Lecture 3: Exponential Complexity

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Hyperbolic normal forms

Recall a simple example from advanced calculus.

Let $f(x) = x/2$.

Let $g(x)$ be smooth with $g(0) = 0$, $g'(0) = 1/2$.

Then $g \sim f$ near $x = 0$. That is, for $\delta > 0$ sufficiently small, there is a smooth map $h\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that $h^{-1} \circ g \circ h = f(x)$ for all $|x| < \delta$.

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Moral: *Complexity is Simplicity.*

Foliation complexity

Let $\mathcal{F}$ be a foliation of a compact Riemannian manifold M.

For each $w \in M$ the leaf L_w containing w inherits a Riemannian metric for which L_w is geodesically complete.

Fix L_w and then count the number of points $Gr(L_w, d) = \#\{L_w \cap \mathcal{T}\}$.

The rate of growth of the function $d \mapsto Gr(L_w, d)$ is a measure of the complexity of the leaf.

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L_w has *polynomial growth type* if there exists $m > 0$ and $d_0 \geq 0$ such that

$$Gr(L_w, d) \leq m^k \quad , \quad d \geq d_0$$

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Definition: A foliation $\mathcal{F}$ is *uniformly subexponential* if:

for all $\epsilon > 0$, there exists C_ϵ, d_ϵ so that for all $w \in M$,

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The celebrated theorem of Connes, Feldman and Weiss [1981] implies:

Theorem: If $\mathcal{F}$ is uniformly subexponential, then the equivalence relation it defines on the transversal space $\mathcal{T}$ is amenable, hence hyperfinite.

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Moral: Subexponential complexity often leads to ambiguity.

Expansion growth

We measure exponential complexity for pseudogroup actions, following [Bowen 1971] and [Ghys, Langevin & Walczak 1988].

Let $\epsilon > 0$ and $d > 0$. A subset $\mathcal{E} \subset \mathcal{T}$ is said to be (ϵ, d)-separated if

- for all $w, w' \in \mathcal{E} \cap \mathcal{T}_i$
- there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(g) \subset \mathcal{T}_i$, and $\|g\| \leq d$
- then $d_{\mathcal{T}}(g(w), g(w')) \geq \epsilon$.
- If $w \in \mathcal{T}_i$ and $w' \in \mathcal{T}_j$ for $i \neq j$ then they are (ϵ, d)-separated by default.

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The "expansion growth function" counts the maximum of this quantity:

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon, d) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathcal{T} \text{ is } (\epsilon, d)\text{-separated}\}$$

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If the pseudogroup consists of isometries, for example, then applying elements of $\mathcal{G}_{\mathcal{F}}$ does not help to separate points, so these growth functions remain polynomial as functions of d, for all ϵ.

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Theorem: If $\mathcal{F}$ is defined by a flow ϕ_t then $h(\mathcal{G}_{\mathcal{F}}) = 2 \cdot h_{top}(\phi_1)$.

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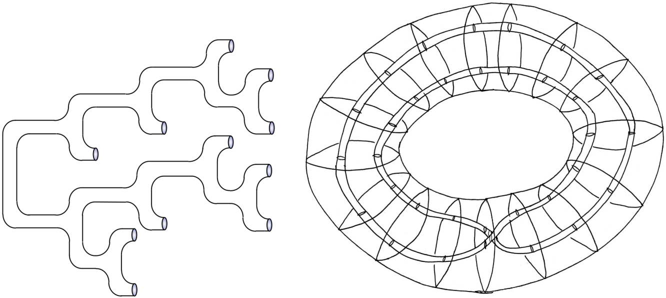
Solution: The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch example is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$.

After d-iterations, the inverse map to $z \mapsto z^{2^d}$ has derivative of norm 2^d so we have a rough estimate

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon, d) \sim (2\pi/\epsilon) \cdot 2^d$$

Orbit growth implies entropy

For the Hirsch example, notice as we wander out the tree-like leaf, we are also wandering around the transversal space $\mathcal{T}$.



Manning's Theorem

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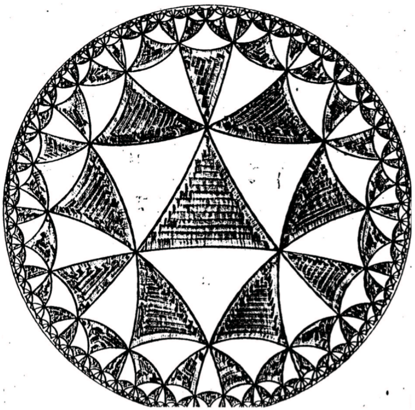
That is, the growth rate of the volume of balls in the universal covering of B equals the entropy. This is actually a theorem about foliation entropy and growth rates of leaves.

Fundamental domains

The assumption that B has non-positive curvature implies that its universal covering $\widetilde{B}$ is a disk, and we can "color" it with fundamental domains:



The proof of Manning's Theorem follows from the picture.

Weak stable foliations

Assume that B has uniformly negative sectional curvatures.

Let $\phi_t \colon M \to M$ be the geodesic flow. Define an equivalence relation on points of M:

$$w \sim_\phi w' \quad \Longleftrightarrow \quad d_M(\phi_t(w), \phi_t(w')) \leq C \text{ for } t \to \infty$$

Then define

$$L_w = \{w' \in M \mid w' \sim_\phi w\}$$

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Theorem: [Pugh-Shub 1974] The sets L_w form the leaves of a C^1-foliation of M. The resulting foliation is called the *weak-stable foliation* for ϕ_t.

1) Each leaf L_w is a C^∞-immersed submanifold of M.

2) The orbits of the geodesic flow $\phi_t(w)$ are contained in the leaves of $\mathcal{F}$.

Entropy for weak stable foliations

Theorem: Let B be a compact manifold of negative curvature, and let $\mathcal{F}$ be the weak stable foliation for the geodesic flow ϕ_t. Then

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The proof that $h(\mathcal{G}_{\mathcal{F}}) =\geq 2 \cdot h_{top}(\phi_1)$ is easy - we use the holonomy along geodesic segments to separate points.

The other estimate requires knowing about the structure of the weak stable foliations - the leaves are obtained by applying the geodesic flow to the strong stable foliations, which are polynomial growth, so do not add any exponential complexity.

Entropy and chaos

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Next time, we discuss the relation between foliation entropy and the existence of hyperbolic invariant measures for the foliation geodesic flow.

Problemos de la diá

Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]:

Friday [7/5/2010]:

References

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