Lecture 5: Foliation minimal sets

Steven Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder/talks/

Minimal sets

$\mathcal{Z} \subset M$ *minimal* $\Longleftrightarrow$ closed and every leaf $L \subset \mathcal{Z}$ is dense.

$\mathcal{W} \subset M$ is *transitive* $\Longleftrightarrow$ closed and there exists a dense leaf $L \subset \mathcal{Z}$

Minimal sets

$\mathcal{Z} \subset M$ *minimal* $\Longleftrightarrow$ closed and every leaf $L \subset \mathcal{Z}$ is dense.

$\mathcal{W} \subset M$ is *transitive* $\Longleftrightarrow$ closed and there exists a dense leaf $L \subset \mathcal{Z}$

M compact then minimal sets for foliations always exist.

Transitive sets are most important for flows – Axiom A attractors, have minimal sets which are periodic orbits in the domain of attraction.

Minimal sets

$\mathcal{Z} \subset M$ *minimal* $\Longleftrightarrow$ closed and every leaf $L \subset \mathcal{Z}$ is dense.

$\mathcal{W} \subset M$ is *transitive* $\Longleftrightarrow$ closed and there exists a dense leaf $L \subset \mathcal{Z}$

M compact then minimal sets for foliations always exist.

Transitive sets are most important for flows – Axiom A attractors, have minimal sets which are periodic orbits in the domain of attraction.

Understanding minimal sets of foliations causes enough trouble.

Shape of minimal sets

$\mathcal{Z} \subset M$ a compact set.

Shape of minimal sets

$\mathcal{Z} \subset M$ a compact set.

Definition: The *shape* of $\mathcal{Z}$ is the equivalence class of any descending chain of open subsets $M \supset V_1 \supset \cdots \supset V_k \supset \cdots \supset \mathcal{Z}$ with $\mathcal{Z} = \bigcap_{k \to \infty} V_k$

Choose a basepoint $w_0 \in \mathcal{Z}$. A minimal set $\mathcal{Z}$ has *stable* shape if the pointed inclusions

$$(V_{k+1}, w_0) \subset (V_k, w_0)$$

are homotopy equivalences for all $k \gg 0$.

Shape of minimal sets

$\mathcal{Z} \subset M$ a compact set.

Definition: The *shape* of $\mathcal{Z}$ is the equivalence class of any descending chain of open subsets $M \supset V_1 \supset \cdots \supset V_k \supset \cdots \supset \mathcal{Z}$ with $\mathcal{Z} = \bigcap_{k \to \infty} V_k$

Choose a basepoint $w_0 \in \mathcal{Z}$. A minimal set $\mathcal{Z}$ has *stable* shape if the pointed inclusions

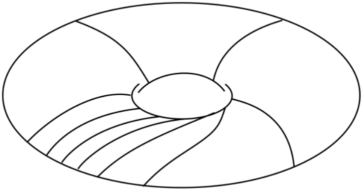
$$(V_{k+1}, w_0) \subset (V_k, w_0)$$

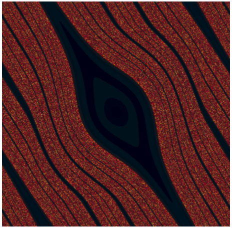
are homotopy equivalences for all $k \gg 0$.

The *shape fundamental group*:

$$\widehat{\pi}_1(\mathcal{Z}, w_0) = \text{inv lim}\{\pi_1(V_k, w_0) \leftarrow \pi_1(V_{k+1}, w_0)\}$$

The "DA process" converts the irrational slope foliation into an exceptional minimal set for a DA map:





These minimal sets are stable: $\widehat{\pi}_1(\mathcal{Z}, w_0) = \pi_1(\mathbb{T}^2 - \{w_1\}) \cong \mathbb{Z} * \mathbb{Z}$.

Foliated shape fundamental group

$\epsilon_L > 0$ is Lebesgue number for an open cover of M by foliation charts.

Given a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ suppose that $d_M(w,z) < \delta < \epsilon_L$.

Then τ defines closed path in $V_\delta = \{x \in M \mid d_M(x, \mathcal{Z}) < \delta\}$, and holonomy map h_τ.

This is called an approximate holonomy orbit, and generalize a similar concept for hyperbolic flows.

Foliated shape fundamental group

$\epsilon_L > 0$ is Lebesgue number for an open cover of M by foliation charts.

Given a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ suppose that $d_M(w,z) < \delta < \epsilon_L$.

Then τ defines closed path in $V_\delta = \{x \in M \mid d_M(x, \mathcal{Z}) < \delta\}$, and holonomy map h_τ.

This is called an approximate holonomy orbit, and generalize a similar concept for hyperbolic flows.

Such holonomy maps define the *shape dynamics* of $\mathcal{Z}$..

Problem: Given a minimal set $\mathcal{Z}$, what can se saw about the "shape dynamics" of $\mathcal{Z}$.

Hyperbolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *hyperbolic* if $\mathcal{Z} \cap M_{\mathcal{H}} \neq \emptyset$, and *uniformly hyperbolic* if $\mathcal{Z} \subset M_{\mathcal{H}}$.

Hyperbolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *hyperbolic* if $\mathcal{Z} \cap M_{\mathcal{H}} \neq \emptyset$, and *uniformly hyperbolic* if $\mathcal{Z} \subset M_{\mathcal{H}}$.

Proposition: If $\mathcal{Z}$ is hyperbolic, then there exists hyperbolic *approximate* holonomy maps for $\mathcal{Z}$. That is, there exists closed orbits defined on arbitrarily small open neighborhoods of $\mathcal{Z}$ along which the normal holonomy has contracting directions.

Hyperbolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *hyperbolic* if $\mathcal{Z} \cap M_{\mathcal{H}} \neq \emptyset$, and *uniformly hyperbolic* if $\mathcal{Z} \subset M_{\mathcal{H}}$.

Proposition: If $\mathcal{Z}$ is hyperbolic, then there exists hyperbolic *approximate* holonomy maps for $\mathcal{Z}$. That is, there exists closed orbits defined on arbitrarily small open neighborhoods of $\mathcal{Z}$ along which the normal holonomy has contracting directions.

Problem: Suppose that $\mathcal{F}$ is a C^2foliation of codimension one, and $\mathcal{F}$ is a hyperbolic *exceptional* minimal set. Must $\mathcal{Z}$ have Lebesgue measure zero?

Hyperbolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *hyperbolic* if $\mathcal{Z} \cap M_{\mathcal{H}} \neq \emptyset$, and *uniformly hyperbolic* if $\mathcal{Z} \subset M_{\mathcal{H}}$.

Proposition: If $\mathcal{Z}$ is hyperbolic, then there exists hyperbolic *approximate* holonomy maps for $\mathcal{Z}$. That is, there exists closed orbits defined on arbitrarily small open neighborhoods of $\mathcal{Z}$ along which the normal holonomy has contracting directions.

Problem: Suppose that $\mathcal{F}$ is a C^2foliation of codimension one, and $\mathcal{F}$ is a hyperbolic *exceptional* minimal set. Must $\mathcal{Z}$ have Lebesgue measure zero?

Problem: Can we even begin to classify the stable exceptional minimal sets for C^1-foliations?

Minimal sets defined by an IFS

Let $K \subset \mathbb{R}^n$ be compact convex set, and $h_\ell \colon K \to K$ affine maps.

Then pseudogroup generated by $\{h_1, \ldots, h_k\}$ on $K \subset \mathbb{R}^n$ is called a Iterated Function System.

For $J = (j_1, j_2, \ldots, j_m)$ set $h_J = h_{j_1} \circ \cdots \circ h_{j_m} \colon K \to K$.

Minimal sets defined by an IFS

Let $K \subset \mathbb{R}^n$ be compact convex set, and $h_\ell \colon K \to K$ affine maps.

Then pseudogroup generated by $\{h_1, \ldots, h_k\}$ on $K \subset \mathbb{R}^n$ is called a Iterated Function System.

For $J = (j_1, j_2, \ldots, j_m)$ set $h_J = h_{j_1} \circ \cdots \circ h_{j_m} \colon K \to K$.

Proposition: If each map h_ℓ is a contraction, then $K_* = \bigcap h_J(K)$ is a hyperbolic minimal set.

Minimal sets defined by an IFS

Let $K \subset \mathbb{R}^n$ be compact convex set, and $h_\ell \colon K \to K$ affine maps.

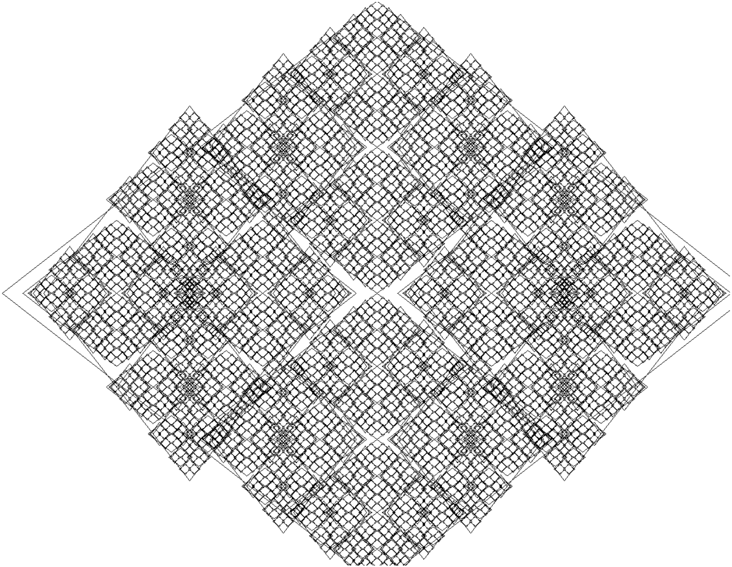
Then pseudogroup generated by $\{h_1, \ldots, h_k\}$ on $K \subset \mathbb{R}^n$ is called a Iterated Function System.

For $J = (j_1, j_2, \ldots, j_m)$ set $h_J = h_{j_1} \circ \cdots \circ h_{j_m} \colon K \to K$.

Proposition: If each map h_ℓ is a contraction, then $K_* = \bigcap h_J(K)$ is a hyperbolic minimal set.

This construction has many generalizations, and leads to a variety of interesting examples.

A hyperbolic minimal set defined by an IFS



Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap M_{\mathcal{H}} = \emptyset$.

Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap M_{\mathcal{H}} = \emptyset$.

Proposition: Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.

Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap M_{\mathcal{H}} = \emptyset$.

Proposition: Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.

Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

What other sorts of parabolic minimal sets are there?

Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap M_{\mathcal{H}} = \emptyset$.

Proposition: Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.

Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

What other sorts of parabolic minimal sets are there?

Proposition: A parabolic minimal set has zero entropy.

Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap M_{\mathcal{H}} = \emptyset$.

Proposition: Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.

Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

What other sorts of parabolic minimal sets are there?

Proposition: A parabolic minimal set has zero entropy.

Question: What are the zero entropy minimal sets?

Solenoidal minimal sets

An *n-dimensional solenoid* is an inverse limit space

$$\mathcal{S} = \varprojlim \; \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps.

Solenoidal minimal sets

An *n-dimensional solenoid* is an inverse limit space

$$\mathcal{S} = \lim_{\leftarrow} \; \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps.

Theorem: [Clark-H 2008] Let $\mathcal{F}_0$ be a C^r-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}$ on M which is C^r-close to $\mathcal{F}_0$, and $\mathcal{F}$ has a solenoidal minimal set contained in U with base L_0. If $\mathcal{F}_0$ is a distal foliation, then $\mathcal{F}$ is also distal.

Solenoidal minimal sets

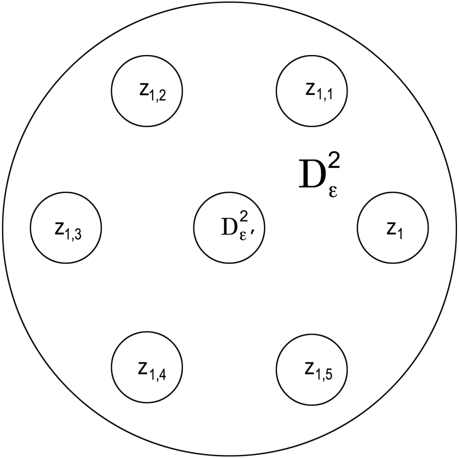
This is a consequence of a general construction:

Theorem: Let L_0 be a closed oriented manifold of dimension n, with $H^1(L_0, \mathbb{R}) \neq 0$. Let $q \geq 2$, $r \geq 1$, and $\mathcal{F}_0$ denote the product foliation of $M = L_0 \times \mathbb{D}^q$. Then there exists a C^r-foliation $\mathcal{F}$ of M which is C^r-close to $\mathcal{F}_0$, such that $\mathcal{F}$ is a volume-preserving, distal foliation, and satisfies

1. L_0 is a leaf of $\mathcal{F}$
2. $\mathcal{F} = \mathcal{F}_0$ near the boundary of M
3. $\mathcal{F}$ has a minimal set $\mathcal{S}$ which is a generalized solenoid with base L_0
4. each leaf $L \subset \mathcal{S}$ is a covering of L_0.

Constructing solenoids

This is a consequence of a general construction:



Problemos de la semana

Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]: Find conditions on the Lyapunov spectrum and invariant measures for the geodesic flow which implies positive entropy.

Friday [7/5/2010]: Characterize the exceptional minimal sets of zero entropy.

References

J.-M. Gambaudo and C. Tresser, *Diffeomorphisms with infinitely many strange attractors*, **J. Complexity**, 6:409–416, 1990.

J.-M. Gambaudo, D. Sullivan and C. Tresser, *Infinite cascades of braids and smooth dynamical systems*, **Topology**, 33:85–94, 1994.

J. Kennedy and J. Yorke, *Bizarre topology is natural in dynamical systems*, **Bull. Amer. Math. Soc. (N.S.)**, 32:309–316, 1995.

A. Bís, S. Hurder, and J. Shive, *Hirsch foliations in codimension greater than one*, in **Foliations 2005**, World Scientific Publishing, 2006: 71–108.

A. Clark and S. Hurder, *Embedding matchbox manifolds*, preprint 2008.

S. Hurder, *Classifying foliations*, **Foliations, Topology and Geometry**, Contemp Math. Vol. 498, AMS 2009, pages 1–65.

A. Biś and S. Hurder, *Markov minimal sets of foliations*, 2010.

S. Hurder and A. Rechtman, *Some remarks on pseudogroups of four generators*, 2010.

A. Clark, S. Hurder & O. Lukina, *Classifying Matchbox Manifolds*, 2010.