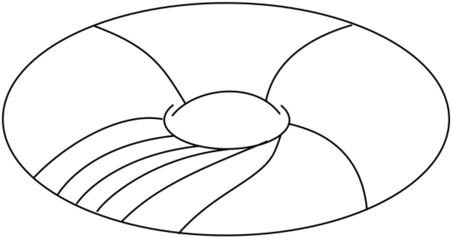
Lecture 1: Derivatives

Steven Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder/talks/

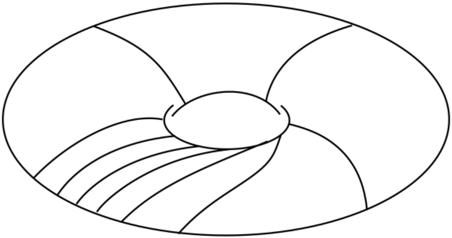
Some basic examples

Many talks on with "foliations" in the title start with this example, the 2-torus foliated by lines of irrational slope:



Some basic examples

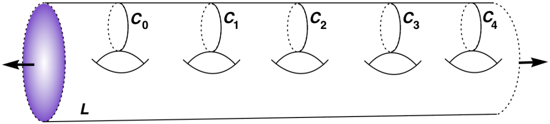
Many talks on with "foliations" in the title start with this example, the 2-torus foliated by lines of irrational slope:



Never trust a talk which starts with this example! It is just too simple.

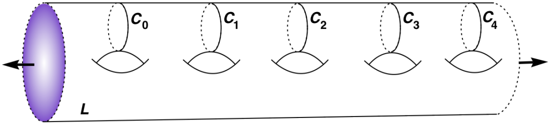
Some basic examples

Although, the example can be salvaged, by considering that the "same example" might have leaves that look like this:



Some basic examples

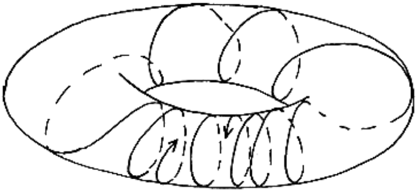
Although, the example can be salvaged, by considering that the "same example" might have leaves that look like this:



The suspension construction an its generalizations are very useful for producing examples.

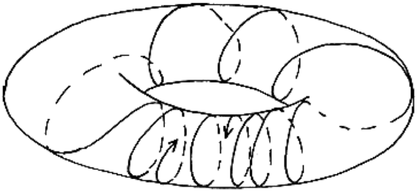
Some basic examples, 2

More interesting are talks which discuss more irregular flows such as this:



Some basic examples, 2

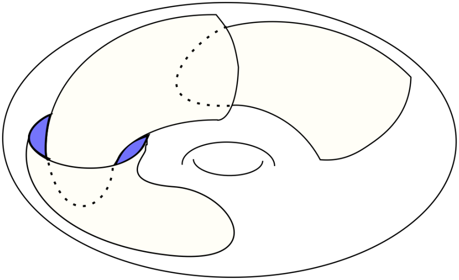
More interesting are talks which discuss more irregular flows such as this:



Every orbit limits into the circle, so at least things have a direction.

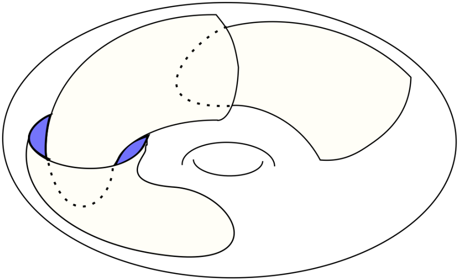
Some basic examples, 3

Earnest foliation talks start with this example, immortalized by Reeb:



Some basic examples, 3

Earnest foliation talks start with this example, immortalized by Reeb:



Now begins the real questions – what does it mean to discuss "foliation dynamics"? What is "dynamic" about this example?

Foliation dynamics

- Study the asymptotic properties of leaves of $\mathcal{F}$ -

  What is the topological shape of minimal sets?

  Invariant measures: can you quantify their rates of recurrence?

Foliation dynamics

- Study the asymptotic properties of leaves of $\mathcal{F}$ -

  What is the topological shape of minimal sets?

  Invariant measures: can you quantify their rates of recurrence?

- Directions of "stability" and "instability" of leaves -

  Exponents: are there directions of exponential divergence?

  Stable manifolds: dynamically defined transverse invariant manifolds?

Foliation dynamics

- Study the asymptotic properties of leaves of $\mathcal{F}$ -

  What is the topological shape of minimal sets?

  Invariant measures: can you quantify their rates of recurrence?

- Directions of "stability" and "instability" of leaves -

  Exponents: are there directions of exponential divergence?

  Stable manifolds: dynamically defined transverse invariant manifolds?

- Quantifying chaos -

  Define a measure of transverse chaos – foliation entropy

  Estimate the entropy using linear approximations

Foliation dynamics

- Study the asymptotic properties of leaves of $\mathcal{F}$ -

  What is the topological shape of minimal sets?

  Invariant measures: can you quantify their rates of recurrence?

- Directions of "stability" and "instability" of leaves -

  Exponents: are there directions of exponential divergence?

  Stable manifolds: dynamically defined transverse invariant manifolds?

- Quantifying chaos -

  Define a measure of transverse chaos – foliation entropy

  Estimate the entropy using linear approximations

- Shape of minimal sets -

  Hyperbolic exotic minimal sets

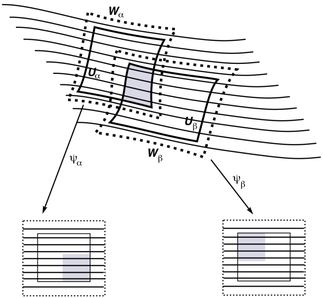
  Distal exceptional minimal sets

First definitions

M is a compact Riemannian manifold without boundary.

$\mathcal{F}$ is a codimension q-foliation, transversally C^r for $r \in [1, \infty)$.

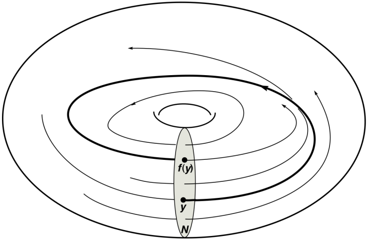
Transition functions for the foliation charts $\varphi_i \colon U_i \to [-1,1]^n \times T_i$ are C^∞ leafwise, and vary C^r with the transverse parameter:



Holonomy - flows

Recall for a flow $\varphi_t \colon M \to M$ the orbits define 1-dimensional leaves of $\mathcal{F}$.

Choose a cross-section $\mathcal{N} \subset M$ which is transversal to the orbits, and intersects each orbit (so $\mathcal{N}$ need not be connected) then for each $x \in \mathcal{T}$ there is some least $\tau_x > 0$ so that $\varphi_{\tau_x}(x) \in \mathcal{N}$ – *the return time* for x.



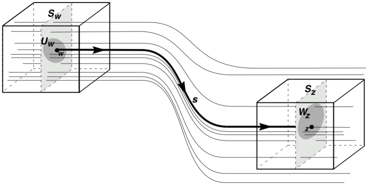
The induced map $f(x) = \varphi_{\tau_x}(x)$ is a *Borel map* $f \colon \mathcal{N} \to \mathcal{N}$ the holonomy of the flow.

Holonomy - foliations

Let L_w be leaf of $\mathcal{F}$ containing w – no such concept as "future" or "past".

Rather, choose $z \in L_x$ and smooth path $\tau_{w,z} \colon [0,1] \to L_w$.

Cover path $\tau_{w,z}$ by foliation charts and slide open subset U_w of transverse disk S_w along path to open subset W_z of transverse disk S_z



Holonomy pseudogroup

Standardize above by choosing finite covering of M by foliation charts, with transversal sections $\mathcal{T} = \mathcal{T}_1 \cup \cdots \mathcal{T}_k \subset M$.

The holonomy of $\mathcal{F}$ defines pseudogroup $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ which is compactly generated in sense of Haefliger.

Given $w \in \mathcal{T}$, $z \in L_w \cap \mathcal{T}$ and path $\tau_{w,z} \colon [0,1] \to L_w$ from w to z, we obtain $h_{\tau_{w,z}} \colon U_w \to W_z$ where now

*) $h_{\tau_{w,z}}$ depends on the leafwise homotopy class of the path

*) maximal sizes of the domain U_w and range W_z depends on $\tau_{w,z}$

*) $\{h_{\tau_{w,z}} \colon U_w \to W_z\}$ generates $\mathcal{G}_{\mathcal{F}}$.

Holonomy pseudogroup

Standardize above by choosing finite covering of M by foliation charts, with transversal sections $\mathcal{T} = \mathcal{T}_1 \cup \cdots \mathcal{T}_k \subset M$.

The holonomy of $\mathcal{F}$ defines pseudogroup $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ which is compactly generated in sense of Haefliger.

Given $w \in \mathcal{T}$, $z \in L_w \cap \mathcal{T}$ and path $\tau_{w,z} \colon [0,1] \to L_w$ from w to z, we obtain $h_{\tau_{w,z}} \colon U_w \to W_z$ where now

*) $h_{\tau_{w,z}}$ depends on the leafwise homotopy class of the path

*) maximal sizes of the domain U_w and range W_z depends on $\tau_{w,z}$

*) $\{h_{\tau_{w,z}} \colon U_w \to W_z\}$ generates $\mathcal{G}_{\mathcal{F}}$.

Proposition: We can assume $\tau_{w,z}$ is a leafwise geodesic path.

Proof: Each leaf L_w is complete for the induced metric.

Transverse differentials

Let $\varphi\colon \mathbb{R} \times M \to M$ be a smooth non-singular flow for vector field $\vec{X}$.

Defines foliation $\mathcal{F}$.

Transverse differentials

Let $\varphi\colon \mathbb{R} \times M \to M$ be a smooth non-singular flow for vector field $\vec{X}$.

Defines foliation $\mathcal{F}$.

For $w = \varphi_t(w)$, the Jacobian matrix $D\varphi_t\colon T_w \to T_z M$.

Group Law $\quad \varphi_s \circ \varphi_t = \varphi_{s+t} \Longrightarrow D\varphi_s(\vec{X}_w) = \vec{X}_z$

Transverse differentials

Let $\varphi\colon \mathbb{R} \times M \to M$ be a smooth non-singular flow for vector field $\vec{X}$.

Defines foliation $\mathcal{F}$.

For $w = \varphi_t(w)$, the Jacobian matrix $D\varphi_t\colon T_w \to T_z M$.

Group Law $\quad \varphi_s \circ \varphi_t = \varphi_{s+t} \Longrightarrow D\varphi_s(\vec{X}_w) = \vec{X}_z$ (. . . boring!)

Transverse differentials

Let $\varphi\colon \mathbb{R} \times M \to M$ be a smooth non-singular flow for vector field $\vec{X}$.

Defines foliation $\mathcal{F}$.

For $w = \varphi_t(w)$, the Jacobian matrix $D\varphi_t \colon T_w \to T_z M$.

Group Law $\quad \varphi_s \circ \varphi_t = \varphi_{s+t} \Longrightarrow D\varphi_s(\vec{X}_w) = \vec{X}_z$ (...boring!)

Normal bundle to flow $Q = TM/\langle \vec{X} \rangle = TM/T\mathcal{F} \subset T\mathcal{F}$.

Riemannian metric on TM induces metrics on Q_w for all $w \in M$.

Measure for norms of maps $D\varphi_t \colon Q_w \longrightarrow Q_z$.

Un poquito de Pesin Theory

Definition: $w \in M$ is hyperbolic point of flow if

$$e_{\mathcal{F}}(w) \equiv \lim_{T\to\infty} \sup_{s \geq T} \{\frac{1}{s} \cdot \log\{\|(D\varphi_t \colon Q_w \to Q_z)^{\pm}\|\} \mid -s \leq t \leq s\} \ > \ 0$$

Lemma: Set of hyperbolic points $\mathcal{H}(\varphi) = \{w \in M \mid e_{\mathcal{F}}(w) > 0\}$ is flow-invariant.

Un poquito de Pesin Theory

Definition: $w \in M$ is hyperbolic point of flow if

$$e_{\mathcal{F}}(w) \equiv \lim_{T \to \infty} \ \sup_{s \geq T} \ \{\frac{1}{s} \cdot \log\{\|(D\varphi_t \colon Q_w \to Q_z)^{\pm}\|\} \mid -s \leq t \leq s\} \ > \ 0$$

Lemma: Set of hyperbolic points $\mathcal{H}(\varphi) = \{w \in M \mid e_{\mathcal{F}}(w) > 0\}$ is flow-invariant.

Pesin Theory of C^2-flows studies properties of the set of hyperbolic points.

Proposition: Closure $\overline{\mathcal{H}(\varphi)} \subset M$ contains an invariant ergodic probability measure μ_* for φ, for which there exists $\lambda > 0$ such that for μ_*-a.e. w,

$$e_{\mathcal{F}}(w) = \lim_{T \to \infty} \{\frac{1}{T} \cdot \log\{\|D\varphi_T \colon Q_w \to Q_z\|\} = \lambda$$

Un poquito de Pesin Theory

Definition: $w \in M$ is hyperbolic point of flow if

$$e_{\mathcal{F}}(w) \equiv \lim_{T\to\infty} \sup_{s \geq T} \{\frac{1}{s} \cdot \log\{\|(D\varphi_t : Q_w \to Q_z)^{\pm}\|\} \mid -s \leq t \leq s\} > 0$$

Lemma: Set of hyperbolic points $\mathcal{H}(\varphi) = \{w \in M \mid e_{\mathcal{F}}(w) > 0\}$ is flow-invariant.

Pesin Theory of C^2-flows studies properties of the set of hyperbolic points.

Proposition: Closure $\overline{\mathcal{H}(\varphi)} \subset M$ contains an invariant ergodic probability measure μ_* for φ, for which there exists $\lambda > 0$ such that for μ_*-a.e. w,

$$e_{\mathcal{F}}(w) = \lim_{T\to\infty} \{\frac{1}{T} \cdot \log\{\|D\varphi_T : Q_w \to Q_z\|\} = \lambda$$

Proof: Just calculus! (plus usual subadditive tricks)

Foliation geodesic flow

Let $w \in M$ and consider L_w as complete Riemannian manifold.

For $\vec{v} \in T_w\mathcal{F} = T_w L_w$ with $\|\vec{v}\|_w = 1$, there is unique geodesic $\tau_{w,\vec{v}}(t)$ starting at w with $\tau'_{w,\vec{v}}(0) = \vec{v}$ Define

$$\varphi_{w,\vec{v}} \colon \mathbb{R} \to M \quad , \quad \varphi_{w,\vec{v}}(w) = \tau_{w,\vec{v}}(t)$$

Let $\widehat{M} = T^1\mathcal{F}$ denote the unit tangent bundle to the leaves, then we obtain the *foliation geodesic flow*

$$\varphi_t^{\mathcal{F}} \colon \mathbb{R} \times \widehat{M} \to \widehat{M}$$

Foliation geodesic flow

Let $w \in M$ and consider L_w as complete Riemannian manifold.

For $\vec{v} \in T_w\mathcal{F} = T_w L_w$ with $\|\vec{v}\|_w = 1$, there is unique geodesic $\tau_{w,\vec{v}}(t)$ starting at w with $\tau'_{w,\vec{v}}(0) = \vec{v}$ Define

$$\varphi_{w,\vec{v}} \colon \mathbb{R} \to M \quad , \quad \varphi_{w,\vec{v}}(w) = \tau_{w,\vec{v}}(t)$$

Let $\widehat{M} = T^1\mathcal{F}$ denote the unit tangent bundle to the leaves, then we obtain the *foliation geodesic flow*

$$\varphi_t^{\mathcal{F}} \colon \mathbb{R} \times \widehat{M} \to \widehat{M}$$

Remark: $\varphi_t^{\mathcal{F}}$ preserves the leaves of the foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$ whose leaves are the unit tangent bundles to leaves of $\mathcal{F}$.

$\Longrightarrow D\varphi_t^{\mathcal{F}}$ preserves the normal bundle $\widehat{Q} \to \widehat{M}$ for $\widehat{\mathcal{F}}$.

Foliation exponents

Definitions:

(H) $\widehat{w} \in \widehat{M}$ is *hyperbolic* if

$$e_{\mathcal{F}}(\widehat{w}) \equiv \lim_{T\to\infty} \sup_{s\geq T} \{\frac{1}{s} \cdot \log\{\|(D\varphi_t^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to Q_{\widehat{z}})^{\pm}\|\} \mid -s \leq t \leq s\} \ > \ 0$$

Foliation exponents

Definitions:

(H) $\widehat{w} \in \widehat{M}$ is *hyperbolic* if

$$e_{\mathcal{F}}(\widehat{w}) \equiv \lim_{T \to \infty} \sup_{s \geq T} \{\frac{1}{s} \cdot \log\{\|(D\varphi_t^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to Q_{\widehat{z}})^{\pm}\|\} \mid -s \leq t \leq s\} \; > \; 0$$

(E) $\widehat{w} \in \widehat{M}$ is *elliptic* if $e_{\mathcal{F}}(\widehat{w}) = 0$, and there exists $\kappa(\widehat{w})$ such that

$$\{\|(D\varphi_t^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to Q_{\widehat{z}})^{\pm}\| \leq \kappa(\widehat{w}) \text{ for all } t \in \mathbb{R}$$

Foliation exponents

Definitions:

(H) $\widehat{w} \in \widehat{M}$ is *hyperbolic* if

$$e_{\mathcal{F}}(\widehat{w}) \equiv \lim_{T\to\infty} \sup_{s\geq T} \{\frac{1}{s} \cdot \log\{\|(D\varphi_t^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to Q_{\widehat{z}})^{\pm}\|\} \mid -s \leq t \leq s\} \ > \ 0$$

(E) $\widehat{w} \in \widehat{M}$ is *elliptic* if $e_{\mathcal{F}}(\widehat{w}) = 0$, and there exists $\kappa(\widehat{w})$ such that

$$\{\|(D\varphi_t^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to Q_{\widehat{z}})^{\pm}\| \leq \kappa(\widehat{w}) \text{ for all } t \in \mathbb{R}$$

(P) $\widehat{w} \in \widehat{M}$ is *parabolic* if $e_{\mathcal{F}}(\widehat{w}) = 0$, and $\widehat{w}$ is not elliptic.

Dynamical decomposition of foliations

Theorem: Let $\mathcal{F}$ be a C^1-foliation of a compact Riemannian manifold M. Then there exists a decomposition of M into $\mathcal{F}$-saturated Borel subsets

$$M = M_{\mathcal{H}} \ \cup \ M_{\mathcal{P}} \ \cup \ M_{\mathcal{E}}$$

where the derivative for the geodesic flow of $\mathcal{F}$ satisfies:

• $D\varphi_t^{\mathcal{F}}$ is "transversally hyperbolic" for $L_w \subset M_{\mathcal{H}}$

• $D\varphi_t^{\mathcal{F}}$ is bounded (in time) for $L_w \subset M_{\mathcal{E}}$

• $D\varphi_t^{\mathcal{F}}$ has subexponential growth (in time), but is not bounded, for $L_w \subset M_{\mathcal{P}}$

Transversally hyperbolic measures

Definition: An invariant probability measure μ_* for the foliation geodesic flow on $\widehat{M}$ is said to be transversally hyperbolic if $e_{\mathcal{F}}(\widehat{w}) = \lambda > 0$ for μ_*-a.e. $\widehat{w}$.

Transversally hyperbolic measures

Definition: An invariant probability measure μ_* for the foliation geodesic flow on $\widehat{M}$ is said to be transversally hyperbolic if $e_{\mathcal{F}}(\widehat{w}) = \lambda > 0$ for μ_*-a.e. $\widehat{w}$.

Theorem: Let $\mathcal{F}$ be a C^1foliation of a compact manifold. If $M_{\mathcal{H}} \neq \emptyset$, then the foliation geodesic flow has at least one transversally hyperbolic ergodic measure.

Transversally hyperbolic measures

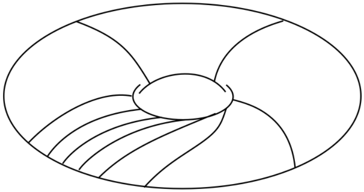
Definition: An invariant probability measure μ_* for the foliation geodesic flow on $\widehat{M}$ is said to be transversally hyperbolic if $e_{\mathcal{F}}(\widehat{w}) = \lambda > 0$ for μ_*-a.e. $\widehat{w}$.

Theorem: Let $\mathcal{F}$ be a C^1foliation of a compact manifold. If $M_{\mathcal{H}} \neq \emptyset$, then the foliation geodesic flow has at least one transversally hyperbolic ergodic measure.

Proof: The proof is technical, but is actually just calculus applied to the foliation pseudogroup.

Standard examples, revisited: 1

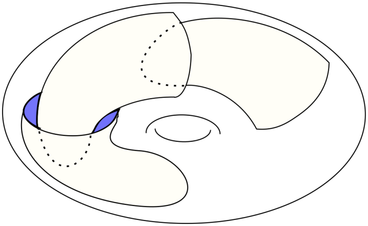
For the linear foliation, every point is elliptic (it is Riemannian!)



However, if $\mathcal{F}$ is a C^1-foliation which is topologically semi-conjugate to a linear foliation, so is a generalized Denjoy example, then $M = M_{\mathcal{P}}$!

Standard examples, revisited: 2

The case of the Reeb foliation on the solid torus is more interesting:



Pick $w \in M$ and a direction, $\vec{v} \in T_w L_w$, then follow the geodesic $\tau_{w,\vec{w}}(t)$. It is asymptotic to the boundary torus, so defines a limiting Schwartzman cycle on the torus for some flow. Thus, it limits on either a circle, or a lamination. This will be a hyperbolic measure if the holonomy of the compact leaf is hyperbolic. The exponent depends on the direction!

References

S. Hurder, *Ergodic theory of foliations and a theorem of Sacksteder*, in **Dynamical Systems: Proceedings, University of Maryland 1986-87**. Lect. Notes in Math. Vol. 1342, pages 291–328, 1988.

P. Walczak, *Dynamics of the geodesic flow of a foliation*, **Ergodic Theory Dynamical Systems**, 8:637–650, 1988.

L. Barreira and Ya.B. Pesin, **Lyapunov exponents and smooth ergodic theory**, University Lecture Series, Vol. 23, AMS, 2002.