Stable Cantor dynamics

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Three approaches to the same subject:

- Equicontinuous Cantor actions

  Dynamical systems approach

- Group actions on rooted trees

  Geometric group theory approach

- Clopen subset chains for profinite groups

  Descriptive set theory approach

Each approach has its own language.

The approach of Lukina & myself is a sort of "creole", combining language and techniques from each of these three areas.

Cantor actions

- Γ is a finitely generated group
- $\mathfrak{X}$ is a Cantor space
- $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is a minimal continuous action

- A Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x,y) < \delta \implies d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \epsilon \quad \text{for all } g \in \Gamma.$$

When $G = \mathbb{Z}$ then a minimal equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is conjugate to a classical odometer.

For a general group Γ, the action is called a (sub) odometer in

- [Cortez & Petite, *J. London Math Soc.*, 2008]
- [Cortez & Medynets, *J. London Math Soc.*, 2016]
- [Li, *Ergodic Theory Dynamical Systems*, 2018]

General approach via group chains: $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$

$\Gamma_{\ell+1} \subset \Gamma_\ell$ is proper inclusion with finite index.

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \cdots$$

Each quotient $X_\ell = \Gamma/\Gamma_\ell$ is a finite set with left Γ-action.

The Cantor space is

$$\mathfrak{X} = \varprojlim \, \{\Gamma_0/\Gamma_{\ell+1} \longrightarrow \Gamma_0/\Gamma_\ell\} \;\; \subset \;\; \prod_{\ell>0} X_\ell$$

Induces left Γ-action Φ on $\mathfrak{X}$ which is minimal and equicontinuous.

Problem: How to organize and classify Cantor actions?

Topology of Cantor space $\mathfrak{X}$ is generated by clopen subsets: U is closed and open. Non-empty clopen $U \subset \mathfrak{X}$ is adapted if the return times to U form a subgroup

$$\Gamma_U = \{g \in \Gamma \mid \Phi(g)(U) = U\} \subset \Gamma$$

Definition: $(\mathfrak{X}, \Gamma, \Phi)$ a Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is said to be an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action Φ, if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 0$ with $\cap_{\ell>0} U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.

Proposition: Let $(\mathfrak{X}, \Gamma, \Phi)$ be a Cantor action. Given $x \in \mathfrak{X}$, there exists an adapted neighborhood basis $\mathcal{U}$ at x for the action Φ.

The defining group chain is given by $\mathcal{G} = \{\Gamma_\ell \equiv \Gamma_{U_\ell} \mid \ell \geq 0\}$.

Isomorphism (Rigidity Theory)

Definition: Cantor actions $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ are said to be isomorphic if there is a homeomorphism $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ and group isomorphism $\Theta \colon \Gamma_1 \to \Gamma_2$ so that

$$\Phi_1(g) = h^{-1} \circ \Phi_1(\Theta(g)) \circ h \in \mathbf{Homeo}(\mathfrak{X}_1) \text{ for all } g \in \Gamma_1 \ . \quad (1)$$

Return equivalence (Dynamical systems)

Definition: Equicontinuous Cantor actions $\Phi_1 \colon \Gamma_1 \times \mathfrak{X}_1 \to \mathfrak{X}_1$ and $\Phi_2 \colon \Gamma_2 \times \mathfrak{X}_2 \to \mathfrak{X}_2$ are <u>return equivalent</u> if there exist adapted clopen subsets $U \subset \mathfrak{X}_1$ and $V \subset \mathfrak{X}_2$, such that the restricted actions $\Phi_{1,U} \colon \Gamma_{1,U} \times U \to U$ and $\Phi_{2,V} \colon \Gamma_{2,V} \times V \to V$ are isomorphic.

This is weaker than isomorphism, even for $U = \mathfrak{X}$.

It loses information about the kernel of the action map $\Phi_0 \colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$

Continuous orbit equivalence (C^*-algebras)

Definition: Let $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ be Cantor actions. A continuous orbit equivalence between the actions is a homeomorphism $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ which is an orbit equivalence, and satisfies the locally constant properties:

1. for each $x \in \mathfrak{X}_1$ and $g \in \Gamma_1$, there exists $\alpha(g, x) \in \Gamma_2$ and an open set $x \in U_x \subset \mathfrak{X}_1$ s.t. $\Phi_2(\alpha(g, x)) \circ h|U_x = h \circ \Phi_1(g)|U_x$;
2. for each $y \in \mathfrak{X}_2$ and $k \in \Gamma_2$, there exists $\beta(k, y) \in \Gamma_1$ and an open set $y \in V_y \subset \mathfrak{X}_2$ s.t. $\Phi_1(\beta(k, y)) \circ h|V_y = h \circ \Phi_2(k)|V_y$.

The functions $\alpha\colon \Gamma_1 \times \mathfrak{X} \to \Gamma_2$ and $\beta\colon \Gamma_2 \times \mathfrak{X}_2 \to \Gamma_1$ are continuous, as the groups Γ_1 and Γ_2 have the discrete topology, but need not be cocycles over the actions.

- Renault showed that COE is basic notion for isomorphism of cross-product C^* -algebras with Cartan subalgebra.

Diagonal Odometers

- $\Gamma = \mathbb{Z}^k$ for $k \geq 1$
- Choose sequence of integer vectors $\vec{n}^\ell = (n_1^\ell, \ldots, n_k^\ell)$, $n_i^\ell > 1$
- Set $m_i^\ell = n_1^\ell \cdot n_2^\ell \cdots n_k^\ell$
- $\Gamma_\ell = \{(m_1^\ell n_1, \ldots, m_k^\ell n_k) \mid (n_1, \ldots n_k) \in \mathbb{Z}^k\}$

Random Odometers

- $\Gamma = \mathbb{Z}^k$ for $k \geq 1$
- Choose sequence of integer matrices $A_\ell \in \mathbf{GL}(\mathbf{k}, \mathbb{Z})$, $\det A_\ell > 1$
- $\Gamma_\ell = \{A_\ell A_{\ell-1} \cdots A_1 \cdot \vec{n} \mid \vec{n} \in \mathbb{Z}^k\}$

In both cases, the inverse limit $\mathfrak{X}$ is profinite group.

For $k = 1, 2$ classified by Giordano, Putnam & Skau [2017]

Renormalizable

A countable group Γ is finitely non-co-Hopfian if there exists a self-embedding $\varphi\colon \Gamma \to \Gamma$ whose image is a proper subgroup of finite index. $\varphi\colon \Gamma \to \Gamma$ is called a renormalization of Γ

If Γ admits a renormalization, then it is said to be renormalizable.

The renormalization group chain $\mathcal{G}_\varphi = \{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$.

Yields equicontinuous Cantor action $\Phi_\varphi\colon \Gamma \times X_\varphi \to X_\varphi$

$\mathcal{G}_\varphi$ is a scale for Γ if $K(\mathcal{G}_\varphi) \equiv \bigcap_{\ell > 0} \Gamma_\ell$ is a finite group.

Conjecture: [Benjamini, Nekrashevych & Pete] If Γ admits a renormalization which defines a scale, then Γ is virtually nilpotent.

Heisenberg group:

$$\Gamma = \left\{ [x,y,z] = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{Z} \right\}$$

Example 1:

- Let $m > 1$. Define $\varphi_1[x,y,z] = [mx, my, m^2 z]$
- X_{φ_1} is a profinite Heisenberg group,
- Φ_{φ_1} is left multiplication by Γ.

Example 2:

- $p, q > 1$ distinct primes. Define $\varphi_2[x,y,z] = [px, qy, pqz]$
- $X_{\varphi_2} \cong \{[x,y,z] \mid x \in \widehat{\mathbb{Z}}_p\ ,\ y \in \widehat{\mathbb{Z}}_q\ ,\ z \in \widehat{\mathbb{Z}}_{pq}\}$.
- Φ_{φ_2} is left multiplication by Γ.

- Equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$, same as homomorphism $\Phi_0\colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$
- $\widehat{\Gamma} = \overline{\Phi_0(\Gamma)} \subset \mathbf{Homeo}(\mathfrak{X})$ - closure in uniform topology
- $\widehat{\Gamma}$ is separable profinite group, compact and totally disconnected
- $\widehat{\Phi}\colon \widehat{\Gamma} \times \mathfrak{X} \to \mathfrak{X}$ is transitive equicontinuous action
- For $x \in \mathfrak{X}$, define $\mathcal{D} = \{\widehat{g} \in \widehat{\Gamma} \mid \widehat{\Phi}(\widehat{g})(x) = x\}$
- $\mathcal{D}$ is called the discriminant of the action Φ
- Isomorphism class of $\mathcal{D}$ is independent of choice of x and invariant of isomorphism of actions.

- If $\mathcal{D}$ is the trivial group, then $\mathfrak{X} \cong \widehat{\Gamma}$ is a profinite group, and the action of Φ on $\mathfrak{X}$ is given by left multiplication on $\widehat{\Gamma}$.

- For Γ abelian, $\mathcal{D}$ is always trivial.

Dynamics of action are studied using structures of profinite groups. If action is effective then it is fixed-point free.

- There are fixed-point free actions for which $\mathcal{D}$ is non-trivial.
- Study properties of equicontinuous actions with $\mathcal{D}$ non-trivial.
- $\mathcal{D}$ acts via adjoint on $\widehat{\Gamma}$: $\mathbf{Ad}(\widehat{h})(\widehat{g}) = \widehat{h}^{-1}\, \widehat{g}\, \widehat{h}$, $\widehat{h} \in \mathcal{D}$, $\widehat{g} \in \widehat{\Gamma}$

Problem: Study properties of adjoint $\mathbf{Ad}\colon \mathcal{D} \times \widehat{\Gamma} \to \widehat{\Gamma}$.

For an equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$

1. (Dyer Thesis 2016) $\mathcal{D}$ can be a Cantor group for a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ when Γ is 3-dimensional Heisenberg group.
2. ([DHL2016]) Every finite group and every separable profinite group can be realized as $\mathcal{D}$ for a Cantor action by a torsion-free, finite index subgroup of $\mathbf{SL}(n, \mathbb{Z})$, $n \geq 3$.
3. ([Lukina2018a,Lukina2018b]) $\mathcal{D}$ can be wide-ranging for arboreal representations of absolute Galois groups of number fields and function fields.

Cantor action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$

- is effective, or *faithful*, if $\Phi_0 \colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$ has trivial kernel.
- is free if for all $x \in \mathfrak{X}$ and $g \in \Gamma$, $g \cdot x = x$ implies that $g = e$
- isotropy group of $x \in \mathfrak{X}$ is $\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\}$
- $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$, and isotropy set

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\ g \in \Gamma\ ,\ g \neq id\ ,\ g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)$$

- is topologically free if $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X}$.

  If $\mathrm{Iso}(\Phi)$ meager, then $\mathrm{Iso}(\Phi)$ has empty interior.

Quasi-analytic (General topological actions)

Definition: An action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$, where

- H is a topological group and
- $\mathfrak{X}$ is a Cantor space

is quasi-analytic if for each clopen set $U \subset \mathfrak{X}$, $g \in H$

- if $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

For H a countable group, this is equivalent to topologically free.

Locally quasi-analytic (Equicontinuous Cantor actions)

Definition: An action $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$, where

- H is a topological group and
- $\mathfrak{X}$ is a Cantor space

is locally quasi-analytic if if there exists $\epsilon > 0$ such that

- for any adapted set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$,
- for any adapted subset $V \subset U$,
- $g \in H$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ acts as the identity on all of U.

Definition: An equicontinuous Cantor action $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$ is stable if the associated profinite action $\widehat{\Phi} \colon \widehat{\Gamma} \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic. The action is said to be wild otherwise.

Equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ defined by group chain $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$, $\mathfrak{X} = \varprojlim \, \{\Gamma_0/\Gamma_{\ell+1} \longrightarrow \Gamma_0/\Gamma_\ell\}$.

For $k \geq 0$, define

$$\begin{aligned} U_k &= \{(g_0, g_1, g_2, \ldots) \in \mathfrak{X} \mid g_i = e \text{ for } 0 \leq i \leq k\} \\ &= \varprojlim \, \{p_{\ell+1}\colon \Gamma_k/\Gamma_{\ell+1} \to \Gamma_k/\Gamma_\ell \mid \ell \geq k\} \ , \end{aligned}$$

which is a clopen subset of X adapted to the action Φ, with stabilizer subgroup $\Gamma_{U_k} = \Gamma_k$. Define

$$\widehat{U}_k = \{\widehat{g} \in \widehat{\Gamma} \mid \widehat{\Phi}(\widehat{g})(U_k) = U_k$$

which is a clopen set in $\widehat{\Gamma}$, and $\mathcal{D} = \bigcap_{\ell > 0} \widehat{U}_k$.

$$\mathcal{K}^{(k)} = \ker \left\{ \widehat{\Phi}^{(k)} \colon \mathcal{D} \to \mathcal{D}^{(k)} \subset \mathbf{Homeo}(U_k) \right\}$$

$$C(\mathcal{G}) = \mathcal{K}^{(0)} \subset \mathcal{K}^{(1)} \subset \mathcal{K}^{(2)} \subset \mathcal{K}^{(3)} \subset \cdots$$

Proposition: Φ is stable iff the chain has an upper bound;
i.e. there exists $k_0 \geq 0$ such that $k > k_0$ implies $\mathcal{K}^{(k)} = \mathcal{K}^{(k_0)}$.

Definition: If Φ is stable, then $\mathcal{D}^s \equiv \widehat{\Phi}^{(k)}(\mathcal{D}) \cong \mathcal{D}/\mathcal{K}_k$ for $k \geq k_0$.
This is called the stable discriminant for the action.

Definition: An equicontinuous Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is wild if the chain $\mathcal{K} = \{\mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots\}$ does not stabilize.

- The examples of group actions on trees generated by automata studied by Nekrashevych, Bartholdi, Grigorchuk *et al* typically induce wild actions on the boundary of the trees.

- A torsion-free, finite index subgroup $\Gamma \subset \mathbf{SL}(n, \mathbb{Z})$, $n \geq 3$ has uncountably many wild equicontinuous Cantor actions which are pairwise not return equivalent - see [HL2017].

Idea is to use isomorphism

$$\widehat{\mathbf{SL}(n, \mathbb{Z})} \cong \mathbf{SL}(n, \widehat{\mathbb{Z}}) \cong \prod_{p \ prime} \mathbf{SL}(n, \widehat{\mathbb{Z}}_p)$$

- The dynamical properties of equicontinuous Cantor actions are a lot more mysterious than one might expect.

- Analyze the cases:
  - ⋆ action is stable;
  - ⋆ Γ is virtually nilpotent;
  - ⋆ Γ is renormalizable.

- Classification of wild actions is mostly unknown. Closely related to presence of non-Hausdorff elements for the action of $\widehat{\Gamma}$ on $\mathfrak{X}$.

Stable actions

Theorem [HL2018]: The stable property is invariant under continuous orbit equivalence.

Theorem [HL2019b]: Let $(\mathfrak{X}, \Gamma, \Phi)$ be a nilpotent Cantor action. Then the action is stable.

Theorem [HLvW2020]: The Cantor action $\Phi_\varphi \colon \Gamma \times X_\varphi \to X_\varphi$ associated to a renormalization $\varphi \colon \Gamma \to \Gamma$ is quasi-analytic. Hence, if the action Φ_φ is also effective, then it is topologically free.

Rigidity

Theorem [HL2018,HL2019b]: Let $\Phi_1 \colon \Gamma_1 \times \mathfrak{X}_1 \to \mathfrak{X}_1$ be a stable equicontinuous Cantor action. Let $\Phi_2 \colon \Gamma_2 \times \mathfrak{X}_2 \to \mathfrak{X}_2$ be a Cantor action which is continuously orbit equivalent to Φ_1.

Then the action Φ_2 is equicontinuous and stable, and the actions Φ_1 and Φ_2 are return equivalent.

- The dynamics of a stable equicontinuous Cantor action is essentially preserved by continuous orbit equivalence.

Nilpotent actions

A finitely generated torsion-free nilpotent group Γ has many special properties:

- ⋆ Γ is Noetherian (ascending group chains stabilize)
- ⋆ The profinite completion $\widehat{\Gamma}$ is nilpotent and torsion-free.

Theorem [HL2019b]: Let $\Phi_1 \colon \Gamma_1 \times \mathfrak{X}_1 \to \mathfrak{X}_1$ be an equicontinuous Cantor action with Γ virtually nilpotent. Suppose that the action is continuously orbit equivalent to a Cantor action $\Phi_2 \colon \Gamma_2 \times \mathfrak{X}_2 \to \mathfrak{X}_2$. Then

- The actions Φ_1 and Φ_2 are return equivalent.
- If the action Φ_2 is effective, then Γ_2 is virtually nilpotent.
- The stable discriminants of the two actions are isomorphic.

Renormalizable groups

Recall that Γ is renormalizable if there exists a self-embedding $\varphi\colon \Gamma \to \Gamma$ whose image is a proper subgroup of finite index.

This induces a very strong type of self-symmetry for the action $\Phi_\varphi\colon \Gamma \times X_\varphi \to X_\varphi$.

Theorem [HLvL2020]: Let φ be a renormalization of the finitely generated group Γ. Suppose that

$$K(\mathcal{G}_\varphi) = \bigcap_{\ell>0} \varphi^\ell(\Gamma) \subset \Gamma \quad , \quad \mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi$$

are both finite groups, then

- Γ is virtually nilpotent,
- If both groups are trivial, then Γ is nilpotent.

Key point in the proof is to use results by Colin Reid on self-embeddings of profinite groups:

Endomorphisms of profinite groups, **Groups Geom. Dyn.**, 8:553–564, 2014.

Extensive literature of profinite subgroups with self-embeddings.

One expects there are many more ideas in these works to exploit.

Question: What are the implications of the above results for the C^*-algebras associated to equicontinuous Cantor actions?

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Thank you for your attention!