# Solenoidal minimal sets in foliations

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# Laminations - back to the 60's

A $p$-dimensional lamination $\mathcal{L}$ is a compact foliated space $\mathbb{X}$ modeled transversally on a continua: there is a compact metric space $\mathcal{T}$ and an open covering of $\mathbb{X}$ by flow-boxes

$$\{\phi_\ell \colon (-1,1)^p \times V_\ell \to U_\ell \subset \mathbb{X} \mid 1 \leq \ell \leq k\} \ , \ V_\ell \subset \mathcal{T}$$

so that if $U_k \cap U_\ell \neq \emptyset$, then $\phi_{\ell k} = \phi_\ell \circ \phi_k^{-1}$ maps open subsets of the "horizontal" slices $(-1,1)^p \times \{x\}$ to horizontal slices, where defined.

The leaf of $\mathcal{L}$ through $x \in \mathbb{X}$ is the connected component $L_x$ of $\mathbb{X}$ containing $x$, where $\mathbb{X}$ is given the fine topology generated by the open subsets of the form $\phi_\ell(W \times \{y\})$ where $W \subset (-1,1)^p$ is open and $y \in V_\ell$.

$\mathcal{L}$ is *minimal* if every leaf $L_x$ is dense in $\mathbb{X}$ for the metric topology.

# Minimal sets in foliations

Let $\mathcal{F}$ be a codimension $q$, $C^r$-foliation of a closed manifold $M$, for $r \geq 0$.

For each leaf $L$ of $\mathcal{F}$, its closure $\mathbb{K} = \overline{L}$ is a minimal set for $\mathcal{F}$ if for each $y \in \mathbb{K}$, the closure of the leaf through $y$ is all of $\mathbb{K}$ : $\overline{L_y} = \mathbb{K}$.

Zorn's Lemma $\Longrightarrow$ closure of any leaf $L$ contains a minimal set.

**Basic Fact:** Let $\mathbb{K}$ be a minimal set for $\mathcal{F}$ then the restriction $\mathcal{F}|\mathbb{K}$ is a minimal lamination.

# Basic Questions

**Basic Problem:** Can you characterize the minimal laminations which arise as minimal sets of $C^r$-foliations, $r \geq 0$?

**Remark:** For a $\mathbb{K}$ which is the minimal set of a foliation, the model space $\mathcal{T}$ has natural local embeddings into Euclidean space $\mathbb{R}^q$ where $q$ is the codimension of $\mathcal{F}$.

For a codimension-one foliation $\mathcal{F}$, the model space $\mathcal{T}$ must either be Euclidean space, or a Cantor set. But do not have a quasi-isometric characterization of the possible Cantor sets.

**Open Question:** If $\mathcal{F}$ is codimension-one $C^2$-foliation and $\mathbb{K}$ is a minimal set modeled on a Cantor set, must $\mathbb{K}$ have Lebesgue measure zero?

**Starter Problem:** Given a class of laminations, construct foliations which have minimal sets homeomorphic to laminations in this class.

## Special laminations: Solenoids

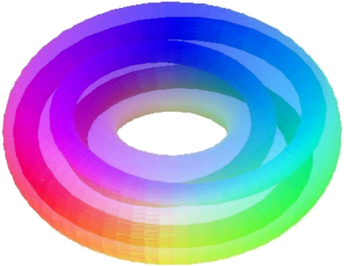
Let $L_\ell = L$ be a closed $p$-dimensional manifold for all $\ell \geq 0$.

Let $f \colon L \to L$ be a non-trivial covering map.

Set $f_\ell = f \colon L_\ell \to L_{\ell-1}$ for $\ell > 0$.

**Definition:** $\mathcal{S} = \varprojlim \{f_\ell \colon L_\ell \to L_{\ell-1}\}$ is the solenoid defined by $f \colon L \to L$.

**Example:** $f \colon \mathbb{S}^1 \to \mathbb{S}^1$, where $f(z) = z^k$ for some integer $k > 1$.



# Generalized solenoids

In the most general case, let $f_\ell \colon L_\ell \to L_{\ell-1}$ be a sequence of non-trivial covering maps of closed (branched) manifolds, for $\ell \geq 1$. Then the (generalized) solenoid defined by this data is

$$\mathcal{S} = \varprojlim \{ f_\ell \colon L_\ell \to L_{\ell-1} \}$$

**Problem:** Given a generalized solenoid $\mathcal{S}$ with $p$-dimensional leaves, when does there exists a $C^r$-foliation $\mathcal{F}$ of a compact manifold such that $\mathcal{S}$ is homeomorphic to a minimal set for $\mathcal{F}$?

# Reeb–Thurston–Stowe Stability Theorems

Let $\mathcal{F}$ be a $C^r$-foliation of a smooth compact manifold $M$, for $r \geq 1$.

**Theorem:** (Reeb [1952]) Let $L$ be a compact leaf of a codimension one foliation $\mathcal{F}$ such that $\pi_1(L, x) = 0$. Then there exists an open saturated neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation.

**Theorem:** (Thurston [1974]) Let $L$ be a compact leaf of a codimension one foliation $\mathcal{F}$ such that $H^1(L, \mathbb{R}) = 0$. Then there exists an open saturated neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation.

**Theorem:** (Stowe [1983]) Let $L$ be a compact leaf of a codimension $q$ foliation $\mathcal{F}$ such that $H^1(L, \mathbb{V}) = 0$ for all flat finite-dimensional vector bundles associated to a representation of $\pi_1(L, x)$. Then there exists an open saturated neighborhood $L \subset U$ such that if $\mathcal{F}'$ is a sufficiently $C^1$ close to $\mathcal{F}$, then $\mathcal{F}' \mid U$ is a product foliation.

# Instability of leaves

**Theorem 1:** (Clark & Hurder [2006]) $\mathcal{F}$ is a codimension $q$ $C^1$-foliation. Let $L$ be a compact leaf with $H^1(L, \mathbb{R}) \neq 0$, and $L \subset U$ is a saturated open neighborhood for which $\mathcal{F} \mid U$ is a product foliation. Then there exists a $C^1$-foliation $\mathcal{F}'$ arbitrarily $C^1$-close to $\mathcal{F}$ such that

- $U$ is saturated for $\mathcal{F}'$,
- $\mathcal{F} = \mathcal{F}'$ on $M - U$,
- $\mathcal{F}' \mid U$ contains a compact minimal set $\mathbb{K} \subset U$ with

$$\mathbb{K} \cong \mathcal{S} = \varprojlim \{f_\ell \colon L_\ell \to L_{\ell-1}\}$$

where $L_0 = L$, and each $L_\ell$ is a non-trivial covering of $L_{\ell-1}$.

For example, if $p = 2$ and $L$ is an oriented surface with $H^1(L; \mathbb{R}) \neq 0$, then there exists $C^1$-perturbations of $\mathcal{F}$ for which the leaf $L$ has nearby solenoidal minimal sets.

# Realization of solenoids

The construction technique used to prove the Theorem 1 has other applications. For example:

Fix an integer $N \geq 2$. Let $\Gamma_\ell$ be a sequence of finite groups, for $\ell \geq 1$, such that their orders satisfy a uniform bound $2 \leq |\Gamma_\ell| \leq N$.

Let $\mathbf{K} = \Pi_{\ell=1}^{\infty} \Gamma_\ell$ be the product (Cantor) space, with the sequence metric.

**Theorem 2:** There exists a $C^0$-foliation $\mathcal{F}$ of a compact manifold $M$ with leaf dimension $p = 2$ and codimension $q = N!$ which has a compact leaf $L_0 = \Sigma_N$ (surface of genus $N$) and a solenoidal minimal set $\mathbb{K}$ near to $L_0$ so that the transverse geometry of $\mathbb{K}$ is quasi-isometric to $\mathbf{K}$.

# Flat bundles

Choose a basepoint $x \in L$, and set $\Gamma = \pi_1(L, x)$.

$\Gamma$ acts on the right as deck transformations of the universal cover $\widetilde{L} \to L$.

Let $\rho \colon \Gamma \to \mathbf{SO(q)}$ be an orthogonal representation.

$\Gamma$ acts on the left as isometries of $\mathbb{R}^q$ by $\gamma \cdot \vec{v} = \rho(\gamma)\vec{v}$.

Define a flat $\mathbb{R}^q$-bundle with holonomy $\rho$ by

$$\mathbb{E}^q_\rho = (\widetilde{L} \times \mathbb{R}^q)/(\widetilde{y} \cdot \gamma, \vec{v}) \sim (\widetilde{y}, \gamma \cdot \vec{v}) \ \longrightarrow \ L$$

The most familiar example is for $L = \mathbb{S}^1$ and $\Gamma = \pi_1(\mathbb{S}^1, x) = \mathbb{Z} \to \mathbf{SO(2)}$. Then $\mathbb{E}^2_\rho$ is the flat vector bundle over $\mathbb{S}^1$ with the foliation by lines of slope $\rho(1) = \exp(2\pi\sqrt{-1}\,\alpha)$.

In general, the bundle $\mathbb{E}^q_\rho \to L$ need not be a product vector bundle.

# Trivializing flat bundles

**Proposition:** Suppose that there exists a 1-parameter family of representations $\{\rho_t \colon \Gamma \to \mathbf{SO(q)}\}$ such that $\rho_0$ is the trivial map, and $\rho_1 = \rho$, then $\{\rho_t\}$ induces a vector bundle trivialization, $\mathbb{E}^q_\rho \cong L \times \mathbb{R}^q$.

**Proof:** The family of representations defines a family of flat bundles $\mathbb{E}^q_{\rho_t}$ over the product space $L \times [0,1]$. This defines an isotopy between the bundles $\mathbb{E}^q_{\rho_0}$ and $\mathbb{E}^q_{\rho_1}$, which induces a bundle isomorphism between them. The initial bundle $\mathbb{E}^q_{\rho_0}$ is a product, hence the same holds for $\mathbb{E}^q_{\rho_1}$.

In the case of the example above over $\mathbb{S}^1$ the product structure can be written down explicitly, so that we can "view" the resulting 2-torus with foliation having lines of slope $\alpha$.

A key point is that the bundle isomorphism between $\mathbb{E}^q_{\rho_0}$ and $\mathbb{E}^q_{\rho_1}$ depends smoothly on the path $\rho_t$.

# Abelian representations

$k =$ the greatest integer such that $2k \leq q$.

$\mathbb{T}^k \subset \mathbf{SO(q)}$ a maximal embedded $k$-torus.

$\xi = (\xi_1, \ldots, \xi_k) \colon \Gamma \to \mathbb{R}^k$ a representation. Define

$$\begin{array}{rcl} \rho_t^{\xi} \colon \Gamma & \to & \mathbf{SO(q)} \\ \rho_t^{\xi}(\gamma) & = & [\exp(2\pi t\sqrt{-1}\xi_1(\gamma), \ldots, \exp(2\pi t\sqrt{-1}\xi_k(\gamma))] \end{array}$$

This is a 1-parameter family of orthogonal representations. Set

$$\begin{array}{rcl} \mathbb{D}_{\epsilon}^{q} & = & \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 < \epsilon\} \subset \mathbb{R}^q \\ \mathbb{B}_{\epsilon}^{q} & = & \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 \leq \epsilon\} \subset \mathbb{R}^q \\ \mathbb{S}_{\epsilon}^{q-1} & = & \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 = \epsilon\} \subset \mathbb{R}^q \end{array}$$

# Realizing abelian representations

**Proposition:** $\xi\colon \Gamma \to \mathbb{R}^k$ defines a flat bundle foliation $\mathcal{F}_\xi$ of $L \times \mathbb{S}^{q-1}$ whose leaves cover $L$. Moreover, if the image of $\xi$ is contained in the rational points $\mathbb{Q}^k \subset \mathbb{R}^k$, then all leaves of $\mathcal{F}_\xi$ are compact.

**Proof:** $\rho_t^\xi$ is an isotopy from $\xi$ to the trivial representation. $\square$

**Basic Observation:** Given a path $\lambda\colon [0,\epsilon] \to \mathbf{Rep}(\Gamma, \mathbf{SO(q)})$ of representations with $\lambda(\epsilon)$ the trivial representation, we obtain a foliation $\mathcal{F}_\lambda$ of $L \times \mathbb{D}^q_\epsilon$ so that

- the restriction of $\mathcal{F}_\lambda$ to the spherical fiber $L \times \mathbb{S}^{q-1}_s$ is $\mathcal{F}_{\lambda(s)}$
- the restriction of $\mathcal{F}_\lambda$ to the spherical fiber $L \times \mathbb{S}^{q-1}_\epsilon$ is the product foliation.

# The basic plug

Suppose we are given a codimension-$q$, $C^1$-foliation $\mathcal{F}$, a compact leaf $L$ with $H^1(L,\mathbb{R}) \neq 0$, and $L \subset U$ is a saturated open neighborhood for which $\mathcal{F} \mid U$ is a product foliation. We can assume that $U = L \times \mathbb{D}^q_\epsilon$.

Fix a non-trivial representation $\xi_1 \colon \Gamma \to \mathbb{Q}^k$ which exists as $H^1(L,\mathbb{R}) \neq 0$.

Let $0 < \epsilon/2 < \epsilon_1 < \epsilon$, and set $\epsilon_1' = (\epsilon_1 + \epsilon)/2$. Choose a monotone decreasing smooth function $\mu_1 \colon [0,\epsilon] \to [0,1]$ such that

$$\mu_1(s) = \begin{cases} 1 & \text{if } \ 0 \leq s \leq \epsilon_1, \\ 0 & \text{if } \ \epsilon_1' \leq s \leq \epsilon \end{cases}$$

Set $\rho^{\xi_1}_{1,s} = \rho^{\mu_1(s)\xi_1} \colon \Gamma \to \mathbf{SO(q)}$. Use this family of representations to define a foliation $\mathcal{F}_1$ of $N_1 = L \times \mathbb{D}^q_\epsilon$.

Note that $\mathcal{F}_1$ is the product foliation outside of $L \times \mathbb{D}^q_{\epsilon_1'}$, and has all leaves compact inside $L \times \mathbb{B}^q_{\epsilon_1}$ and outside of $L \times \mathbb{B}^q_{\epsilon_1'}$

# Iterating the plug

Let $L_1$ be a generic leaf of $\mathcal{F}_1$ contained in $L \times \mathbb{S}^{q-1}_{\epsilon_1/2}$.

By construction, $L_1 \longrightarrow L$ is the compact covering associated to the kernel $\Gamma_1 \subset \Gamma$ of the homomorphism $\rho^{\xi_1} \colon \Gamma \to \mathbf{SO(q)}$.

Next choose $0 < \epsilon_2 < \epsilon$ sufficiently small so that $\mathcal{F}_1$ restricted to the $\epsilon_2$-disk bundle $N_2$ about $L_1$ is a product foliation.

We now repeat the construction: choose a non-trivial map $\xi_2 \colon \Gamma_1 \to \mathbb{Q}^k$ and maps $\mu_2$ as before.

# Iterating the plug 2

Iterate for all $n \geq 2$. This yields:

- A descending sequence of subgroups $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$
- An increasing sequence of open saturated subsets, $V_1 \subset V_2 \subset \cdots$ where

$$V_n = L \times \mathbb{D}^q_\epsilon - L_{n+1} \times \mathbb{B}^q_{\epsilon_{n+1}}$$

- $\mathbf{K}_n = L \times \mathbb{D}^q_\epsilon - V_n$ forms a nested sequence of compact sets, $\mathbf{K}_n \supset \mathbf{K}_{n+1}$
- Smooth foliations $\mathcal{F}'_n$ of $L \times \mathbb{D}^q_\epsilon$, such that all leaves of the restriction $\mathcal{F}'_n | \mathbf{K}_n$ are coverings of $L$ that have increasingly high order, bounded below by the orders of the subgroups $\Gamma_n$.

# The perturbation $\mathcal{F}'$

**Proposition:** If the maps $\xi_n$ are suitably chosen (i.e., the images of the generators of $\Gamma$ approach 0 in $\mathbb{Q}^k$ sufficiently rapidly) then:

1. the foliations $\mathcal{F}'_n$ converge to a $C^r$-foliation $\mathcal{F}'$ of $L \times \mathbb{D}^q_\epsilon$.
2. $\mathbf{K} = \cap_{n=1}^{\infty} \mathbf{K}_n$ is a saturated compact set.
3. $\mathcal{F}' \mid \mathbf{K}$ is a solenoid.

**Question:** If $\mathcal{F}$ is $C^1$, must the orders of the quotient groups $\Gamma_n / \Gamma_{n+1}$ be unbounded?

**Remark:** The key property used above is that there is a representation $\rho \colon \Gamma = \pi_1(L) \to \mathbf{SO(q)}$ that is connected to the identity. This suggest the dichotomy: either $\Gamma = \pi_1(L)$ is a Kahzdan group, or there exists perturbations of $\mathcal{F}$ with solenoidal minimal sets. No idea how to do this.