Foliations, Fractals and Cohomology

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Abstract

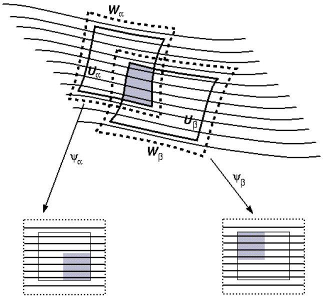
We give a brief introduction to each of the three themes: Foliations, Fractals and Cohomology. By cohomology, we mean in particular the Cheeger-Simons classes of vector bundles.

The goal of the talk will be to show how the combination of the three subjects leads to new questions about dynamics, and the "wild" topological sets that naturally arise in dynamical systems.

This leads to a new understanding of one of the "mysterious" results of foliation theory, the so-called Bott-Heitsch Theorem which dates from 1972. This new understanding raises as many questions as it answers.

Definition of foliation

A foliation $\mathcal{F}$ of dimension p on a manifold M^m is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds of codimension q: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.

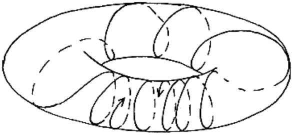


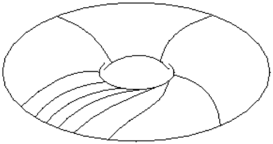
A leaf of $\mathcal{F}$ is a connected component of the manifold M in the "fine" topology induced by charts.

A natural example

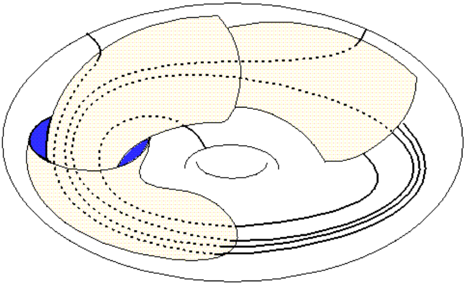
As first seen in a seminar at Rice University, Spring 1972.

Examples in 2-dimensions





Foliations by surfaces



Reeb Foliation of the solid torus

Fundamental problems

Problem: "Classify" the foliations on a given manifold M

Two classification schemes have been developed since 1970: using either "homotopy properties" or "dynamical properties".

Question: How are the homotopy and cohomology invariants of a foliation related to its dynamical behavior?

These are long-term research topics, dating from the 1970's.

See "Classifying foliations", to appear in Proceedings of Rio de Janeiro Conference in 2007, *Foliations, Topology and Geometry*, Contemp. Math., American Math. Soc., 2009. Or, download from website.

Foliation dynamics

- A *continuous dynamical system* on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x,t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$. The trajectories of the points of M are necessarily points, circles or lines immersed in M, and the study of their aggregate and statistical behavior is the subject of ergodic theory for flows.

- In foliation dynamics, we replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the *dynamics* of $\mathcal{F}$ asks for properties of the limiting and statistical behavior of the collection of its leaves.

Limit sets

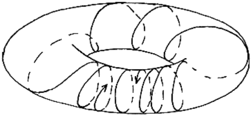
A basic property of a dynamical system is to ask for the sets where the orbits accumulate. These are called limit sets.

Let $\varphi_t \colon M \to M$ be a flow on a compact manifold M, and $x \in M$, then

$$\omega_x(\varphi) = \bigcap_{n=1}^{\infty} \overline{\{\varphi_t(x) \mid t \geq n\}}$$

is a compact set which is a union of flow lines for φ.

x is *recurrent* if $x \in \omega_x(\varphi)$.



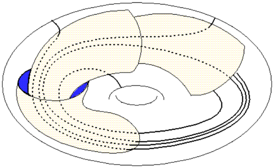
Circle is only recurrent orbit

A leaf $L \subset M$ of a foliation $\mathcal{F}$ inherits a quasi-isometry class of Riemannian metric from TM, and a metric topology.

The limit set of the leaf L_x through x is

$$\omega_x(\mathcal{F}) = \bigcap_{\substack{Y \subset L_x \\ Y \text{ compact}}} \overline{L_x - Y}$$

L_x is *recurrent* if $x \in \omega(L_x) \ \Rightarrow \ L_x \subset \omega(L_x)$.



Boundary torus is only recurrent leaf

Minimal sets

A closed subset $Z \subset M$ is *minimal* if

- Z is a union of leaves,
- each leaf $L \subset Z$ is dense

Z minimal $\Rightarrow$ $\omega_x(L_x) = Z$ for all $x \in Z$

The study of the minimal sets for a foliation is the first approximation to understanding foliation dynamics.

For a minimal set, ask about its shape, and the dynamics of the foliation restricted to it.

Shape of minimal sets

For each $x \in Z$, there is an open neighborhood $U_x \subset Z$

$$U_x \cong (-1,1)^p \times K$$

where $K \subset \mathbb{R}^q$ is a closed. If K is not finite, then K is perfect.

If K has no interior points, then Z is called *exceptional*.

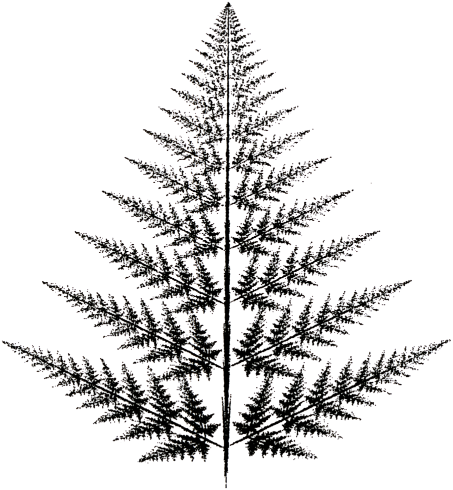
For $\mathcal{F}$ codimension $q = 1$, Z exceptional $\Leftrightarrow$ K is a Cantor set.

For $\mathcal{F}$ codimension $q \geq 2$, the possibilities for K include:

- Compact submanifolds of $\mathbb{R}^q$
- Fractals defined by *Iterated Function Systems*
- Julia sets of Rational Polynomials & Holomorphic Dynamics
- Limit sets of Schottky Groups

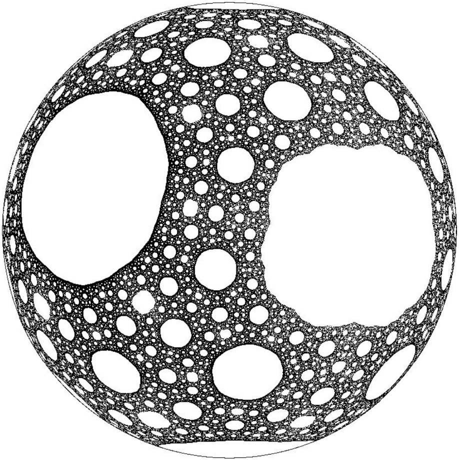
Each of these categories of "wild topology" for K is more complicated than we can hope to fully understand.

Wild Topology, I



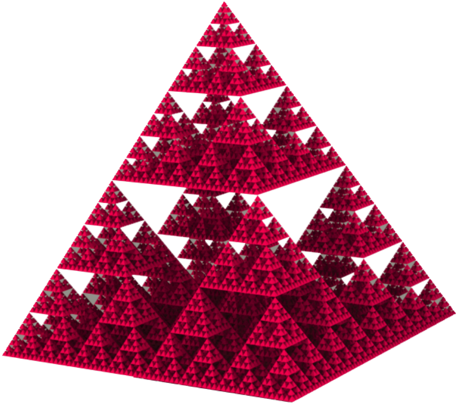
Fractal Tree

Wild Topology, II



Limit set of Schottky Group

Wild Topology, III



Sierpinski Pyramid

Infinite constructions

Common to all of these examples is a finite set of defining data for K:

- Compact connected set $X \subset \mathbb{R}^q$, where $K \subset X$
- Open sets $U_1, \ldots, U_k$ with $X \subset \overline{U_1} \cup \cdots \cup \overline{U_k}$
- smooth weak-contractions $\varphi_i \colon U_i \to U_j$ where $j = \nu(i)$

Theorem: K is characterized by $K = \bigcup_{i=1}^{k} \varphi_i(\overline{U_i} \cap K)$

Proof: The data defines a contraction mapping on the space $\mathcal{M}$ of compact subsets of $\mathbb{R}^q$ such that K is the unique fixed point.

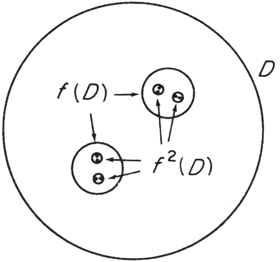
Moran (1946) → Hutchinson (1981) → Dekking (1982) → Hata (1985).

The foliation $\mathcal{F}$ is defined by the suspension of extensions $\widetilde{\varphi}_i \colon \mathbb{S}^q \to \mathbb{S}^q$.

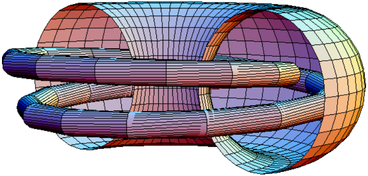
K appears as a slice of a minimal set Z of $\mathcal{F}$ by a transverse submanifold.

Wild Topology, IV

Solenoids occur very naturally in the dynamics of Hamiltonian flows and diffeomorphisms. [Bob Williams (1967,1974), many others...]



Cantor Set $\longrightarrow$ *Solenoid*



The leaves of $\mathcal{F}$ give a "twist" to the points of K. Because it is also a foliation, there is also a "twist" imparted to every open neighborhood of K.

We want to measure this twist with cohomology.

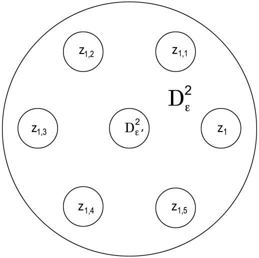
A construction

Solenoids are traditionally associated with dynamics of flows. But they are also part of the dynamics of foliations.

Theorem: [Clark & Hurder (2008)] For $n \geq 1$ and $q \geq 2n$, there exists commuting diffeomorphisms $\varphi_i \colon \mathbb{S}^q \to \mathbb{S}^q$, $1 \leq i \leq n$, so that the suspension of the induced action $\mathbb{Z}^n$ on $\mathbb{S}^q$ yields a smooth foliation $\mathcal{F}$ with solenoidal minimal set $\mathcal{S}$, such that:

- The leaves of $\mathcal{F}$ restricted to $\mathcal{S}$ are all isometric to $\mathbb{R}^n$.
- The action of $\mathbb{Z}^n$ on the Cantor set $K = \mathcal{S} \cap \mathbb{S}^q$ has a unique invariant probability measure.
- Periodic domains of the action of $\mathbb{Z}^n$ on $\mathbb{S}^q$ contained in every open neighborhood of K.
- K is a "p-adic" completion of $\mathbb{Z}^n$.

The standard construction makes use of infinitely repeated iteration of embeddings, disks inside of disks:



First stage of inductive construction

At each stage of the iteration:

- Keep a center disk $\mathbb{D}^q_{\epsilon'}$ on which the action is a rotation about center
- View actions of $\mathbb{Z}^n$ as deformations of finite representations into $SO(q)$
- View process as sequence of inductive surgeries on suspended foliations

See "Solenoidal minimal sets for foliations", Clark & Hurder 2008

Local invariants

In our construction, we force the existence of periodic disks for the action of the group $\mathbb{Z}^n$. Periodic orbits are important in classical dynamics!

Every open neighborhood of the minimal Cantor set K for the $\mathbb{Z}^n$ action has infinitely many essential *local actions* of finite groups whose orders tends to infinity. This has to be useful!

Exploiting this information about the dynamics of neighborhoods of fractal minimal sets, takes us down a well-trodden path, but in a new direction for the study of fractals.

Shape invariants

$Z \subset M$ a minimal set of $\mathcal{F}$ always has a "neighborhood system"

$$K \subset \cdots U_i \subset \cdots \subset U_1 \quad , \quad K = \bigcap_{i=1}^{\infty} U_i$$

where the U_i are open. The system defines the *shape* of K.

Each U_i inherits a foliation $\mathcal{F} \mid U_i$.

At each point $x \in U_i$ there is a germ of holonomy of the leaf L_x. This defines a groupoid $\Gamma_{\mathcal{F}|U_i}$ with morphisms given by germs of local diffeomorphisms of $\mathbb{R}^q$.

More accurately, $\mathcal{F} \mid U_i$ defines a *Topos* whose models are transversal submanifolds to $\mathcal{F}$ in U_i. See "Classifying Toposes and Foliations", by Ieke Moerdijk, Ann. Inst. Fourier (1991).

Classifying spaces

"Milnor join" construction $\Rightarrow$ classifying space BG of a Lie group G

Generalized by [Haefliger (1970), Segal (1975)] to a classifying space for a groupoid Γ. The space $B\Gamma \equiv \|\Gamma\|$ is the "semi-simplicial fat realization" of the groupoid Γ.

In general, the space $B\Gamma$ is as obscure as the nomenclature suggests.

Most well-known: the "universal classifying space" of codimension-q foliations, $B\Gamma_q$ introduced by [Haefliger (1970)]. The objects of Γ_q are points of $\mathbb{R}^q$, and morphisms are germs of local diffeomorphisms of $\mathbb{R}^q$.

The foliation $\mathcal{F}$ on M has a well-defined homotopy class $h_{\mathcal{F}} \colon M \to B\Gamma_q$.

For any open set $U \subset M$, the restriction $\mathcal{F} \mid U$ defines a groupoid $\Gamma_{U|\mathcal{F}}$.

There is a natural map of its realization $B\Gamma_{U|\mathcal{F}} \to B\Gamma_q$.

Exotic cohomology

A neighborhood system of a minimal set $Z \subset M$ yields directed system of spaces $\{B\Gamma_{U_{i+1}|\mathcal{F}} \to B\Gamma_{U_i|\mathcal{F}}\}_{i=1}^{\infty}$.

Theorem: Each minimal set Z of $\mathcal{F}$ has a well-defined space of cohomology invariants

$$\mathcal{H}^*(Z, \mathcal{F}) \equiv \varinjlim \; \{H^*(B\Gamma_{U_i|\mathcal{F}}) \to H^*(B\Gamma_{U_{i+1}|\mathcal{F}})\}$$

Example: Suppose that Z is a periodic orbit of a flow, defined by the suspension of an effective $\Gamma = \mathbb{Z}/p\mathbb{Z}$-action on a disk $\mathbb{D}^2$ fixing the origin, then

$$\mathcal{H}(Z, \mathcal{F}; \mathbb{Z}/p\mathbb{Z}) \;=\; H^*(B\Gamma; \mathbb{Z}/p\mathbb{Z}) \;\cong\; (\mathbb{Z}/p\mathbb{Z})[e_1]$$

is a polynomial ring generated by the Euler class e_1 of degree 2. This is just the "Borel construction" for the finite group action. See also Heitsch & Hurder, "Coarse cohomology for families", Illinois J. Math (2001).

Application

Each open neighborhood U_i of the minimal set Z yields is a natural map $B\Gamma_{U_i|\mathcal{F}} \to B\Gamma_q$, hence an induced map of the limit space

$$h_Z\colon \widehat{Z} \equiv \varprojlim \; \{B\Gamma_{U_{i+1}|\mathcal{F}} \to B\Gamma_{U_i|\mathcal{F}}\} \longrightarrow B\Gamma_q$$

Theorem: [Hurder (2008)] Let $\mathcal{S}$ be the solenoidal minimal set above. Then the homotopy class of the induced map $h_{\mathcal{S}}\colon \widehat{\mathcal{S}} \to B\Gamma_q$ is non-trivial:

$$h_{\mathcal{S}}^*\colon H^{4\ell-1}(B\Gamma_q;\mathbb{R}) \to \mathcal{H}^{4\ell-1}(\mathcal{S},\mathcal{F};\mathbb{R}) \text{ is non-trivial for } \ell > q/2$$

Proof: The Cheeger-Simons classes derived from $H^*(BSO(q);\mathbb{R})$ are in the image of $h_{\mathcal{S}}^*$.

The "cycle" $h_{\mathcal{S}}\colon \widehat{\mathcal{S}} \to B\Gamma_q$ is determined by a finite set of holonomy maps $\{\varphi_1,\ldots,\varphi_n\}$ obtained from the generators of the action of $\mathbb{Z}^n$. It realizes the inverse limit of the Bott-Heitsch *torsion classes* for finite flat bundles.

Cheeger-Simons invariants – elliptic invariants

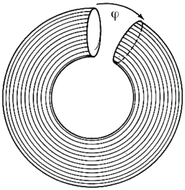
Suppose $A \in GL(\mathbb{R}^2)$ is rotation by θ radians in plane. Set

$$\widehat{e}(A) = \theta/2\pi \quad \mod (1) \in \mathbb{R}/\mathbb{Z}$$

Form vector bundle $\mathbb{E} \to \mathbb{S}^1$ by "suspension construction"

$$\mathbb{E} = \mathbb{R} \times \mathbb{R}^2/(t+1, \vec{x}) \sim (t, \varphi(\vec{x})) \to \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$$

There is a foliation $\mathcal{F}_\varphi$ of $\mathbb{E}$, foliated as follows:



Fact: $\widehat{e}(A)$ is the first "Cheeger-Simons invariant" for $\mathcal{F}_\varphi$.

Cheeger-Simons classes

Let $p\colon \mathbb{E} \to M$ be an oriented $\mathbb{R}^{2m}$-vector bundle over manifold M.

$\nu_{\mathbb{E}}\colon M \to B\Gamma_{2m}^{+} \to BSO(2m)$ is the classifying map for $E_{\mathcal{F}}$.

For example, $m = 1$ then $BSO(2) \cong \mathbb{S}^\infty/\mathbb{S}^1$.

$H^*(BSO(2m); \mathbb{R})$ is generated as algebra by:

$$
\begin{aligned}
e \in H^{2m}(BSO(2m); \mathbb{R}) \quad &- \quad \text{The Euler Class} \\
p_\ell \in H^{4\ell}(BSO(2m); \mathbb{R}) \quad &- \quad \text{The Pontrjagin Classes}
\end{aligned}
$$

Set $e(\mathbb{E}) = \nu_{\mathbb{E}}^*(e) \in H^{2m}(M; \mathbb{R})$ and $p_\ell(\mathbb{E}) = \nu_{\mathbb{E}}^*(p_\ell) \in H^{4\ell}(M; \mathbb{R})$

Consider the Bockstein maps:

$$\cdots \to H^{*-1}(M; \mathbb{R}/\mathbb{Z}) \to H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{R}) \to \cdots$$

$C \in \ker\{H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{R})\}$, "preimage" $\widehat{C} \in H^{*-1}(M; \mathbb{R}/\mathbb{Z})$

Let $J = (j_1 \leq j_2 \leq \cdots \leq j_k)$ and set $p_J = p_{j_1} p_{j_2} \cdots p_{j_k}$.

$|J| = j_1 + \cdots j_k$ and then $\deg p_J = 4|J|$.

In the case where $\mathbb{E}$ has a foliation $\mathcal{F}$ transverse to the fibers, and $C = p_J(\mathbb{E})$ with $|J| > m$, we have:

Bott Vanishing Theorem: [1970] If $\mathbb{E}$ has a foliation $\mathcal{F}$ transverse to the fibers of $p \colon \mathbb{E} \to M$, then

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{R}) \to H^*(M; \mathbb{R}) \text{ is trivial for } * > 2q = 4m.$$

The Bott Vanishing Theorem implies there exists pre-images $\widehat{p_J(\mathbb{E})} \in H^{4|J|-1}(M; \mathbb{R}/\mathbb{Z})$. If $\mathbb{E}$ is a trivial bundle, then these "Bockstein classes" lift to the *Cheeger-Simons classes* for $\mathcal{F}$:

$$\begin{aligned} T(p_J) &\in H^{4|J|-1}(M; \mathbb{R}) \\ T(e_m) &\in H^{2m-1}(M; \mathbb{R}) \end{aligned}$$

Generalized winding numbers

For $m = 1$, $\mathcal{F}$ foliation transverse to $\mathbb{D}^2$-bundle $\mathbb{E} \to M$, then have

$$T(e_1^\ell) \in H^{2\ell-1}(M;\mathbb{R}) \ , \ \ell > 2$$

Each class $T(e_1^\ell)$ is a "generalized winding invariant" for the holonomy of the foliation $\mathcal{F}$ on the fibers of $\mathbb{E} \to M$.

For $m > 1$, $\mathcal{F}$ foliation transverse to $\mathbb{D}^{2m}$-bundle $\mathbb{E} \to M$, then for each $p \in H^*(BSO(2m);\mathbb{Z})$ there is a "generalized non-commutative winding invariant"

$$T(p) \in H^{*-1}(M;\mathbb{R}) \ , \ * > 4m$$

Heitsch Thesis (1970)

Theorem: The Bott Vanishing Theorem is false for $\mathbb{Z}$ coefficients!

Example, continued: Let $(\mathbb{Z}/p\mathbb{Z})^m$ act on $\mathbb{D}^{2m}$ via rotations $\{\varphi_1, \ldots, \varphi_m\}$ with period p on each of the m-factors of $\mathbb{D}^2$.

Form the suspension flat bundle

$$\mathbb{E} = \mathbb{S}^\infty \times \mathbb{D}^{2m}/\varphi$$

Then the composition

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{Z}/p\mathbb{Z}) \to H^*(B\Gamma_{2m}^+; \mathbb{Z}/p\mathbb{Z}) \to H^*(\mathbb{E}; \mathbb{Z}/p\mathbb{Z})$$

is injective. Let $p \to \infty$.

Cheeger-Simons classes and solenoids

Question: How can these higher dimensional classes Cheeger-Simons classes be non-zero for a solenoid defined by a flow on a 3-manifold?

Answer: Follow the idea of the Borel construction, that a fixed-point for a group action is not really a point; in homotopy theory, it is BG_x where G_x is the isotropy subgroup of the fixed-point x.

For a solenoid $\mathcal{S}$ defined by an action of $G = \mathbb{Z}^n$ on $\mathbb{D}^q$, and an open neighborhood $\mathcal{S} \subset U$, the realization $B\Gamma_{U|\mathcal{F}}$ contains a copy of a Borel space BG_x for each fixed point $x \in U$ with finite group action germ. That is, the neighborhood of $\mathcal{S}$ in the foliated manifold M contains all the ingredients needed for the Heitsch proof. In the limit, we obtain $\mathbb{R}$-valued Cheeger-Simons classes supported on the limit $\widehat{\mathcal{S}}$, which is a new object:

$\widehat{\mathcal{S}}$ is a *semi-simplical measured lamination* equipped with a foliated microbundle structure, carrying non-trivial cohomology classes of $B\Gamma_q$.

Shape Cycles

Definition: A *shape cycle* for $B\Gamma_q$ is a foliated lamination $\mathcal{L}$ equipped with a foliated microbundle $\nu \to \mathcal{L}$ such that the classifying map $h^*_{\mathcal{L}} \colon H^*(B\Gamma_q; \mathbb{R}) \to \mathcal{H}^*(\mathcal{L}, \mathcal{F}; \mathbb{R})$ is non-trivial for some $* > 0$.

Typical "shape cycle"

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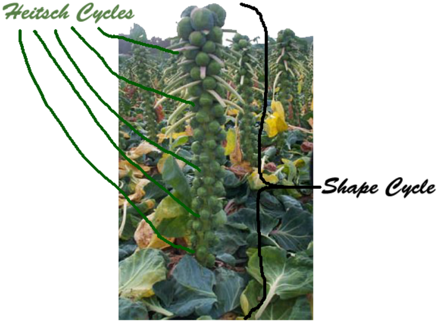

Typical "shape cycle"

Remarks

- New procedure for obtaining foliated cycles in $B\Gamma_q$.

Previous results were constructions in 1970's of minimal foliations on compact manifolds, for which some of the Cheeger-Simons classes are non-trivial in the ranges of degrees $2q + 1 \leq * \leq 2q + q^2$.

- The "shape cycles" which are used to detect this new homotopy structure are derived from the simplest type of wild constructions, the solenoids defined via an infinite sequence of finite coverings of $\mathbb{T}^n$. The embedding into a foliation is a key part of the data.

Question: Do the analogous "shape cycles" obtained from Julia sets of holomorphic dynamics yield non-trivial cycles, for a similar construction with the classifying spaces of complex analytic groupoids? [Asuke, Ghys]