Critical point theory for foliations

Steven Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder

TTFPP 2007– Bedlewo, Poland

Critical points with symmetry

Let $f\colon M \to \mathbb{R}$ be a C^0-function, where M is a Riemannian manifold.

Study the critical points of f which respect some additional symmetry:

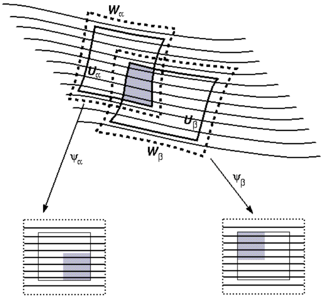
- with respect to finite group Γ acting on M;
- with respect to connected Lie group G acting on M;
- with respect to a foliation $\mathcal{F}$ of M, where we require that f be a foliated map.

Definition: $f\colon M \to M'$ is a foliated map, if $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are foliated spaces and f maps leaves of F to leaves of $\mathcal{F}'$.

Last example is the most general, and includes the previous cases.

Definition of foliation:

A foliation $\mathcal{F}$ of dimension p on a manifold M is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.



If the dimensions of the leaves are not constant, then we say $\mathcal{F}$ is a *singular foliation*. Many of the results of this talk apply also to singular foliations.

Foliations arise in the study of many subjects:

1. Partial differential equations (Reeb, Godbillon, Sacksteder)
2. Representation theory - cocycles, co-orbit spaces, W^* & C^*-algebras (Murray – von Neumann, Mackey, Kasparov)
3. Generalized dynamical systems (Anosov, Smale, Hirsch, Shub, Hector)
4. Topology of classifying spaces (Bott, Haefliger, Gelfand-Fuks, Mather, Morita, Thurston)
5. Geometry - open book decompositions and laminations of manifolds (Lawson, Winkelnkemper, Thurston, Gabai)
6. Physics & Non-Commutative Geometry (Bellisard, Connes, et al)

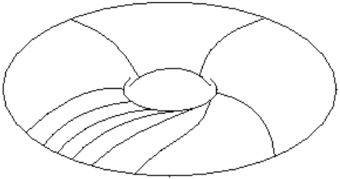
Examples: compact foliations

We say that $\mathcal{F}$ is a *compact foliation* if every leaf L of $\mathcal{F}$ is a compact submanifold. $\mathcal{F}$ is a *compact Hausdorff foliation* if every leaf is compact and the quotient space $M' = M/\mathcal{F}$ is Hausdorff. Here are three examples:

- Let $\pi \colon M \to M'$ be a fibration with M compact. The fibers $L = \pi^{-1}(x')$ for $x' \in M'$ define the leaves.
- Let M^3 be a Seifert fibered 3-manifold, fibered by circles, with base space B an orbifold.
- G a compact connected Lie group and $G \times M \to M$ locally free.

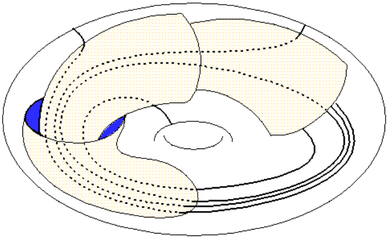
Not every compact foliation is compact Hausdorff, so even in this simplest class of foliations, their study is complicated.

Examples: non-commutative tori



Lines fill up the 2-torus $\mathbb{T}^2$

Examples: Reeb foliation of $\mathbb{S}^3$



Planes fill up the solid 2-torus.
Two copies of the torus glue together to give $\mathbb{S}^3$.

Examples: Lie group actions

Let G be a connected Lie group and $\mathbf{K}$ a compact topological space, with a continuous action $\varphi\colon G \times \mathbf{K} \to \mathbf{K}$. If all orbits of φ have the same dimension, then the action defines a *lamination* of $\mathbf{K}$.

Examples include:

- Locally free action of a Lie group on a compact manifold M
- Hull closure $\mathbf{K}$ of a quasi-periodic symbol on $\mathbb{R}^p$ (or more generally on connected Lie group G.)

If the orbits of G have varying dimensions, then we get a singular foliation.

Examples: discrete group actions

Let Γ be a finitely-generated group and N a compact manifold of dimension q, with a smooth action $\alpha\colon \Gamma \times N \to N$.

Then there exists a compact $q+2$-dimensional manifold M with foliation $\mathcal{F}_\alpha$ having 2-dimensional leaves, such that the global holonomy of $\mathcal{F}_\alpha$ is conjugate to the representation α.

The point is that the geometry (more precisely, the holonomy) of $\mathcal{F}$ captures all of the information about the given group action.

(The construction of $\mathcal{F}$ uses a sequence of "twisted surgeries" on $\mathbb{S}^2 \times N$, one for each generator of Γ.)

Elementary properties:

- $X \subset M$ is *saturated* if $L \cap X \neq \emptyset \implies L \subset X$
- $\mathbf{K} \subset M$ is *transitive* if $\mathbf{K}$ is closed, saturated (non-empty) and there exists $L \subset X$ whose closure is all of $\mathbf{K}$
- $\mathbf{K} \subset M$ is *minimal* if $\mathbf{K}$ is closed, saturated (non-empty) and every leaf $L \subset \mathbf{K}$ is dense.
- X compact, saturated subset then there exists a minimal set $\mathbf{K} \subset X$.

Lemma: If $f \colon M \to \mathbb{R}$ is a foliated map for the point foliation $\mathcal{F}'$ on $\mathbb{R}$, then for all $c \in \mathbb{R}$ the inverse image $X_c = f^{-1}(c)$ is a closed, saturated subset.

Corollary: Assume that $f \colon M \to \mathbb{R}$ is proper, then for each critical value $c \in \mathbb{R}$ of f, there exists a minimal set $\mathbf{K}_c \subset X_c$ with $f(\mathbf{K}_c) = c$.

Counting critical points

We want to study the critical behavior of foliated maps, where $\mathcal{F}$ is assumed to have leaves which are possibly non-compact.

The classical case: let $f : M \to \mathbb{R}$ be a C^1-function on a closed Riemannian manifold M.

Theorem: (Lusternik-Schnirelmann [1934])

$$\#\{x \mid x \in M \text{ is critical for } f\} \geq \mathrm{cat}(M)$$

where $\mathrm{cat}(M)$ is the Lusternik-Schnirelmann category of M, which is defined as the least number of *open* sets $\{U_1, \ldots, U_k\}$ required to cover M such that each U_ℓ is contractible in M to a point.

Counting critical orbits

If $f\colon M \to \mathbb{R}$ satisfies a symmetry condition, then can require that each categorical open set U and its contracting homotopy $H_t\colon U \to M$ be invariant (or saturated) for the symmetry.

The equivariant Lusternik-Schnirelmann category $\mathrm{cat}_G(M)$ of an action by a Lie group G on a manifold M was introduced by Marzantowicz [1989] for compact G. He proved that

$$\#\{G\cdot x \mid G\cdot x \text{ is critical orbit for } f\} \geq \mathrm{cat}_G(M)$$

Ayala, Lasheras and Quintero [2001] generalized the Marzantowicz results to proper group actions.

Colman defined transverse LS-category for foliations in her Thesis [1998].

Transverse LS category of foliations

Let $(M, \mathcal{F})$ be a foliated manifold, and $U \subset M$ and open saturated subset.

If $\mathcal{F}$ is defined by the action of a Lie group G, then we are requiring that U be G-invariant.

Definition: U is (transversally) categorical if there is a foliated homotopy $H_t \colon U \to M$, where H_0 is the inclusion, and H_1 has image in a leaf of $\mathcal{F}$.

Definition: (Colman) The transverse LS category $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is the least number of transversely categorical open saturated sets required to cover M. If no such covering exists, then set $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

Transverse LS category of foliations – examples

Example: Let $M \to M'$ be a fibration with compact fibers which defines the foliation $\mathcal{F}$ on M. Then $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \mathrm{cat}(M')$.

Theorem: (Colman [1998]) If $\mathcal{F}$ is compact Hausdorff, then $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ is finite. Moreover, the Lusternik-Schnirelmann estimate holds for counting the number of critical leaves:

$$\#\{L \mid L \subset M \text{ is critical for } f\} \geq \mathrm{cat}_{\pitchfork}(M, \mathcal{F})$$

Transverse LS category of foliations – more examples

Example: Let $M = \mathbb{T}^2$ be the linear foliation of the 2-torus, with all leaves dense. Then $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

Example: Let $M = \mathbb{S}^3$ with the 2-dimensional Reeb foliation. Then $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

In both examples above, we see the problem arises from the properties of the leaf closures of $\mathcal{F}$.

Theorem: (Hurder [2000]) Let $k = \mathrm{cat}_{\pitchfork}(M, \mathcal{F}) < \infty$. Given a transversally categorical covering of M, $\{H_{\ell,t} \colon U_\ell \to M \mid 1 \leq \ell \leq k\}$ with $H_{\ell,1}(U_\ell) \subset L_\ell$, then each L_ℓ is a compact leaf.

Corollary: $\mathcal{F}$ has no compact leaves $\Rightarrow \mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

Counting critical minimal sets

The basic observation is that compact leaves are just a special case of compact minimal sets.

• Given a foliated map $f\colon M \to \mathbb{R}$, the goal should not be to count the critical leaves of $\mathcal{F}$, but rather the critical minimal sets (or possibly the critical transitive sets.)

Problem: How to count critical minimal sets?

Solution: Modify the definition of transversally categorical set.

Essential transverse LS category of foliations

Let $(M, \mathcal{F})$ be a foliated manifold, and $U \subset M$ and open saturated subset.

Definition: U is essentially transversally categorical if there is a foliated homotopy $H_t \colon U \to M$, where H_0 is the inclusion, and H_1 has image in a minimal set of $\mathcal{F}$.

Definition: The essential transverse LS category $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is the least number of essentially transversely categorical open saturated sets required to cover M. If no such covering exists, then set $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F}) = \infty$.

Remark: $\mathcal{F}$ a compact foliation $\Longrightarrow \mathrm{cat}^e_{\pitchfork}(M, \mathcal{F}) = \mathrm{cat}_{\pitchfork}(M, \mathcal{F})$.

Riemannian foliations

Definition: $\mathcal{F}$ is a Riemannian foliation if there is a Riemannian metric on TM so that the restriction to the normal bundle $Q = T\mathcal{F}^{\perp}$ is invariant under the leafwise parallelism.

Equivalently, the induced metric on Q is locally projectable: for any open set $U \subset M$ such that $\mathcal{F} \mid U$ is defined by a fibration $\pi_U \colon U \to B_U$ then the map π_U is a local Riemannian submersion.

- $\mathcal{F}$ defined by locally free action of compact connected Lie group $\Longrightarrow$ $\mathcal{F}$ is Riemannian.
- $\mathcal{F}$ compact Hausdorff $\Longrightarrow \mathcal{F}$ is Riemannian.
- For each leaf L, the closure $\overline{L}$ is a minimal set of $\mathcal{F}$.

Main Theorem

Theorem: (Hurder-Töben [2006]) Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M. Then:

- the essential transverse category $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F})$ is finite;
- if the transverse category $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ is finite, then

$$\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F}) = \mathrm{cat}_{\pitchfork}(M, \mathcal{F});$$

- Let $f : M \to \mathbb{R}$ be a foliated map. Then we have a generalized form of the Lusternik-Schnirelmann Theorem:

$$\#\{\mathbf{K}_c \mid \mathbf{K}_c \subset M \text{ is critical for } f\} \geq \mathrm{cat}^e_{\pitchfork}(M, \mathcal{F})$$

The proof actually gives much more information.

Molino Theory

Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M.

Let $\widehat{M}$ denote the bundle of orthonormal frames to $\mathcal{F}$ – $\pi\colon \widehat{M} \to M$ is an $\mathbf{O}(q)$-fibration with a right action of $\mathbf{O}(q)$.

Theorem: (Molino [1982])

- The foliation $\mathcal{F}$ "lifts" to a Riemannian foliation $\widehat{\mathcal{F}}$ of $\widehat{M}$ whose leaves cover those of $\mathcal{F}$
- $\widehat{\mathcal{F}}$ is $\mathbf{O}(q)$-equivariant.
- For each leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$, the closure $\overline{\widehat{L}}$ is a submanifold of $\widehat{M}$ (and a minimal set for $\widehat{\mathcal{F}}$.)
- The closures of the leaves of $\widehat{\mathcal{F}}$ form a compact foliation $\overline{\widehat{\mathcal{F}}}$ of $\widehat{M}$
- The leaf space $\widehat{W} = \widehat{M}/\overline{\widehat{\mathcal{F}}}$ is a manifold, and the quotient map $\widehat{\Upsilon}\colon \widehat{M} \to \widehat{W}$ is an $\mathbf{O}(q)$-equivariant Riemannian submersion.

Equivariant foliated LS category

A foliated C^r-map $f\colon M \to \mathbb{R}$ induces an $\mathbf{O}(q)$-invariant map $\widehat{f}\colon \widehat{W} \to \mathbb{R}$.

Proposition: Critical minimal sets of $f \iff$ critical orbits of $\widehat{f}$

$$\begin{array}{ccccccc} & \mathbf{O}(q) & = & \mathbf{O}(q) & \\ & \downarrow & & \downarrow & \\ & \widehat{M} & \xrightarrow{\widehat{\Upsilon}} & \widehat{W} = \widehat{M}/\overline{\widehat{\mathcal{F}}} & \\ \pi & \downarrow & & \downarrow & \widehat{\pi} \\ & M & \xrightarrow{\Upsilon} & W = M/\overline{\mathcal{F}} & \end{array}$$

Theorem: (Hurder-Töben [2006]) Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M. Then $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F}) = \mathrm{cat}_{\mathbf{O}(q)}(\widehat{W})$.

Corollary: Let $f\colon M \to \mathbb{R}$ be a foliated map.

$$\#\{\mathbf{K}_c \mid \mathbf{K}_c \subset M \text{ is critical for } f\} \geq \mathrm{cat}_{\mathbf{O}(q)}(\widehat{W})$$

Hence, one can use the full-force of equivariant LS category theory to calculate $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F})$ and estimate the number of critical minimal sets.

Polar actions

Definition: Let G Lie group acting smoothly by isometries on a complete Riemannian manifold M. A *section* for the G-action is an isometrically immersed complete submanifold $i : \Sigma \to M$ which meets every orbit and always orthogonally.

The dimension of Σ is equal to the cohomogeneity of the action, denoted by q. Note that for any $g \in G$, the map $g \circ i \colon g\Sigma \to M$ is again a section.

Definition: A *polar action* is a G-action with a section. If Σ is a flat submanifold, then the action is called *hyperpolar*.

The geometry of polar actions has been extensively studied by Kostant [1973], Szenthe [1984], Dadok [1985], Palais & Terng [1988], Thorbergsson [1999], Kollross [2002], and others.

Examples of polar actions

1. Isometric cohomogeneity one actions. The sections are the normal geodesics of a regular orbit. These have been classified in special cases by Kollross and Berndt & Tamaru, although remains an open problem to classify all such actions.

2. A compact Lie group G with bi-invariant metric acting on itself by conjugation. The maximal tori are the sections.

3. Let N be a symmetric space. The identity component of the isometry group, $G = I(N)_0$, acts transitively on N. We can write $N = G/K$, where $K = G_p$ for some point $p \in N$, and (G, K) is called a *symmetric pair*. Then the isotropy action

$$K \times G/K \to G/K \ ; \quad (k, gK) \mapsto kgK$$

and its linearization $K \times (T_{[K]}G/K) \to T_{[K]}G/K$ at the the tangent space to the point [K], are hyperpolar. The sections are the maximal flat submanifolds through $[K]$, and their tangent spaces in $[K]$, respectively. These are called *s-representations*.

Weyl group

Let G a Lie group acting smoothly by isometries on a complete Riemannian manifold M, and assume the action is polar with section $i : \Sigma \to M$. Let

$$\begin{aligned} N := N_G(\Sigma) &= \{g \in G \mid g(i(\Sigma)) = i(\Sigma)\} \\ Z := Z_G(\Sigma) &= \{g \in G \mid gi(x) = i(x) \text{ for any } x \in \Sigma\} \end{aligned}$$

Definition: The Weyl group is

$$W_G(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$$

In the case of Example (2) above, this is just the usual Weyl group.

Category for polar actions

Theorem: (Hurder-Töben [2007]) Let G be a Lie group with a proper polar action on M, $i : \Sigma \to M$ a section, and $W = N_G(\Sigma)/Z_G(\Sigma)$ the generalized Weyl group acting on Σ. Then

$$\mathrm{cat}_G(M) \leq \mathrm{cat}_W(\Sigma) \tag{1}$$

Proof uses ideas and techniques developed for the study of the transverse LS category of Riemannian foliations (especially the lifting of foliated homotopies via the Ehresmann connection on Riemannian submersions.)

As an application, we obtain a well-known result of Wilhelm Singhoff:

Theorem: (Singhoff [1975]) The LS-categories of the unitary and the special unitary groups are $\mathrm{cat}(SU(n)) = n$ and $\mathrm{cat}(U(n)) = n + 1$.

Some open problems / works in progress

1. Develop relations between $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ and $\mathrm{cat}(M)$ for other Lie group actions.
2. Extend the proof of the Main Theorem to *Singular Riemannian Foliations*, and arbitrary isometric actions of connected Lie groups.
3. Classify the Riemannian foliations for which the action $\mathbf{O}(q)$ on $\widehat{W}$ is polar, or hyperpolar.
4. Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold. Relate $\mathrm{cat}^{e}_{\pitchfork}(M, \mathcal{F})$ to the transverse Euler characteristic (Hopf index) of $\mathcal{F}$.