Lecture 5: Foliation minimal sets

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Minimal sets

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Understanding minimal sets of foliations causes enough trouble.

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Choose a basepoint $w_0 \in \mathcal{Z}$. A minimal set $\mathcal{Z}$ has *stable* shape if the pointed inclusions

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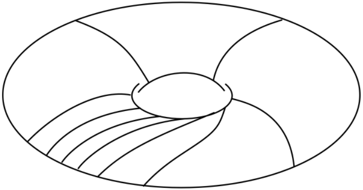
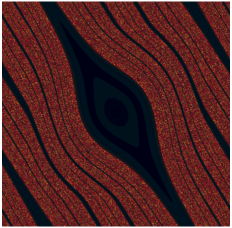
The *shape fundamental group*:

$$\widehat{\pi}_1(\mathcal{Z}, w_0) = \text{inv lim}\{\pi_1(V_k, w_0) \leftarrow \pi_1(V_{k+1}, w_0)\}$$

DA examples

The "DA process" converts the irrational slope foliation into an exceptional minimal set for a DA map:





These minimal sets are stable: $\widehat{\pi}_1(\mathcal{Z}, w_0) = \pi_1(\mathbb{T}^2 - \{w_1\}) \cong \mathbb{Z} * \mathbb{Z}$.

Foliated shape fundamental group

$\epsilon_L > 0$ is Lebesgue number for an open cover of M by foliation charts.

Given a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ suppose that $d_M(w,z) < \delta < \epsilon_L$.

Then τ defines closed path in $V_\delta = \{x \in M \mid d_M(x, \mathcal{Z}) < \delta\}$, and holonomy map h_τ.

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Such holonomy maps define the *shape dynamics* of $\mathcal{Z}$..

Problem: Given a minimal set $\mathcal{Z}$, what can se saw about the "shape dynamics" of $\mathcal{Z}$.

Hyperbolic minimal sets

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Problem: Can we even begin to classify the stable exceptional minimal sets for C^1-foliations?

Minimal sets defined by an IFS

Let $K \subset \mathbb{R}^n$ be compact convex set, and $h_\ell \colon K \to K$ affine maps.

Then pseudogroup generated by $\{h_1, \ldots, h_k\}$ on $K \subset \mathbb{R}^n$ is called a Iterated Function System.

For $J = (j_1, j_2, \ldots, j_m)$ set $h_J = h_{j_1} \circ \cdots \circ h_{j_m} \colon K \to K$.

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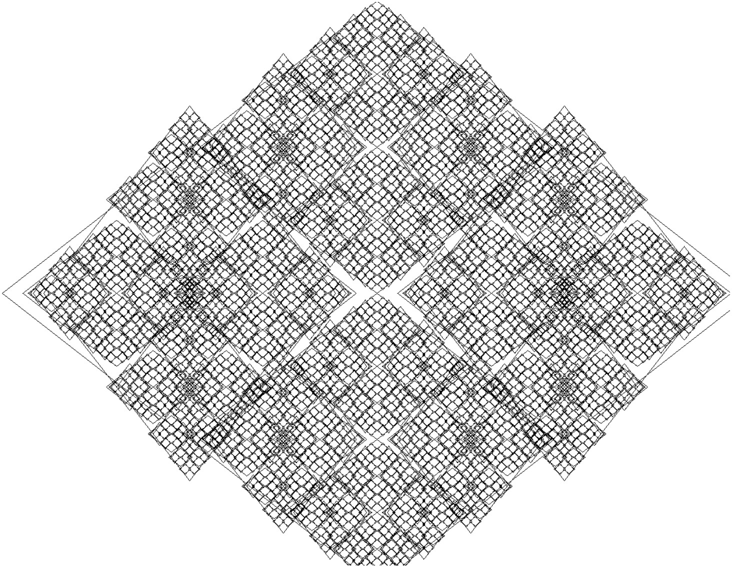
This construction has many generalizations, and leads to a variety of interesting examples.

A hyperbolic minimal set defined by an IFS



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Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

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Question: What are the zero entropy minimal sets?

Solenoidal minimal sets

An *n-dimensional solenoid* is an inverse limit space

$$\mathcal{S} = \lim_{\leftarrow} \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps.

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Theorem: [Clark-H 2008] Let $\mathcal{F}_0$ be a C^r-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}$ on M which is C^r-close to $\mathcal{F}_0$, and $\mathcal{F}$ has a solenoidal minimal set contained in U with base L_0. If $\mathcal{F}_0$ is a distal foliation, then $\mathcal{F}$ is also distal.

Solenoidal minimal sets

This is a consequence of a general construction:

Theorem: Let L_0 be a closed oriented manifold of dimension n, with $H^1(L_0, \mathbb{R}) \neq 0$. Let $q \geq 2$, $r \geq 1$, and $\mathcal{F}_0$ denote the product foliation of $M = L_0 \times \mathbb{D}^q$. Then there exists a C^r-foliation $\mathcal{F}$ of M which is C^r-close to $\mathcal{F}_0$, such that $\mathcal{F}$ is a volume-preserving, distal foliation, and satisfies

1. L_0 is a leaf of $\mathcal{F}$
2. $\mathcal{F} = \mathcal{F}_0$ near the boundary of M
3. $\mathcal{F}$ has a minimal set $\mathcal{S}$ which is a generalized solenoid with base L_0
4. each leaf $L \subset \mathcal{S}$ is a covering of L_0.

Constructing solenoids

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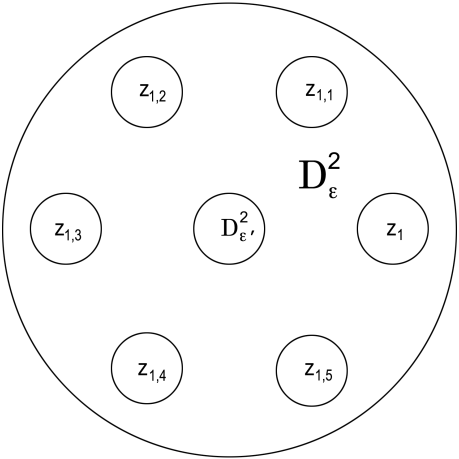

Problemos de la semana

Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]: Find conditions on the Lyapunov spectrum and invariant measures for the geodesic flow which implies positive entropy.

Friday [7/5/2010]: Characterize the exceptional minimal sets of zero entropy.

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