Classification problems for laminations

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A *generalized lamination* is a compact connected metric space $\mathfrak{M}$, which admits a covering by open sets, each homeomorphic to a product of a totally disconnected space with an open subset of euclidean space $\mathbb{R}^n$. The path components of $\mathfrak{M}$ define the leaves of dimension n and form a foliation $\mathcal{F}$ of $\mathfrak{M}$.

- A lamination is a hybrid of a compact manifold and a profinite group or groupoid structure.

- We call these *matchbox manifolds* in our works, following usage introduced by Aarts & Oversteegen.

- These arise in the study of aperiodic tilings, minimal sets for dynamical systems, the study of complex rational maps, ...

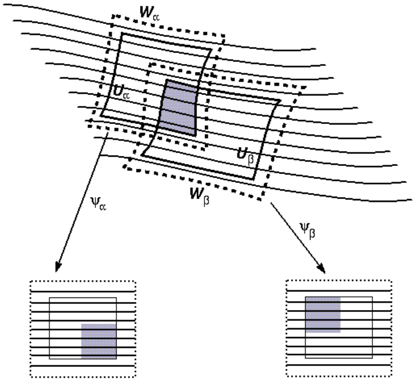
Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

- $\mathfrak{M}$ is a continuum $\equiv$ a compact, connected metric space;
- $\mathfrak{M}$ admits a covering by foliated coordinate charts
  $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$;
- each $\mathfrak{X}_i$ is a *clopen* subset of a *totally disconnected* space $\mathfrak{X}$;
- plaques $\mathcal{P}_i(z) = \varphi_i^{-1}([-1,1]^n \times \{z\})$ are connected, $z \in \mathfrak{X}_i$;
- for $U_i \cap U_j \neq \emptyset$, each plaque $\mathcal{P}_i(z)$ intersects at most one plaque $\mathcal{P}_j(z')$, and change of coordinates along intersection is smooth diffeomorphism;

$+$ some other technicalities.

The path connected components of $\mathfrak{M}$ are the leaves of the foliation $\mathcal{F}$.

The charts $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i$ on the open covering satisfy a compatibility condition on their overlaps.



Problem: Given matchbox manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of the same leaf dimension $n \geq 1$, find invariants which are sufficient to imply that the spaces are homeomorphic.

"Manifold invariants" $+$ "algebraic invariants" $=$ "Classification"

Problem: Given a matchbox manifold $\mathfrak{M}$, what are the properties of the group of self-homeomorphisms **Homeo**$(\mathfrak{M})$.

- $\mathfrak{M}$ is *homogeneous* if **Homeo**$(\mathfrak{M})$ acts transitively on $\mathfrak{M}$.

Problem: [Bing] Characterize the homogeneous laminations.

Theorem: [Hagopian (1977), Aarts, Hagopian & Oversteegen (1991)] The homogeneous 1-dimensional matchbox manifolds are precisely the Vietoris solenoids.

Theorem: [Fokkink (1991), Aarts and Oversteegen (1995)] Two orientable, minimal, 1–dimensional matchbox manifolds are homeomorphic if and only if they are return equivalent.

Theorem: [Clark & Hurder (2013)] The homogeneous n-dimensional matchbox manifolds are precisely the regular solenoids.

Classic example: Vietoris solenoid (1927), defined by tower of coverings:

$$\mathcal{P} \equiv \cdots \longrightarrow \mathbb{S}^1 \xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

where each p_ℓ is a covering map of degree $n_\ell > 1$.

$\mathcal{P}$ is called a presentation. Set $n_{\mathcal{P}} = \{n_1, n_2, n_3, \ldots\}$. Can assume that each $n_i > 1$ is prime, otherwise randomly chosen.

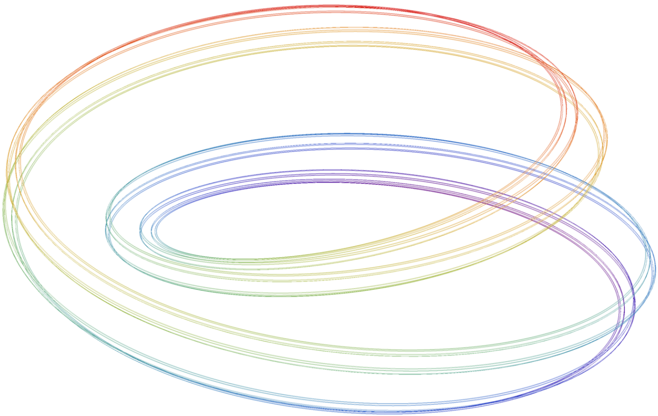
$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell \geq 0} \mathbb{S}^1$$

$\mathcal{S}_{\mathcal{P}}$ is given the (relative) product topology.

Proposition: The space $\mathcal{S}_{\mathcal{P}}$ is a matchbox manifold, for which every leaf is diffeomorphic to $\mathbb{R}$, and dense in $\mathcal{S}_{\mathcal{P}}$.

Van Dantzig - Vietoris solenoid

The case where $n_i = 2$ for all $i \geq 1$



Proposition: The homeomorphism type of $\mathcal{S}_{\mathcal{P}}$ depends only on the set of integers $n_{\mathcal{P}}$.

More is true. Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations, and let P be the infinite set of prime factors of the integers in the set $n_{\mathcal{P}}$, included with multiplicity, and Q the same of $n_{\mathcal{Q}}$.

Theorem: [Bing (1960), McCord (1965), Aarts & Fokkink (1991)] The solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic, if and only if there is a *bijection* between a cofinite subset of $P = (p_1, p_2, \ldots)$ with a cofinite subset of $Q = (q_1, q_2, \ldots)$.

Conclusion: no matter how chaotic the choice of the sequence of primes, the classification reduces to an analytic problem in the sense of descriptive set theory.

A weak solenoid $\mathcal{S}_{\mathcal{P}}$ is obtained by considering a closed connected n-manifold M, and a *presentation*

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$$

which is a collection of maps satisfying:

- M_ℓ is a connected compact manifold of dimension n;
- each *bonding map* $p_{\ell+1}$ is a proper covering map. Then set

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \; \subset \prod_{\ell \geq 0} M_\ell$$

- $\mathcal{S}_{\mathcal{P}}$ is given the restriction of the product topology.

Proposition:[McCord (1965)] A weak solenoid is a matchbox manifold.

Examples: 2-*dimensional matchbox manifolds.*

M_0 is the 2-torus, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Suppose that $A \in GL(2,\mathbb{Z}) \subset GL(2,\mathbb{R})$ is a 2×2 invertible integer matrix, then $\Gamma_A = A \cdot \mathbb{Z}^2$ is a subgroup of finite index in $\Gamma_0 = \mathbb{Z}^2$.

Then there is an induced proper covering map $\phi_A \colon \mathbb{T}^2 \to \mathbb{T}^2$, where the degree of the covering is the index of Γ_A in Γ_0, which equals the determinant $\det(A) \in \mathbb{Z}$.

Given an infinite collection $\mathcal{A} \equiv \{A_\ell \in GL(2,\mathbb{Z}) \mid \ell = 1, 2, \ldots\}$ set $p_\ell = \phi_{A_\ell}$ and we obtain a presentation

$$\mathcal{P}_\mathcal{A} = \{p_\ell \colon \mathbb{T}^2 \to \mathbb{T}^2 \mid \ell = 1, 2, \ldots\}$$

which defines a solenoid $\mathcal{S}(\mathcal{A})$.

Fact: The solenoids of the form $\mathcal{S}(\mathcal{A})$ are not classifiable.

Examples: 2-*dimensional matchbox manifolds.*

Let $M_0 = \Sigma_g$ be a Riemann surface of genus $g \geq 2$, pick a basepoint $x_0 \in M_0$ and let $\Gamma_0 = \pi_1(M_0, x_0)$.

Γ_0 is residually finite, and each subgroup $\Gamma \subset \Gamma_0$ of finite index defines a proper covering $\pi \colon \Sigma_\Gamma \to \Sigma_g$. Given an infinite descending chain of subgroups with Γ_{i+1} finite index in Γ_i,

$$\mathcal{G} \equiv \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

Let $M_\ell = \Sigma_{\Gamma_\ell}$ and $p_{\ell+1} \colon M_{\ell+1} \to M_\ell$ be the induced proper covering maps. Then this defines a presentation $\mathcal{P}_\mathcal{G}$ and a corresponding solenoid $\mathcal{S}(\mathcal{G})$.

Note that the intersection $\bigcap_{n \geq 0} \Gamma_n$ need not be the trivial group.

Fact: The solenoids of the form $\mathcal{S}(\mathcal{G})$ are not classifiable.

Examples: 3-*dimensional matchbox manifolds.*

Let $\widetilde{M}_0 = \mathbb{H}$ be the real Heisenberg group, presented in the form $\mathbb{H} = (\mathbb{R}^3, *)$ with the group operation $*$ given by $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$. This operation is standard addition in the first two coordinates, and addition with a twist in the last coordinate. Let $\mathcal{H} = (\mathbb{Z}^3, *)$ be the integer lattice subgroup, so that $M_0 = \mathbb{H}/\mathcal{H}$ is a compact 3-manifold.

Consider subgroups of $\mathcal{H}$ which can be written in the form $\Gamma = M\mathbb{Z}^2 \times m\mathbb{Z}$ where $M = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is a 2-by-2 matrix with non-negative integer entries and $m > 0$ is an integer. Then $\gamma \in \Gamma$ is of the form $\gamma = (ix + jy, kx + ly, mz)$ for some $x, y, z \in \mathbb{Z}$. A straightforward computation gives the following lemma.

Theorem: [Dyer, Hurder, Lukina (2015)] Let $A_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$, p and q are distinct primes. Define the group chain $G_0 = \mathcal{H}$, $\{G_n\}_{n \geq 1} = \{A_n \mathbb{Z}^2 \times p^n \mathbb{Z}\}_{n \geq 1}$. Then the weak solenoid $\mathcal{S}(\{A_n\})$ defined by the coverings of M_0 associated to this chain is not homogeneous.

Note that the intersection $\bigcap_{n \geq 0} G_n = \{0\}$.

This implies the leaves of the foliation $\mathcal{F}$ on $\mathcal{S}(\{A_n\})$ are all isometric to the real Heisenberg group $\mathbb{H}$.

By a result of Clark, Hurder & Lukina (2013), the classification problem for the Heisenberg solenoids reduces to an algebraic problem, as we discuss next.

Definition: Let G be a finitely generated group. A *group chain* $\mathcal{G} = \{G_i \mid i \geq 0\}$, with $G_0 = G$, is a properly descending chain of subgroups of G, such that $|G : G_i| < \infty$ for every $i \geq 0$.

Definition: [Rogers & Tollefson (1971)] Let G be a finitely generated group, and $\{G_i\}_{i \geq 0}$ and $\{H_i\}_{i \geq 0}$ be group chains with $G_0 = H_0 = G$. We say they are *equivalent*, if and only if, there is a group chain $\{K_i\}_{i \geq 0}$ and infinite subsequences $\{G_{i_k}\}_{k \geq 0}$ and $\{H_{j_k}\}_{k \geq 0}$ such that $K_{2k} = G_{i_k}$ and $K_{2k+1} = H_{j_k}$ for $k \geq 0$.

$\{G_i\}_{i \geq 0}$ and $\{H_j\}_{j \geq 0}$ are *weakly equivalent* if there exists $i_0 \geq 0$ and $j_0 \geq 0$ such that the subchains $\{G_i\}_{i \geq i_0}$ and $\{H_j\}_{j \geq j_0}$ are equivalent.

Definition: [Fokkink & Oversteegen (2002)] Let G be a finitely generated group, and $\{G_i\}_{i\geq 0}$ and $\{H_i\}_{i\geq 0}$ be group chains with $G_0 = H_0 = G$. We say they are *conjugate equivalent*, if and only if, there exists a collection $(g_i) \in G$, such that the group chains $\{g_i G_i g_i^{-1}\}_{i\geq 0}$ and $\{H_i\}_{i\geq 0}$ are equivalent. Here $g_i G_i = g_j G_i$ for all $i \geq 0$ and all $j \geq i$.

$\{G_i\}_{i\geq 0}$ and $\{H_j\}_{j\geq 0}$ are *weakly conjugate equivalent* if there exists $i_0 \geq 0$ and $j_0 \geq 0$ such that the subchains $\{G_i\}_{i\geq i_0}$ and $\{H_j\}_{j\geq j_0}$ are conjugate equivalent.

- G abelian $\Longrightarrow$ conjugate equivalence $\equiv$ equivalence.

Let G be a finitely generated group. Denote by $\mathfrak{G}$ the collection of *all possible* nested proper chains of subgroups of finite index.

Proposition: (Weak) *equivalence* and *conjugate equivalence* of group chains form equivalence relations on $\mathfrak{G}$.

Problem: Let G be a finitely generated group. Determine the (weak) equivalence and conjugate equivalence classes in $\mathfrak{G}$.

Let G be a finitely generated group, and $\mathcal{G} \in \mathfrak{G}$ a group chain.

$$X = G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$$

is a Cantor set, and there is a minimal action $\phi\colon G \to \mathbf{Homeo}(X)$ given by left multiplication on each factor.

This is a *Cantor minimal system*, and is equicontinuous.

Equicontinuous $\Leftrightarrow$ orbits of pairs of points stay uniformly close.

Expansive $\Leftrightarrow$ orbits of pairs of points spread uniformly apart.

Theorem: [Auslander, Glasner & Weiss (2007)] A Cantor minimal system (X, G, ϕ) is either equicontinuous or expansive. That is, there are no distal Cantor minimal systems.

Definition: Let (X_1, G_1, ϕ_1) and (X_2, G_2, ϕ_2) be a Cantor minimal systems. A homeomorphism $h \colon X_1 \to X_2$ is

- a *conjugacy* if
$h(\phi_1(g))(x) = \phi_2(g)(h(x))$, $\forall\, g \in G_1,\, x \in X_1$

- an *orbit equivalence* if
$h\left(\{\phi_1(g)(x) \mid g \in G_1\}\right) = \{\phi_2(g)(h(x)) \mid g \in G_2\}$, $\forall\, x \in X_1$.

The classification of Cantor minimal systems is a major industry, especially for the case where $G = \mathbb{Z}^n$ and the action is expansive.

Giordano, Matui, Putnam, Skau in a series of papers classified these actions *up to orbit equivalence*.

Definition: $Aut(X, G, \phi)$ is the group of automorphisms of the Cantor minimal system (X, G, ϕ).

Definition: A Cantor minimal system (X, G, ϕ) is:

- *regular* if the action of $Aut(X, G, \phi)$ on X is transitive;
- *weakly regular* if the action of $Aut(X, G, \phi)$ decomposes X into a finite collection of orbits;
- *irregular* if the action of $Aut(X, G, \phi)$ decomposes X into an infinite collection of orbits.

Let (X, G, ϕ) be a Cantor minimal system. Each $g \in G$ defines an element $\widehat{\phi}(g) \in X^X$, and let

$$\widehat{G} = \{\widehat{\phi}(g) \mid g \in G\} \subset X^X$$

Endow the space $\widehat{G}$ with the topology of pointwise convergence.

The closure $E(X, G, \phi)$ is called the *enveloping Ellis semi-group*. There is a natural embedding $G \subset E(X, G, \phi)$.

For an expansive Cantor minimal system, $E(X, G, \phi)$ is a monster. Though for G amenable, it has been studied extensively.

Theorem: [Ellis (1960)] If (X, G, ϕ) is equicontinuous, then the topology on $E(X, G, \phi)$ is separable, and $E(X, G, \phi)$ is a Cantor group which acts transitively on X.

Identify the action of G on X with the action of G on $E(X, G, \phi)/\mathcal{H}_x$ where $\mathcal{H}_x$ is the isotropy subgroup at x.

Let G be a finitely generated group, and $\mathcal{G} \in \mathfrak{G}$ a group chain.

$$X = G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$$

Define $C_i = \bigcap_{g \in G} gG_ig^{-1}$ which is the maximal normal subgroup of G_i called the *core* of G_i in G.

G_i finite index in $G \implies C_i$ has finite index in G.

The normal subgroups $\{C_i\}_{i \geq 0}$ form a nested chain, and the inverse limit space is a Cantor group,

$$C_\infty(\phi) = \varprojlim \{G/C_i \to G/C_{i-1}\}$$

Proposition: There is a natural topological isomorphism $C_\infty \cong E(X, G, \phi)$.

For each $x \in X$, there is a natural map $q_x : C_\infty \to G_\infty$ which is the quotient of C_∞ by the action of the subgroup

$$\mathcal{D}_x = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}.$$

$\mathcal{D}_x$ is called the *discriminant group* of (X, G, Φ) at x.

Proposition: There is a natural topological isomorphism $\mathcal{D}_x \cong \mathcal{H}_x$.

Thus, for an equicontinuous Cantor minimal system (X, G, ϕ) defined by the group chain $\mathcal{G} \in \mathfrak{G}$, the abstract group $\mathcal{H}_x$ can be calculated in terms of the quotient group chain $\{G_i/C_i \mid i \geq 0\}$.

Theorem: [Dyer, Hurder & Lukina (2015)] Let (X, G, Φ) be an equicontinuous Cantor minimal system defined by a group chain $\{G_i\}_{i \geq 0}$ at x. Then

- The action (X, G, Φ) is regular if and only if $\mathcal{D}(\phi)_x$ is trivial for some $x \in X$.
- The action (X, G, Φ) is weakly regular if $\mathcal{D}(\phi)_x$ is finite for some $x \in X$.

Corollary: If the equicontinuous Cantor minimal system (X, G, Φ) defined by the group chain $\{G_i\}_{i \geq 0}$ is irregular, then $\mathcal{D}(\phi)_x$ is a Cantor group for all $x \in X$.

Theorem: [Dyer (2015)] The discriminant group $\mathcal{D}(\phi)_x$ for the Cantor minimal system defined by the group chain $G_0 = \mathcal{H}$, $\{G_n\}_{n\geq 1} = \{A_n\mathbb{Z}^2 \times p^n\mathbb{Z}\}_{n\geq 1}$ in the discrete Heisenberg group $\mathcal{H}$ is a Cantor group.

Thus, when we pass from the abelian case, where $G = \mathbb{Z}^n$, to the next most complicated groups, the Heisenberg groups, the structure of $Aut(X, G, \phi)$ becomes non-trivial.

Correspondingly, the study of the conjugacy problems for weak solenoids defined by such chains is much more delicate.

Problem: Explore the relations between the dynamics of an equicontinuous Cantor minimal system (X, G, Φ) and its discriminant groups $\mathcal{D}(\phi)_x$ in the cases where this is Cantor group.

Problem: Explore the relations between the dynamics of an expansive Cantor minimal system (X, G, Φ) and the Ellis enveloping semi-group $E(X, G, \Phi)$.

Thank you for your attention!

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