Lecture 3: Exponential Complexity

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Hyperbolic normal forms

Recall a simple example from advanced calculus.

Let $f(x) = x/2$.

Let $g(x)$ be smooth with $g(0) = 0$, $g'(0) = 1/2$.

Then $g \sim f$ near $x = 0$. That is, for $\delta > 0$ sufficiently small, there is a smooth map $h\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that $h^{-1} \circ g \circ h = f(x)$ for all $|x| < \delta$.

Fact: Exponentially contracting (or simply hyperbolic) maps have a single parameter (their derivative $g'(0)$ at the fixed point) for their germinal conjugacy class.

For maps which are "completely flat" at the origin, where $g(0) = 0$, $g'(0) = 1$, $g^k(0) = 0$ for all $k > 1$, no such classification exists.

Moral: *Complexity is Simplicity.*

Foliation complexity

Let $\mathcal{F}$ be a foliation of a compact Riemannian manifold M.

For each $w \in M$ the leaf L_w containing w inherits a Riemannian metric for which L_w is geodesically complete.

Fix L_w and then count the number of points $Gr(L_w, d) = \#\{L_w \cap \mathcal{T}\}$.

The rate of growth of the function $d \mapsto Gr(L_w, d)$ is a measure of the complexity of the leaf.

L_w has *exponential growth type* if there exists $\lambda > 0$ and $d_0 \geq 0$ such that

$$Gr(L_w, d) \geq \exp\{\lambda \cdot d\} \quad , \quad d \geq d_0$$

L_w has *polynomial growth type* if there exists $m > 0$ and $d_0 \geq 0$ such that

$$Gr(L_w, d) \leq m^k \quad , \quad d \geq d_0$$

Subexponential complexity

Definition: A foliation $\mathcal{F}$ is *uniformly subexponential* if:

for all $\epsilon > 0$, there exists C_ϵ, d_ϵ so that for all $w \in M$,

$$Gr(L_w, d) \leq C_\epsilon \cdot \exp\{\epsilon \cdot d\} \quad , \quad d \geq d_\epsilon$$

The celebrated theorem of Connes, Feldman and Weiss [1981] implies:

Theorem: If $\mathcal{F}$ is uniformly subexponential, then the equivalence relation it defines on the transversal space $\mathcal{T}$ is amenable, hence hyperfinite.

The pseudogroup action $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ is *hyperfinite* if it is *measurably orbit equivalent* to an action of the integers $\mathbb{Z}$ on the interval $[0, 1]$.

Moral: Subexponential complexity often leads to ambiguity.

Expansion growth

We measure exponential complexity for pseudogroup actions, following [Bowen 1971] and [Ghys, Langevin & Walczak 1988].

Let $\epsilon > 0$ and $d > 0$. A subset $\mathcal{E} \subset \mathcal{T}$ is said to be (ϵ, d)-separated if

- for all $w, w' \in \mathcal{E} \cap \mathcal{T}_i$
- there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(g) \subset \mathcal{T}_i$, and $\|g\| \leq d$
- then $d_{\mathcal{T}}(g(w), g(w')) \geq \epsilon$.
- If $w \in \mathcal{T}_i$ and $w' \in \mathcal{T}_j$ for $i \neq j$ then they are (ϵ, d)-separated by default.

The "expansion growth function" counts the maximum of this quantity:

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon, d) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathcal{T} \text{ is } (\epsilon, d)\text{-separated}\}$$

If the pseudogroup consists of isometries, for example, then applying elements of $\mathcal{G}_{\mathcal{F}}$ does not help to separate points, so these growth functions remain polynomial as functions of d, for all ϵ.

Foliation geometric entropy

The function $d \mapsto h(\mathcal{G}_{\mathcal{F}}, \epsilon, d)$ measures *expansion growth* at distance d - sort of an *integrated total exponent*.

Define:

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon) = \limsup_{d \to \infty} \frac{\ln\left\{\max\{\#\mathcal{E} \mid \mathcal{E} \text{ is } (\epsilon, d)\text{-separated}\}\right\}}{d}$$

$$h(\mathcal{G}_{\mathcal{F}}) = \lim_{\epsilon \to 0} \; h(\mathcal{G}_{\mathcal{F}}, \epsilon)$$

Theorem: [GLW 1988] The quantity $h(\mathcal{G}_{\mathcal{F}})$ is finite if $\mathcal{F}$ is a C^1-foliation.

Moreover, the property $h(\mathcal{G}_{\mathcal{F}}) > 0$ is independent of all choices.

Theorem: If $\mathcal{F}$ is defined by a flow ϕ_t then $h(\mathcal{G}_{\mathcal{F}}) = 2 \cdot h_{top}(\phi_1)$.

Doubling maps have entropy $\ln(2) > 0$

Exercise: The Hirsch foliations always have positive geometric entropy.

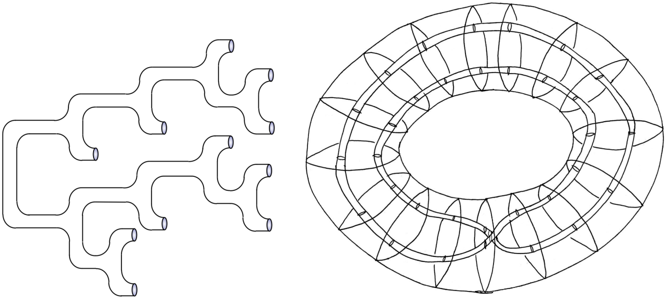
Solution: The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch example is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$.

After d-iterations, the inverse map to $z \mapsto z^{2^d}$ has derivative of norm 2^d so we have a rough estimate

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon, d) \sim (2\pi/\epsilon) \cdot 2^d$$

Orbit growth implies entropy

For the Hirsch example, notice as we wander out the tree-like leaf, we are also wandering around the transversal space $\mathcal{T}$.



Manning's Theorem

Let B be a compact manifold of non-positive curvature.

Let $M = T^1 B$ denote the unit tangent bundle to B.

Let $\phi_t \colon M \to M$ be the geodesic flow of B.

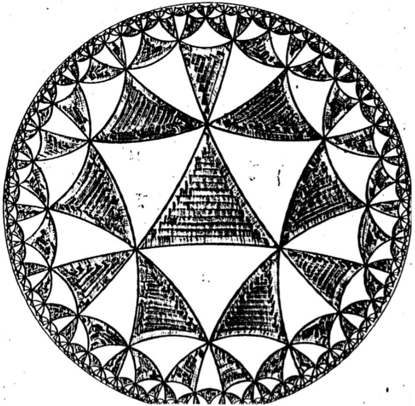
Theorem: [Manning 1976] $h_{top}(\phi) = Gr(\pi_1(B, b_0))$

That is, the growth rate of the volume of balls in the universal covering of B equals the entropy.

This is actually a theorem about foliation entropy and growth rates of leaves.

Fundamental domains

The assumption that B has non-positive curvature implies that its universal covering $\widetilde{B}$ is a disk, and we can "color" it with fundamental domains:



The proof of Manning's Theorem follows from the picture.

Weak stable foliations

Assume that B has uniformly negative sectional curvatures.

Let $\phi_t \colon M \to M$ be the geodesic flow. Define an equivalence relation on points of M:

$$w \sim_\phi w' \quad \Longleftrightarrow \quad d_M(\phi_t(w), \phi_t(w')) \leq C \text{ for } t \to \infty$$

Then define

$$L_w = \{w' \in M \mid w' \sim_\phi w\}$$

Theorem: [Pugh-Shub 1974] The sets L_w form the leaves of a C^1-foliation of M. The resulting foliation is called the *weak-stable foliation* for ϕ_t.

1) Each leaf L_w is a C^∞-immersed submanifold of M.

2) The orbits of the geodesic flow $\phi_t(w)$ are contained in the leaves of $\mathcal{F}$.

Entropy for weak stable foliations

Theorem: Let B be a compact manifold of negative curvature, and let $\mathcal{F}$ be the weak stable foliation for the geodesic flow ϕ_t. Then

$$h(\mathcal{G}_{\mathcal{F}}) = 2 \cdot h_{top}(\phi_1)$$

The proof that $h(\mathcal{G}_{\mathcal{F}}) =\geq 2 \cdot h_{top}(\phi_1)$ is easy - we use the holonomy along geodesic segments to separate points.

The other estimate requires knowing about the structure of the weak stable foliations - the leaves are obtained by applying the geodesic flow to the strong stable foliations, which are polynomial growth, so do not add any exponential complexity.

Entropy and chaos

Question: When is $h(\mathcal{G}_{\mathcal{F}}) > 0$?

- Expanding holonomy (Hirsch examples)
- Weak stable foliations (for Anosov flows)
- Ping-pong games (Resilient leaves in codimension one)

Are there other cnanoical situations where we can expect positive entropy?

For example, if $\mathcal{F}$ has leaves of exponential growth, doses there always exist a C^1-close perturbation of $\mathcal{F}$ with positive entropy?

Next time, we discuss the relation between foliation entropy and the existence of hyperbolic invariant measures for the foliation geodesic flow.

Problemos de la diá

Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]:

Friday [7/5/2010]:

References

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H.B. Lawson, Jr., **The Quantitative Theory of Foliations**, NSF Regional Conf. Board Math. Sci., Vol. 27, 1975.

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S. Hurder, *Entropy and Dynamics of C^1 Foliations*, preprint 2000.

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