms of asymptotic behavior of gap widths.

Herman, McDuff (1981), Norton (1999), Iglesias & Portela (2010) and many others.

Given a sequence of smooth *bonding maps* p_ℓ of degree > 1,

$$\xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \mathbb{S}^1 \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

the *Vietoris solenoid* is defined as the inverse limit

$$\mathcal{S} = \lim_{\leftarrow} \; \{p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \;\subset\; \prod_{\ell=0}^{\infty} \mathbb{S}^1$$

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Question: [Folklore] Given such a solenoid, when does there exists a C^r-flow, for $r \geq 1$, with $\mathcal{S}$ as an invariant set?

$r = 1$ case is easy; $r = 2$ is already problematic.

Smale (1967), Gambaudo, Tressier, et al in 1990's.

Also: what does "with $\mathcal{S}$" mean? The space $\mathcal{S}$ has a huge homeomorphism group, and which one is what we see embedded?

The mash-up

Question 1: Let $K \subset \mathbb{S}^1$ be a Cantor set, and Γ a finitely generated group. When does there exists a $C^{1+\alpha}$ action of $\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ for which K is invariant?

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Question 1': Let $\mathcal{F}$ be a codimension-one $C^{1+\alpha}$-foliation of a compact manifold M, and $Z \subset M$ an exceptional minimal set for $\mathcal{F}$. What restrictions are imposed on the "transverse metric geometry"? – e.g. the Hausdorff dimension of transversals for Z.

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These have partial solutions – some facts are known, but not complete solutions.

Not so simple in higher codimension

Question 2: Let Γ be a finitely generated group, M a closed manifold of dimension $q \geq 2$, and $\Gamma \times M \to M$ a $C^{1+\alpha}$ action. If $K \subset M$ is an invariant minimal (or transitive) set, what can be said about the geometry of K?

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Question 2': Let $\mathcal{F}$ be a codimension-q C^r-foliation of a compact manifold M, for $q \geq 2$, and $Z \subset M$ a minimal set with no interior. What restrictions are imposed on its "transverse metric geometry"? – e.g. the Hausdorff dimension of transversals for Z, or the "writhing" of Z as an embedded space?

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Very little is known about such questions. Thus, we can ask even more general questions, such as what homeomorphism types can occur? What sort of dynamics are allowed? and so forth.

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$\Rightarrow \mathfrak{M}$ be a continuum. $\Leftrightarrow \mathfrak{M}$ is compact, connected, metrizable.

Each $x \in \mathfrak{M}$ has an open neighborhood homeomorphic to $(-1,1)^n \times \mathfrak{T}_x$, where $\mathfrak{T}_x$ is a totally disconnected clopen subset of some Polish space $\mathfrak{X}$. $\Longrightarrow$ arc-components are locally Euclidean.

Classical case in this generality is a continuum, all of whose arc-components are interval-like.

Matchbox manifolds

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* $\Longleftrightarrow$ $\mathfrak{M}$ admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$ where the $\mathfrak{T}_i$ are clopen subsets of a totally disconnected metric space $\mathfrak{X}$.

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All *exceptional* minimal sets for foliations of compact manifolds are matchbox manifolds.

Some examples

The MM concept is much more general than minimal sets for foliations: they also appear in study of tiling spaces, subshifts of finite type, graph constructions, generalized solenoids, pseudogroup actions on totally disconnected spaces, and so forth.

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Minimal $\mathbb{Z}^n$-actions on Cantor set K or symbolic space:

• Adding machines (minimal equicontinuous systems)

•Toeplitz subshifts over $\mathbb{Z}^n$

• Minimal subshifts over $\mathbb{Z}^n$

• Sturmian subshifts

All of these examples are realized as Cantor bundles over base $\mathbb{T}^n$.

More examples

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All matchbox manifolds have dynamics defined by a *pseudogroup* action on a Cantor set.

Covering of $\mathfrak{M}$ by foliation charts $\Longrightarrow$ transversal $\mathcal{T} \subset \mathfrak{M}$ for $\mathcal{F}$

Pseudogroups

Holonomy of $\mathcal{F}$ on $\mathcal{T}$ $\Longrightarrow$ compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$:

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$ such that $\langle \Gamma \rangle = \mathcal{G}_{\mathcal{F}} | \mathcal{T}_0$;
- $g_i : D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$, $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Dynamical properties of $\mathcal{F}$ formulated in terms of $\mathcal{G}_{\mathcal{F}}$; e.g.,

$\mathcal{F}$ has no leafwise holonomy if for $g \in \mathcal{G}_{\mathcal{F}}$, $x \in Dom(g)$, $g(x) = x$ implies $g|V = Id$ for some open neighborhood $x \in V \subset \mathcal{T}$.

Topological dynamics

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(h_{\mathcal{I}}) \text{ with } d_{\mathcal{T}}(x, x') < \delta \quad \Longrightarrow \quad d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(c')) < \epsilon$$

Theorem: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold.

Then $\mathfrak{M}$ is minimal.

This is folklore for group actions, apparently.

C & H give a proof for pseudogroups.

Topological dynamics of pseudogroups

Can also define and study pseudogroup dynamics which are distal, expansive, proximal, etc. See

• **Lectures on Foliation Dynamics: Barcelona 2010**
S. H., [2011 arXiv]

• **Dynamics of foliations, groups and pseudogroups**,
P. Walczak, [2004, Birkhäuser, 2004]

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• Can you "count" the matchbox manifolds? How do you distinguish one from another? with K-Theory invariants? using cohomology invariants? systems of approximations?

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Give three theorems that address a small part of these questions.

Structure theory: Weak solenoids

Let B_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The p_ℓ are the *bonding maps* for the weak solenoid

$$\mathcal{S} = \lim_{\longleftarrow} \, \{p_\ell \colon B_\ell \to B_{\ell-1}\} \, \subset \, \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: $\mathcal{S}$ has natural structure of a matchbox manifold, with every leaf dense.

McCord solenoids

Basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(B_\ell, x_\ell)$.

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon B_\ell \longrightarrow B_0$.

Definition: $\mathcal{S}$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of G_{ℓ_0}.

The *Vietoris solenoids* have $B_\ell = \mathbb{S}^1$ for all $\ell \geq 0$.

Classifying weak solenoids

A weak solenoid is determined by the base manifold B_0 and the tower equivalence of the descending chain

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

Theorem: [Pontryagin 1934; Baer 1937] For $G_0 \cong \mathbb{Z}$, the homeomorphism types of McCord solenoids is uncountable.

Theorem: [Kechris 2000; Thomas2001] For $G_0 \cong \mathbb{Z}^k$ with $k \geq 2$, the homeomorphism types of McCord solenoids is not classifiable, *in the sense of Descriptive Set Theory*.

The number of such is not just huge, but indescribably large.

Structure theory: Generalized solenoids

A generalized solenoid is defined by a tower of *branched manifolds*

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The *bonding maps* p_ℓ are assumed to be locally smooth cellular maps, and we set

$$\mathcal{S} = \varprojlim \; \{p_\ell \colon B_\ell \to B_{\ell-1}\} \; \subset \; \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: If the local degrees of the maps p_ℓ tend to ∞, then the inverse limit $\mathcal{S}$ has natural structure of a matchbox manifold.

These are more general than the Williams solenoids, but same idea.

Theorem: [Clark, H, Lukina 2011] Let $\mathfrak{M}$ be a minimal matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a generalized solenoid.

• For $\mathfrak{M}$ a tiling space of an aperiodic tiling of finite local complexity on $\mathbb{R}^n$, Anderson & Putnam (1991) showed $\mathfrak{M}$ is an inverse limit of branched manifolds.

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• For $\mathfrak{M}$ with foliation defined by free G-action and tiling on orbits, as in Benedetti & Gambaudo, same as their result.

• For general $\mathfrak{M}$, the problem is to find good local product structures, which are stable under transverse perturbation.

The difficulties depends on the dimension:

The leaves are not assumed to have flat structures, so this adds an extra level of difficulty, as compared to the methods in paper of Giordano, Matui, Hiroki, Putnam, & Skau: "Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems", Invent. Math. 179 (2010)

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In terms of *leaf dimensions*, we have the fundamental observation:

$$1 \ll 2 \ll 3 < n$$

Embeddings

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The criteria for embedding depend on the degree of smoothness required, and the tower of subgroups of the fundamental group.

See "Embedding solenoids in foliations", **Topology Appl.**, 2011. The problem is wide open in general.

Application

This is a type of "Reeb Instability" result:

Theorem: Let $\mathcal{F}_0$ be a C^∞-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some saturated open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}_M$ on M which is C^∞-close to $\mathcal{F}_0$, and $\mathcal{F}_M$ has an uncountable set of solenoidal minimal sets $\{\mathcal{S}_\alpha \mid \alpha \in \mathcal{A}\}$, which are *pairwise non-homeomorphic*.

Homeomorphisms

Let $\mathfrak{M}$ be a matchbox manifold of dimension *n*.

Lemma: A homeomorphism $\phi \colon \mathfrak{M} \to \mathfrak{M}'$ of matchbox manifolds must map leaves to leaves $\Rightarrow$ is a foliated homeomorphism.

Proof: Leaves of $\mathcal{F} \iff$ path components of $\mathfrak{M}$

Corollary: $\textbf{Homeo}(\mathfrak{M}) = \textbf{Homeo}(\mathfrak{M}, \mathcal{F})$ – all homeomorphisms are leaf preserving.

Theorem: [McCord 1965] Let B_0 be an oriented smooth closed manifold. Then a McCord solenoid $\mathcal{S}$ is an orientable, homogeneous, equicontinuous smooth matchbox manifold.

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Proofs vary in their degrees of "abstractness", suggesting:

Bing Conjecture: Suppose that $\mathfrak{M}$ is homogeneous continuum, and $\mathfrak{M}$ is a matchbox manifold of dimension $n \geq 1$. Then either $\mathfrak{M}$ is homeomorphic to a compact manifold, or to a McCord solenoid.

Theorem: [C & H, 2010] Bing Conjecture is true for all $n \geq 1$.

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The key to the proof is to show:

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Corollary: Let $\mathfrak{M}$ be a equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to the suspension of an minimal action of a countable group on a Cantor space $\mathbb{K}$.

More problems

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Problem: Study $\textbf{Out}(\mathfrak{M})$.

$\textbf{Out}(\mathfrak{M})$ captures many aspects of the space $\mathfrak{M}$ – its topological, dynamical and algebraic properties.