Classification of weak solenoids

Steve Hurder, University of Illinois at Chicago
Joint work with
Olga Lukina, University of Vienna

A *Riemannian foliation* is one whose holonomy pseudogroup is generated by local isometries of a Riemannian manifold.

Bruce Reinhart: "*Foliated manifolds with bundle-like metrics*", **Ann. of Math**, 69:119–132, 1959

Pierre Molino: **Feuilletages riemanniens**, Montpellier, 1983 & **Riemannian Foliations**, Birkhauser, 1988.

An *equicontinuous foliated space* is one whose holonomy pseudogroup is equicontinuous $\sim$ isometric.

Problem: How much of the theory of Riemannian foliations in Molino's book can be extended to equicontinuous foliated spaces?

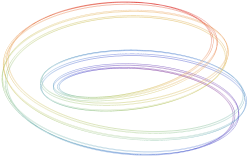
This question has been studied in the works of Jesús Álvarez López:

- ⋆ (with Alberto Candel),
  "*Equicontinuous foliated spaces*",
  Math. Z., 263 (2009), 725–774.

- ⋆ (with Manuel Moreira Galicia),
  "*Topological Molino's theory*",
  Pacific J. Math., 280 (2016), 257–314.

- ⋆ (with Ramón Barral Lijó),
  "*Molino's description and foliated homogeneity*",
  Topology Appl., 260 (2019), 148–177.

Our interest is the special case where the foliated spaces are transversally totally disconnected:



The objects of study are called various names in the literature:

- *Generalized laminations*, [Ghys, Lyubich & Minsky]
- *Matchbox manifolds*, [Aarts & Martens, Clark & Hurder]
- *Solenoidal manifolds*, [Sullivan]

All are foliated spaces as introduced in the book

- Moore & Schochet, **Global Analysis on Foliated Spaces**, 1988.

Theorem [Clark-Hurder, 2013] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a *weak solenoid*.

If $\mathfrak{M}$ is homogeneous space, then the weak solenoid is a profinite group fibration over a compact manifold.

If $\mathfrak{M}$ is not homogeneous, then it is homeomorphic to a quotient of a profinite group fibration by a non-trivial closed subgroup.

Alex Clark & S.H., "*Homogeneous matchbox manifolds*", **Transactions A.M.S.**, 365 (2013), 3151-3191.

Weak Solenoids

- M is compact manifold without boundary
- $G = \pi_1(M, x_0)$ is finitely generated group.

$$M = M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} M_3 \cdots$$

Choose $x_\ell \in M_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(M_\ell, x_\ell)$

Inclusion maps $q_\ell \colon G_\ell \subset G_{\ell-1}$, descending chain of groups

$$G = G_0 \xleftarrow{q_1} G_1 \xleftarrow{q_2} G_2 \xleftarrow{q_3} G_3 \cdots$$

Tower of coverings is *normal* if each $G_\ell \subset G_0$ is a normal subgroup.

Example: Vietoris solenoid is given coverings of $\mathbb{S}^1$, so is determined by a chain of normal subgroups of $\mathbb{Z}$.

Inverse limit space for a tower of coverings:

$$\begin{aligned} M_\infty &= \varprojlim \ \{p^{\ell}_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \\ &= \ \{(y_0, y_1, y_2, \ldots) \mid p^{\ell}_{\ell+1}(y_{\ell+1}) = y_\ell \mid \ell \geq 0\} \\ &\subset \ \prod_{\ell \geq 0} M_\ell \end{aligned}$$

is a compact connected metrizable space called a *(weak) solenoid*.

For each $\ell > 0$, there is a fibration map $\Pi_\ell \colon M_\infty \to M_\ell$.

For fixed $x_\ell \in M_\ell$ the fiber $\mathfrak{X}_\ell = \Pi_\ell^{-1}(x_\ell) \subset \mathfrak{X}_0$ is a Cantor space.

- The path connected components of M_∞ are manifolds,
- leaves are non-compact covering spaces of M_0,
- M_∞ is a foliated space with Cantor transversals.

The monodromy action on the fiber, $\Phi\colon G_0 \times \mathfrak{X}_0 \to \mathfrak{X}_0$

Fundamental group $G_0 = \pi_1(M_0, x_0)$ acts on the fiber $\mathfrak{X}_0$ via lifts of paths in M_0 to the leaves of $\mathcal{F}_{\mathfrak{M}}$.

This action is

- *minimal* = the orbit of each point is dense in $\mathfrak{X}_0$.
- *equicontinuous*: for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x,y) < \delta \implies d_{\mathfrak{X}}(\varphi(g)(x), \varphi(g)(y)) < \epsilon \quad \text{for all } g \in G.$$

Cantor action $(\mathfrak{X}, G, \Phi) \equiv$ minimal & equicontinuous

Conclusion of works with Clark & Lukina can be summarized:

Analyze/Classify weak solenoids $\Leftrightarrow$ *Analyze/Classify Cantor actions*

Profinite model for Cantor action $(\mathfrak{X}, G, \Phi)$.

Definition: $\mathfrak{G}(\Phi) = \overline{H_\Phi}$ = closure of $H_\Phi = \Phi(G) \subset \mathbf{Homeo}(\mathfrak{X})$ in the *uniform topology on maps*. $\mathfrak{G}(\Phi)$ is profinite group.

For $x \in \mathfrak{X}$, $\mathcal{D}_x = \{\widehat{h} \in \overline{H_\Phi} \mid \widehat{h} \cdot x = x\}$ (*isotropy group*)

Lemma: Left action of $\mathfrak{G}(\Phi)$ on $\mathfrak{X}$ is transitive. Hence

- $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathcal{D}_x$
- $\mathcal{D}_x$ independent of the choice of basepoint x.

The normal core of G_ℓ is $C_\ell = \bigcap_{g \in G} gG_\ell g^{-1} \subset G_\ell$

Theorem [Dyer-Hurder-Lukina, 2016]

$$\mathcal{D}_x \cong \varprojlim \{\pi_{\ell+1} \colon G_{\ell+1}/C_{\ell+1} \to G_\ell/C_\ell \mid \ell \geq 0\} .$$

1. $\mathcal{D}_x$ is trivial for Cantor action $(\mathfrak{X}, G, \Phi)$ with G abelian.
2. $\mathcal{D}_x$ can be a Cantor group for a Cantor action $(\mathfrak{X}, G, \Phi)$ when G is 3-dimensional Heisenberg group.
3. Every finite group and every separable profinite group can be realized as $\mathcal{D}_x$ for a Cantor action by a torsion-free, finite index subgroup of $\mathbf{SL}(n, \mathbb{Z})$, $n \geq 3$.
4. $\mathcal{D}_x$ can be wide-ranging for arboreal representations of absolute Galois groups of number fields and function fields.
5. Every Cantor action by a finitely generated group G can be realized by a tower of finite coverings of a closed surface.

Problem: Can one "hear" $\mathcal{D}_x$ in the spectrum of leafwise elliptic operators (e.g. Lapacians) on weak solenoids?

The inverse limit of a weak solenoid can begin at any level of the tower of coverings:

$$\begin{aligned} M_\infty &= \varprojlim \, \{p^\ell_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \\ &\cong \varprojlim \, \{p^\ell_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq k\} \end{aligned}$$

Conclusion: Dynamical invariants for weak solenoids must be unchanged upon passing to restrictions to clopen subsets which are adapted to the action of the monodromy.

⋆ Study dynamics up to *return equivalence*.

For Cantor action $(\mathfrak{X}, G, \Phi)$, $U \subset \mathfrak{X}$ is *adapted* to the action Φ

- U is a non-empty clopen subset,
- for any $g \in G$, $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$.

Translates of U form a partition of the Cantor set $\mathfrak{X}$.

The set of “return times” to U,

$$G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \emptyset\}$$

is a subgroup of finite index in G, called the *stabilizer* of U.

Definition: Let $\Phi_i \colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be Cantor actions, for $i = 1, 2$.

Then Φ_1 is return equivalent to Φ_2 if there exist

- for $i = 1, 2$ a clopen subset $U_i \subset \mathfrak{X}_i$ adapted for action Φ_i
- homeomorphism $h \colon U_1 \to U_2$
- isomorphism $\alpha_h \colon H_1 \to H_2$ of the action groups, induced by h,

where $H_i = \Phi_i(G_{U_i}) \subset \mathbf{Homeo}(U_i)$

Remark: When $U_i = \mathfrak{X}_i$ for $i = 1, 2$, and the actions are effective, this reduces to the notion of isomorphism, or just topological conjugacy of the actions, where $\alpha_h \colon G_1 \to G_2$ intertwines them.

Problem: Find return equivalence invariants of Cantor actions.

Let $(\mathfrak{X}, G, \Phi)$ be Cantor action. Fix basepoint $x \in \mathfrak{X}$ and $\epsilon > 0$.

There exists an adapted clopen set $U \subset \mathfrak{X}$ with $x \in U$ and $\mathrm{diam}(U) < \epsilon$. Iterating this construction, for a given basepoint x, one can always construct the following:

Definition: A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is an *adapted neighborhood basis* at $x \in \mathfrak{X}$ for the action Φ if

- $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\cap \ U_\ell = \{x\}$,
- each U_ℓ is adapted to the action Φ , set $G_\ell = G_{U_\ell}$

We obtain a sequence of localized Cantor actions $(U_\ell, H_\ell, \Psi_\ell)$:

$H_\ell = \Phi_\ell(G_\ell) \subset \mathbf{Homeo}(U_\ell)$, $\Psi_\ell \colon H_\ell \times U_\ell \to U_\ell$

A Cantor action $(\mathfrak{X}, G, \Phi)$ is either stable or wild.

Depends on whether the "sheaf" of local actions is stable, or not.

Let $\mathcal{D}(\Psi_\ell) \subset \mathbf{Homeo}(U_\ell)$ be discriminant group of $(U_\ell, H_\ell, \Psi_\ell)$.

There is surjective homomorphism $\rho_\ell \colon \mathcal{D}_x = \mathcal{D}(\Psi_0) \to \mathcal{D}(\Psi_\ell)$.

Set $K_\ell \equiv \ker\{\rho_\ell\}$ for $\ell \geq 1$. Then $K_1 \subset K_2 \subset \cdots$

Theorem [Hurder-Lukina, 2019] The isomorphism class of the direct limit group

$$\Upsilon(\Phi) = \varinjlim \{K_\ell \subset K_{\ell+1} \mid \ell \geq 1\}$$

is a well-defined conjugacy invariant of a Cantor action $(\mathfrak{X}, G, \Phi)$.

A Cantor action $(\mathfrak{X}, G, \Phi)$ is:

- stable if the chain $\{K_\ell \mid \ell \geq 1\}$ is bounded.

That is, if there exists ℓ_0 so that $K_\ell = K_{\ell+1}$ for $\ell \geq \ell_0$.

- wild if the chain $\{K_\ell \mid \ell \geq 1\}$ is unbounded.

Theorem [Hurder-Lukina, 2019]: The wild property for a Cantor action is invariant under continuous orbit equivalence.

Example: The examples of group actions on trees generated by automata studied by Nekrashevych, Bartholdi, Grigorchuk *et al* typically induce wild actions on the boundary of the trees.

Theorem [Lukina, 2019]: Let p and d be distinct odd primes, let $K = \mathbb{Q}_p$ be the field of p-adic numbers. Let $f(x) = (x + p)^d - p$. Then the action of $\mathrm{Gal}_\infty(f)$ is stable.

Theorem [Lukina, 2018]: Let $f(x)$ be a quadratic polynomial with critical point c. If the post-critical set P_C is infinite, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$, and so of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is wild.

There is a geometric interpretation of the stable/wild property.

Definition: A topological action $\Phi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is *locally quasi-analytic (LQA)* if there exists $\epsilon > 0$ such that for any open set $U \subset \mathfrak{X}$ with $diam(U) < \epsilon$, and for any open $V \subset U$ and $g_1, g_2 \in G$ if

$$\text{if } \Phi(g_1)|V = \Phi(g_2)|V \text{ then } \Phi(g_1)|U = \Phi(g_2)|U \ .$$

Alternatively, the action is locally quasi-analytic if and only if for all $g \in G$ if $\Phi(g)|V = id$, then $\Phi(g)|U = id$, for open sets $V \subset U$.

Theorem [Hurder-Lukina, 2017]: A Cantor action $(\mathfrak{X}, G, \Phi)$ with G finitely generated is stable, if and only if the pro-finite action $\widehat{\Phi} \colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic.

Here are two further results:

Theorem [Hurder-Lukina, 2018]: Let $\Phi \colon G \times \mathfrak{X} \to \mathfrak{X}$ be a Cantor action with G a finitely-generated *nilpotent* group. Then the action is stable. Moreover, any Cantor action which is continuously orbit equivalent must be return equivalent.

Theorem [Hurder-Lukina, 2018]: There exists uncountably many wild actions of torsion-free finite index subgroups of $\mathbf{SL}(n, \mathbb{Z})$ with distinct pro-isomorphism classes of direct limit groups $\Upsilon(\Phi)$.

Problem: The wild property lurks in the spectrum of the leafwise laplacians for weak solenoids. *Find it*.

References

A. Clark and S. Hurder, *Homogeneous matchbox manifolds*, **Trans. A.M.S.**, 365 (2013), 3151–3191.

J. Dyer, S. Hurder and O. Lukina, *Molino theory for matchbox manifolds*, **Pac. Jour. Math.**, 289 (2017), 91-151.

S. Hurder and O. Lukina, *Wild solenoids*, **Trans. A.M.S.**, 371 (2019), 4493–4533; arXiv:1702.03032

A. Clark, S. Hurder and O. Lukina, *Classifying matchbox manifolds*, **Geometry & Topology**, 23 (2019), 1–27.

S. Hurder and O. Lukina, *Orbit equivalence and classification of weak solenoids*, **Indiana Univ. Math. J.**, 2020; arXiv:1803.02098.

A. Clark, S. Hurder and O. Lukina, *Pro-groups and a generalization of a theorem of Bing*, **Topology and Its Applications**, 2020; arXiv:1811.00288.

S. Hurder and O. Lukina, *Limit group invariants for non-free Cantor actions*, **submitted**; arXiv:1904.11072.

Thank you for your attention!