Lecture 4: Entropy and Exponent

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Positive entropy $\leftrightarrow$ chaos $\leftrightarrow$??

Let $\mathcal{F}$ be a C^r-foliation of a compact manifold M, $r \geq 1$.

Problem 1: If $h(\mathcal{G}_{\mathcal{F}}) > 0$, what conclusions can we reach about the dynamics of $\mathcal{F}$?

Problem 2: What hypotheses on the dynamics of $\mathcal{F}$ are sufficient to imply that $h(\mathcal{G}_{\mathcal{F}}) > 0$?

Problem 3: Are there cohomology hypotheses on $\mathcal{F}$ which would "improve" our understanding of its dynamics? How does leafwise cohomology $H^*(\mathcal{F})$ influence dynamics? Secondary invariants for $\mathcal{F}$?

Foliated dynamics toolbox

There are limited sets of techniques applicable to foliation dynamics.

Principle difficulty has been there is no good groupoid replacement for the notion of "uniform recurrence" in the support of an invariant measure, which we have for flows.

The following result has various applications, especially in codimension one. Best news – it introduces a new technique.

Theorem: [G-L-W 1998] Let M be compact with a C^1-foliation $\mathcal{F}$ of codimension $q \geq 1$. If $h(\mathcal{G}_{\mathcal{F}}) = 0$, then the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ admits an invariant probability measure.

Three Theorems

Theorem: [H 2000] Let M be compact with a C^r-foliation $\mathcal{F}$ of codimension-q. If $q = 1$ and $r \geq 1$, or $q \geq 2$ and $r > 1$, then

$$\mathcal{F} \ \text{distal} \quad \Longrightarrow \quad h(\mathcal{G}_{\mathcal{F}}) = 0$$

Theorem: [H 2000] Let M be compact with a codimension one, C^1-foliation $\mathcal{F}$. Then

$$h(\mathcal{G}_{\mathcal{F}}) > 0 \quad \Longrightarrow \quad \mathcal{F} \ \text{has a resilient leaf}$$

Theorem: [H & Langevin 2000] Let M be compact with a codimension one, C^2-foliation $\mathcal{F}$. Then

$$0 \neq GV(\mathcal{F}) \in H^3(M, \mathbb{R}) \quad \Longrightarrow \quad h(\mathcal{G}_{\mathcal{F}}) > 0$$

Positive exponents

Proposition: Let $\mathcal{F}$ be C^1, and suppose $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there is a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow.

The proof illustrates the tools available.

Assume $h(\mathcal{G}_{\mathcal{F}}) = 2\lambda$. Then for sufficiently small $\epsilon > 0$, for $d \gg 0$ there exists $\mathcal{E} = \{w_1, \ldots, w_\ell\}$ where $\ell > \exp\{d \cdot \lambda\}$, so that for $w_i \neq w_j$ there exists some path $\tau_{i,j} \colon [0,1] \to L_{w_i}$ with

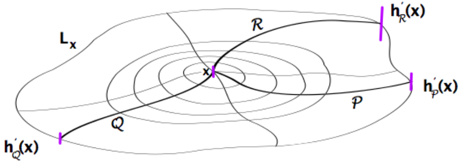
$$d_{\mathcal{T}}(h_{\tau_{i,j}}(w_i), h_{\tau_{i,j}}(w_j)) \geq \epsilon$$

By counting, there is some pair with $d_{\mathcal{T}}(w_i, w_j) < \mathrm{diam}(\mathcal{T})/\ell$

Mean Value Theorem $\Longrightarrow h'_{\tau_{i,j}}(w'_i) > \exp\{d \cdot \lambda\}$

Quivers

Actually, Pigeon Hole Principle implies there are closed neighborhoods $D(w, \delta) \subset \mathcal{T}$ containing an exponential number of such points:



These are call *quivers*.

Hyperbolic measures

If we choose a sequence of such paths, in the limit they define an invariant measure for the geodesic flow, with positive exponent.

Proposition: Let $\mathcal{F}$ be C^1 and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists an exponential collection of leafwise geodesic segments $\{\gamma_\ell \colon [0, \infty) \to M$ along which the normal derivative cocycle Dh_{γ_ℓ} has exponentially decreasing directions.

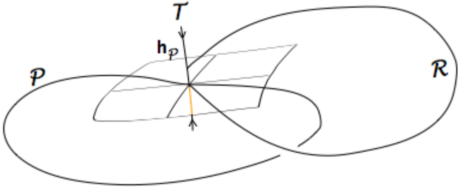
Stable manifolds

Theorem: Let $\mathcal{F}$ be $C^{1+\alpha}$ and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists an exponential collection of leafwise geodesic segments $\{\gamma_\ell \colon [0,\infty) \to M$ with stable transverse manifolds.

Corollary: Let $\mathcal{F}$ be a codimension one, C^1-foliation, or codimension $q > 1$ $C^{1+\alpha}$-foliation. Then $\mathcal{G}_{\mathcal{F}}$ distal implies that $h(\mathcal{G}_{\mathcal{F}}) = 0$.

Ping-pong games

Theorem: Let $\mathcal{F}$ be C^1 and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ admits a "ping-pong game" which implies the existence of a resilient leaf for $\mathcal{F}$.



Problemos de la diá

Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]: Find conditions on the Lyapunov spectrum and invariant measures for the geodesic flow which implies positive entropy.

Friday [7/5/2010]: Characterize the exceptional minimal sets of zero entropy.

References

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