Beyond Kuperberg Flows

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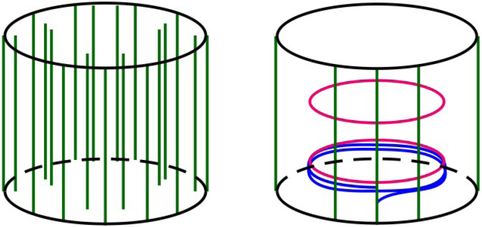
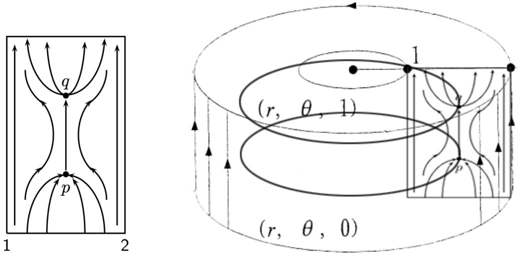
A plug $\mathbb{P} \subset \mathbb{R}^3$ is a 3-manifold with boundary, with a non-vanishing vector field that agrees with the vertical field on the boundary of $\mathbb{P}$:



Mirror Symmetry Property: An orbit entering a plug (from the bottom) either never leaves the plug (it is "trapped"), or exits the plug at the mirror image point at the top of the plug.

[Wilson Plug, 1966] Trap orbits so they limit to two periodic attractors in the plug.



[Schweitzer, 1974] , [Harrison, 1988] Replace the circular orbits of the Wilson Plug with Denjoy minimal sets, so trapped orbits limit to Denjoy minimal sets; but vector field cannot be smooth.

Theorem: [K. Kuperberg 1994] There is a smooth aperiodic plug.

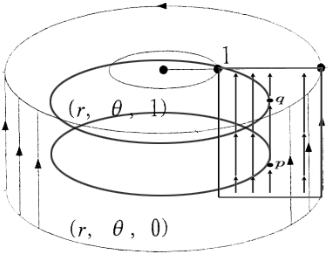
Application: Every closed, oriented 3-manifold M admits a non-vanishing smooth vector field $\mathcal{K}$ without periodic orbits.

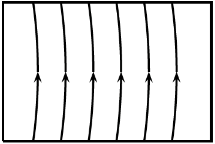
Goals of Talk:

- present construction of Kuperberg plugs
- consider dynamical properties
- consider dependence on parameters of construction

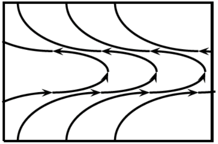
Kuperberg idea: Consider plugs constructed from a modification of the Wilson plug and use “flow surgery”.

Step 1: Modify the Wilson plug.

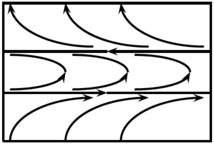




$r \approx 1,3$

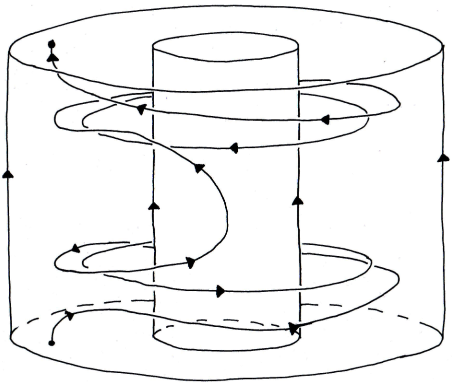


$r \approx 2$



$r = 2$

3d-orbits of $\mathcal{W}$ appear for $r > 2$ appear like:

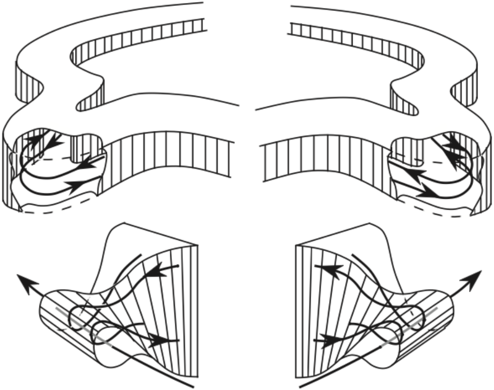


Shigenori Matsumoto's summary of strategy:

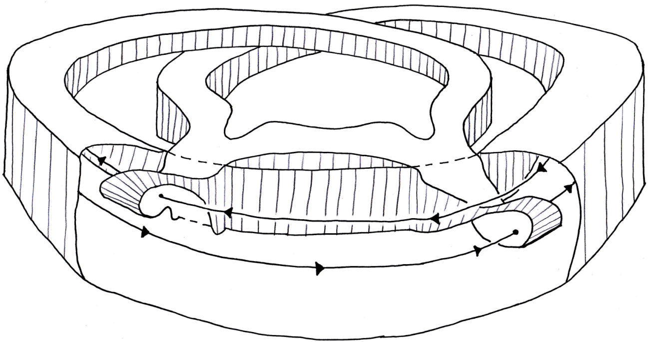
そこで，どうしても W 内のふたつの周期軌道 T_1 と T_2 を予め破壊しておく必要がある．しかしそのために新しい部品を開発するのでは話は振り出しに戻ってしまう．Kuperberg の発想は，W 内の**周期軌道自身で自分達を破壊させる**というものである．敵同士が妨害工作をしあうようにわなを仕掛けた後は，何もせずに黙ってみていればよいということである．

> *We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to let closed orbits demolish themselves. We set up a trap within enemy lines and watch them settle their dispute while we take no active part.*
>
> *(transl. by Kiki Hudson Arai)*

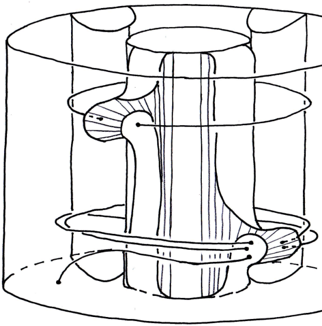
Step 2: Extrude two "horns"



Insert the horns. The vector field $\mathcal{W}$ induces a field $\mathcal{K}$ on the surgered manifold. Then the Kuperberg Plug is pictured as:



Wilson dynamics + insertions = Kuperberg dynamics



This is an aperiodic plug, as only chance for periodic orbit is via the circular Wilson orbits, and they get broken up.

Theorem: [Ghys, Matsumoto, 1994] A Kuperberg flow Φ_t has a *unique minimal set* $\Sigma \subset \mathbb{K}$.

Corollary: Ever orbit either escapes through a face, or limits to Σ.

Problem: *Describe the topological shape of* Σ*, and analyze the dynamics of* Φ_t *restricted to open neighborhoods of* Σ*.*

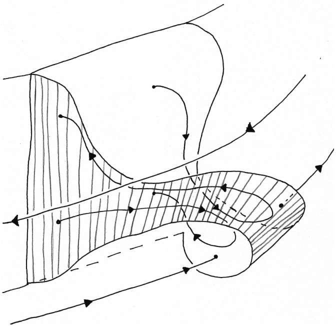
Theorem: [Katok, 1980] If a smooth flow on M^3 has positive topological entropy, then it has periodic orbits.

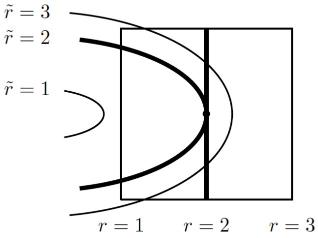
Hence, the Kuperberg flow Φ_t has topological entropy 0.

Problem: *What type of entropy-zero dynamical system does the restricted flow $\Phi_t|\Sigma$ flow yield? For example, is it an odometer?*

Problem: *How do the dynamical properties depend on the construction of the flow?*

Definition: A Kuperberg flow $\mathcal{K}$ is said to be *generic* if the singularities for the vanishing of the vertical part $g(r, \theta, z)\frac{\partial}{\partial z}$ of the Wilson vector field $\mathcal{W}$ are of quadratic type, and each insertion map σ_i for $i = 1, 2$ yields a quadratic radius function.





Theorem: A C^1-Kuperberg flow Φ_t has topological entropy 0.

Vanishing of entropy $h_{top}(\Phi_t) = 0$ is shown using the structure of the flow in the region $\{r \geq 2\}$ which contains the minimal set Σ, and standard but elementary estimates on the norms of derivatives.

The method is valid for C^1-flows.

Theorem: The minimal set for a *generic* smooth Kuperberg flow is a compact stratified lamination, denoted by $\mathfrak{M}$, where:

- 2-strata are open dense manifolds, coarsely isometric to trees;
- 1-strata is dense.

We call these "zippered laminations" as the 1-strata is the boundary seam along which the 2-strata is zipped to itself.

They resemble "fan continua".

Theorem: Let Φ_t be a *generic* smooth Kuperberg flow. Then the *topological shape* of $\mathfrak{M}$ is not *stable*, nor is it *moveable*.

The complexity of the Denjoy continuum is tame, as it is stable and shape equivalent to a wedge of two circles.

The complexity of the Kuperberg minimal set is wild, as its shape is a bouquet of circles which grow exponentially in number as the shape diameter is decreased.

Theorem: Let Φ_t be a *generic* smooth Kuperberg flow.
Let $\mathfrak{C}$ be the transversal Cantor set to $\mathfrak{M}$, and $\mathcal{G}_{\mathfrak{M}}$ the induced pseudogroup acting on $\mathfrak{C}$ by the holonomy of $\mathfrak{M}$. Then the pseudogroup entropy (resp. slow entropy) of $\mathcal{G}_{\mathfrak{M}}$ satisfies:

- $h_{GLW}(\mathcal{G}_{\mathfrak{M}}) = 0$;
- $h^{\alpha}_{GLW}(\mathcal{G}_{\mathfrak{M}}) > 0$ for $\alpha = 1/2$.

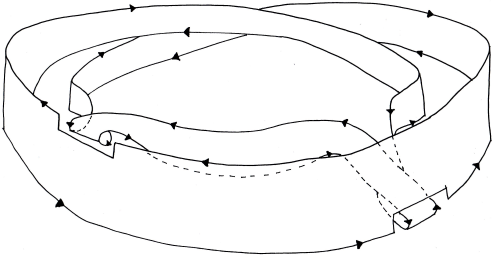
The growth rates of (ϵ, n)-separated sets for the action of $\mathcal{G}_{\mathfrak{M}}$ on $\mathfrak{C}$ is at least $\sim \exp(\sqrt{n})$ for $\epsilon > 0$ small and $n \to \infty$.

Stable manifold

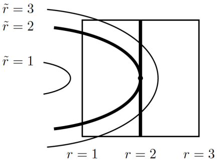
Reeb cylinder for the *Wilson* flow is a type of forward/backward stable manifold

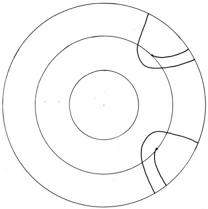
$$\mathcal{R} = \{(2, \theta, z) \mid 0 \leq \theta \leq 2\pi \ \& \ -1 \leq z \leq 1\} \subset \mathbb{W}$$

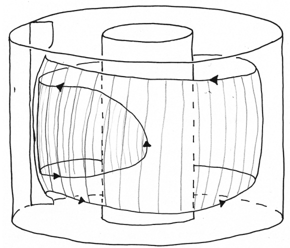
$\mathcal{R}'$ is Reeb cylinder minus insertions to construct Kuperberg plug:



The stable manifold intersect with the faces of insertions is parabolic curve down. Change of coordinates to bottom face gives a parabolic curve upwards. It's Wilson flow is a tongue which wraps around the stable manifold $\mathcal{R}'$ again.

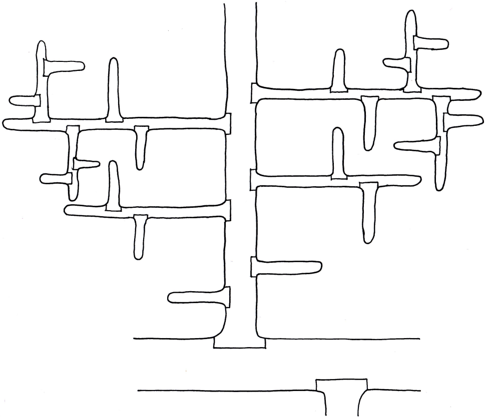






Propellers and levels

Introduce Φ_t-invariant sets $\mathfrak{M}_0 = \bigcup_{t \in \mathbb{R}} \Phi_t(\mathcal{R}')$:



Levels and the lamination $\mathfrak{M}$

$$\mathfrak{M}_0 \;=\; \mathcal{R}' \;\cup\; \mathfrak{M}_0^1 \;\cup\; \mathfrak{M}_0^2 \;\cup\; \cdots \quad ; \quad \mathfrak{M} \equiv \overline{\mathfrak{M}_0} \subset \mathbb{K}$$

The space $\mathfrak{M}_0$ decomposes into unions of disjoint propellers at level $\ell \geq 1$, corresponding to how many insertions are required to generate it. $\mathfrak{M}$ is closed and flow invariant, so $\Sigma \subset \mathfrak{M}$.

Boundary of the propeller $\mathfrak{M}_0$ is the Kuperberg orbit of the periodic orbits in Wilson flow.

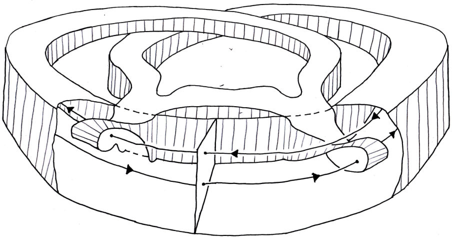
This shows explicitly why the periodic orbit gets opened up. All other orbits with $r = 2$ limit to this infinite orbit.

The transverse section

The main idea is to use a section to the flow Φ_t to convert to the study of a pseudogroup action on the plane.

$$\mathbf{R}_0 = \{(r, \pi, z) \mid 1 \leq r \leq 3 \ \& \ -2 \leq z \leq 2\}$$

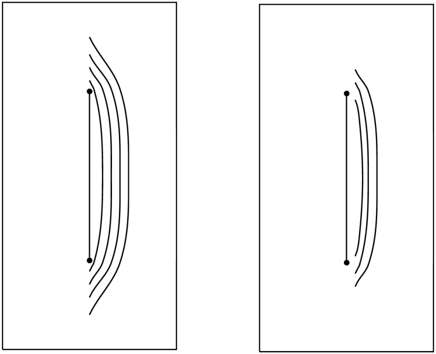
disjoint from insertions. Periodic orbits for Wilson flow intersect $\mathbf{R}_0$ in two special points $\omega_1 = (2, \pi, -1)$ and $\omega_2 = (2, \pi, 1)$.



Transverse pseudogroup

The flow Φ_t induces a pseudogroup $\mathcal{G}_K$ on $\mathbf{R}_0$. There are five generators $\{\psi, \phi_1^{\pm}, \phi_2^{\pm}\}$, which represent basic actions of the induced Wilson flow and the insertions in the dynamical model.

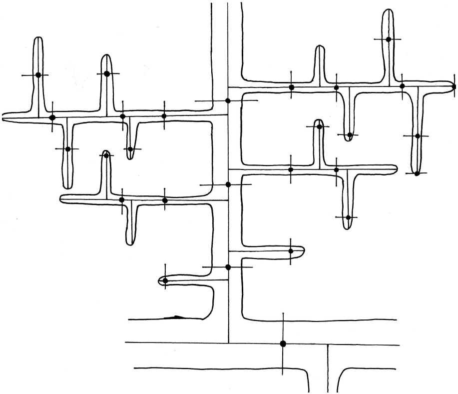
The intersection $\mathfrak{M}_0 \cap \mathbf{R}_0$ of the propellers with the section reveals the orbit structure: (except, the intersection $\mathfrak{M} \cap \mathbf{R}_0$ is a perfect set!)



Cantor transversal

$\mathcal{T} = \{(r, \pi, 0) \mid 1 \leq r \leq 3\} \subset \mathbf{R}_0$ is transverse to $\mathfrak{M}_0$.

Theorem: $\mathfrak{C} = \mathfrak{M} \cap \mathcal{T}$ is a Cantor set, which is a complete transversal for the open leaves $\mathcal{L} \subset \mathfrak{M}$.



Lamination entropy

For $\epsilon > 0$, say that $\xi_1, \xi_2 \in \mathfrak{C}$ are (n, ϵ)-separated if there exists $\varphi \in \mathcal{G}_{\mathfrak{M}}^{(n)}$ with $\xi_1, \xi_2 \in Dom(\varphi)$, and $d_{\mathfrak{C}}(\varphi(\xi_1), \varphi(\xi_2)) \ \geq \ \epsilon$.

A finite set $\mathcal{S} \subset \mathfrak{C}$ is said to be (n, ϵ)-separated if every distinct pair $\xi_1, \xi_2 \in \mathcal{S}$ are (n, ϵ)-separated.

Let $s(\mathcal{G}_{\mathfrak{M}}, n, \epsilon)$ be the maximal cardinality of an (n, ϵ)-separated subset of $\mathfrak{M}$.

The *lamination entropy* of $\mathcal{G}_{\mathfrak{M}}$ is defined by:

$$h(\mathcal{G}_{\mathfrak{M}}) \;=\; \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{1}{n} \ln(s(\mathcal{G}_{\mathfrak{M}}, n, \epsilon)) \right\}.$$

The limit $h(\mathcal{G}_{\mathfrak{M}})$ depends on the generating set, but the fact of being non-zero does not.

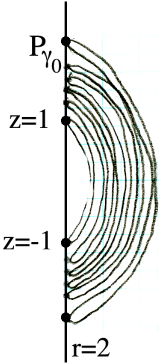
For $0 < \alpha < 1$, the *slow lamination entropy* of $\mathcal{G}_{\mathfrak{M}}$ is defined by:

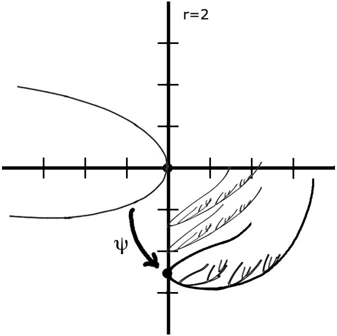
$$h_{\alpha}(\mathcal{G}_{\mathfrak{M}}) \;=\; \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{1}{n^{\alpha}} \ln(s(\mathcal{G}_{\mathfrak{M}}, n, \epsilon)) \right\}.$$

How to generate an (n, ϵ)-separated subset of $\mathcal{S}(n, \epsilon) \subset \mathfrak{M}$?

Use the "geometry" of the pseudogroup action.

The parabolic arcs are one-half of a stretched ellipse, and the action of $\mathcal{G}_K$ maps ellipses into ellipses, so can get a good game of ping-pong up and running. The only catch is that the game runs slow. The n^{th}–volley takes approximately n^2 steps.





Remarks and Problems

Theorem: [H & R] In every C^1-neighborhood of $\mathcal{K}$, there exists a smooth flow Φ'_t on $\mathbb{K}$ with positive entropy, and the associated invariant lamination $\mathfrak{M}'$ is the suspension of a horseshoe. We expect the entropy of this to be positive.

Theorem: [H & R] There exists a piecewise-smooth Kuperberg Plug (with a break at each periodic orbit of Wilson) so that it has positive lamination entropy.

Problem: Give a description of the dynamical properties of flows C^1-close to a generic Kuperberg flow.

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