Perspectives on Kuperberg flows

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joint work with Ana Rechtman

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At the *International Symposium/Workshop on Geometric Study of Foliations* in November 1993, there was an evening seminar on a recent preprint by Krystyna Kuperberg, proving:

Theorem *Let M be a closed, orientable 3-manifold. Then M admits a C^∞ non-vanishing vector field whose flow ϕ_t has no periodic orbits.*

- Krystyna Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.

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Ana Rechtman and I investigate these flows in our manuscript:

- H. & R., *The dynamics of generic Kuperberg flows*, **Astérisque**, Vol. 377, 2016; 250 pages.

Wilson Plug

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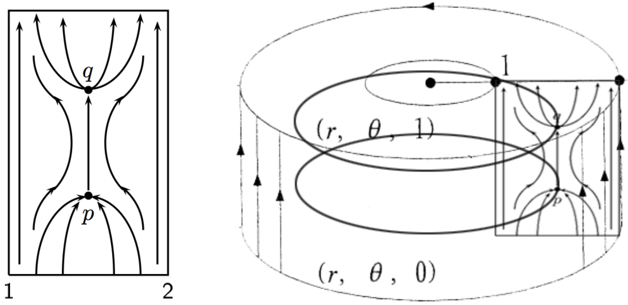
Wilson's fundamental idea in 1966 was the construction of a plug which trapped content, and all trapped orbits have limit set a periodic orbit contained in the plug. The result of repeatedly inserting this plug is a flow with at most two periodic orbits.

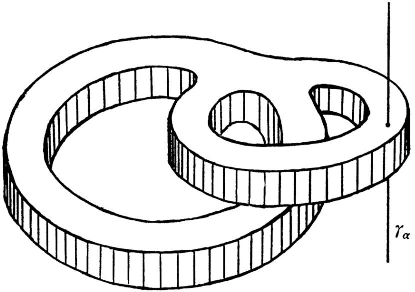


Schweitzer Plug

Paul Schweitzer in 1974 had two deep insights. In the Wilson Plug:

• Each periodic circle can be replaced by a Denjoy minimal set for a flow on a punctured 2-torus;

• the new minimal set does not have to be in a planar flow, but may be contained in a surface flow which embeds in $\mathbb{R}^3$.



Theorem: [Schweitzer, 1974] Every homotopy class of non-singular vector fields on a closed 3-manifold M contains a C^1-vector field with no closed orbits.

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An alternate embedding of the Denjoy minimal set in $\mathbb{R}^3$ was later used by Jenny Harrison in a very difficult proof to show the same result for C^2-flows.

- J.C. Harrison, *C^2 counterexamples to the Seifert conjecture*, **Topology**, 27:249–278, 1988.

So, the reaction to Krystyna's short paper, proving ever so much more, was pure astonishment.

Kuperberg's "Big Idea"

Shigenori Matsumoto's summary:

そこで，どうしても W 内のふたつの周期軌道 T_1 と T_2 を予め破壊しておく必要がある．しかしそのために新しい部品を開発するのでは話は振り出しに戻ってしまう．Kuperberg の発想は，W 内の**周期軌道自身で自分達を破壊させる**というものである． 敵同士が妨害工作をしあうようにわなを仕掛けた後は，何もせずに黙ってみていればよいということである．

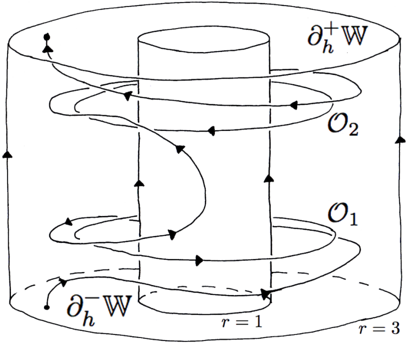
We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to let closed orbits demolish themselves. We set up a trap within enemy lines and watch them settle their dispute while we take no active part.

(transl. by Kiki Hudson Arai)

Modified Wilson Plug $\mathbb{W}$

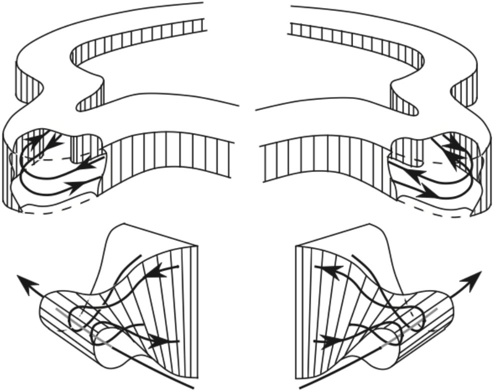
Consider the rectangle $R \times \mathbb{S}^1$ with the vector field $\vec{W} = \vec{W}_1 + f \frac{f}{d\theta}$

f is asymmetric in z and $\vec{W}_1 = g \frac{f}{dz}$ is vertical.

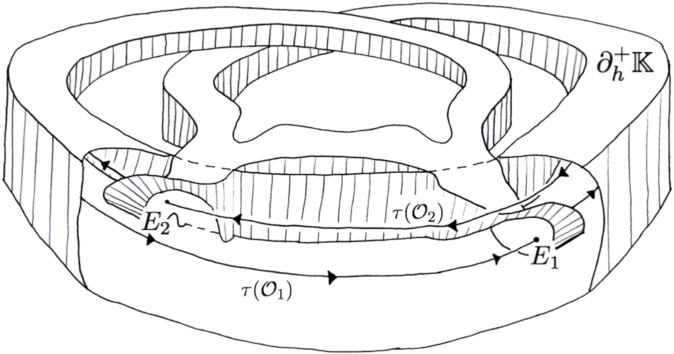


The two periodic orbits are now unstable.

Deform the modified Wilson Plug to have a pair of "horns"



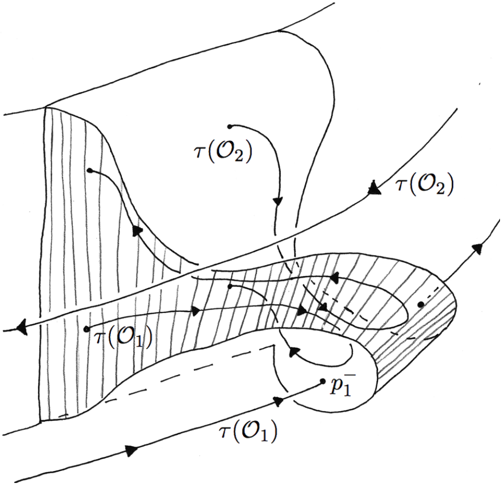
Insert the horns with a twist and a bend, matching the flow lines on the boundaries, to obtain Kuperberg Plug $\mathbb{K}$



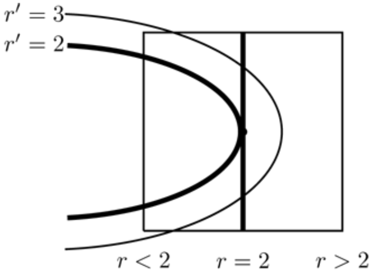
Embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself.

That's it. The resulting flow is aperiodic!

Close up view of the lower embedding σ_1



The insertion map as it appears in the face E_1



Radius Inequality:

For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^{\epsilon}(r', \theta', -2) \in \mathcal{L}_i$, then $r < r'$ unless $x' = (2, \theta_i, -2)$ and then $r = 2$.

Three subsequent papers explored the properties of these flows:

- Étienne Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, **Astérisque**, 227: 283–307, 1995.

- Shigenori Matsumoto, *K.M. Kuperberg's C^∞ counterexample to the Seifert conjecture*, **Sūgaku**, Mathematical Society of Japan, Vol. 47:38–45, 1995. Translation: **Sugaku Expositions**, A.M.S., Vol. 11:39–49, 1998.

- Greg & Krystyna Kuperberg, *Generalized counterexamples to the Seifert conjecture*, **Ann. of Math. (2)**, 144:239–268,, 1996.

These papers also studied the dynamics of Kuperberg flows:

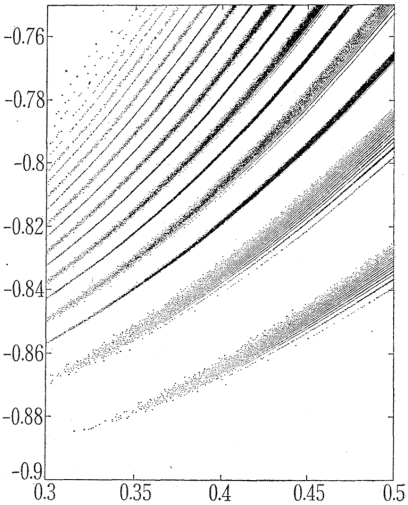
Theorem (Ghys, Matsumoto, 1995) *The Kuperberg flow has a unique minimal set* $\mathcal{Z} \subset M$.

Theorem (Matsumoto, 1995) *The Kuperberg flow has an open set* $\mathfrak{W}$ *of wandering points whose forward orbits limit to the unique minimal set.*

There cannot be an invariant measure equivalent to Lebesgue.

Theorem (A. Katok, 1980) *Let* M *be a closed, orientable 3-manifold. Then an aperiodic flow* ϕ_t *on* M *has zero entropy.*

The other "clue" to the dynamics of Kuperberg flows, was given by the computer model of Bruno Sévennec of the intersection of the unique minimal set with a cross section. Very mysterious!



Problem: Describe the geometric properties of the unique minimal set $\mathcal{Z}$. For example, when is the minimal set 1-dimensional? Or, is it always 2-dimensional, and then what is the growth type of the path connected components of the minimal set?

Problem: Describe the topological shape of the unique minimal set $\mathcal{Z}$. Does $\mathcal{Z}$ have *stable shape*? Is the shape of $\mathcal{Z}$ *movable*, a notion introduced by Karol Borsuk.

- *On movable compacta*, **Fund. Math.**, 66:137–146, 1969.

Karol Borsuk

Karol Borsuk (May 8, 1905 – January 24, 1982) was a Polish mathematician. His main interest was topology.

Borsuk introduced the theory of absolute retracts (AR's) and absolute neighborhood retracts (ANR's), and the cohomotopy groups, later called Borsuk-Spanier cohomotopy groups. He also founded the so-called Shape theory. He has constructed various beautiful examples of topological spaces, e.g. an acyclic, 3-dimensional continuum which admits a fixed point free homeomorphism onto itself; also 2-dimensional, contractible polyhedra which have no free edge. His topological and geometric conjectures and themes stimulated research for more than half a century.

Borsuk received his master's degree and doctorate from Warsaw University in 1927 and 1930, respectively; his Ph.D. thesis advisor was Stefan Mazurkiewicz. He was a member of the Polish Academy of Sciences from 1952. Borsuk's students included Samuel Eilenberg, Jan Jaworowski, *Krystyna Kuperberg*, Włodzimierz Kuperberg, and Andrzej Trybulec.

https : //en.wikipedia.org/wiki/Karol Borsuk

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Problem: Why does a Kuperberg flow have zero entropy? For example, can one calculate the growth rates of ϵ-separated sets for the flow?

Problem: What are the dynamical properties of the flows which are smooth perturbations of a Kuperberg flow? These are called "Derived from Kuperberg" flows, or DK flows.

The question of the shape properties of the unique minimal set $\mathcal{Z}$ was suggested by Alex Clark, and was the motivation for the study by Ana Rechtman and myself, starting in 2010, of the dynamical properties of Kuperberg flows. What we discovered about the shape properties of this minimal set, it would have made Borsuk proud of Krystyna's mathematical legacy!

The first observation of our work was that there are many variations on the Kuperberg construction. A flow is *generic* if it is actually like its illustrations. That is to say, all choices are assumed to be not too pathological.

There are two types of *generic* conditions.

Generic 1: Consider the rectangle $\mathbf{R} = [1,3] \times [-2,2]$, with a vertical vector field $\vec{W}_1 = g\frac{f}{dz}$ where $g(r,z)$ vanishes at $(2,-1)$ and $(2,+1)$. We require that g *vanish to second order with positive definite Jacobian* at these two points.

Then the Wilson field on $\mathbb{W} = \mathbf{R} \times \mathbb{S}^1$ is $\vec{W} = \vec{W}_1 + f\frac{f}{d\theta}$ where f is asymmetric in z, and vanishes near the boundary.

Generic 2: These are conditions on the insertion maps $\sigma_i \colon D_i \to \mathcal{D}_i$. We require that the r-coordinate of the image *depends quadratically* on the θ-coordinate of the domain, for values of r near $r = 2$. Much stronger than the basic radius inequality.

Let Φ_t be a generic Kuperberg flow on a plug $\mathbb{K}$.

Theorem (H & R, 2015) *The unique minimal set $\mathcal{Z}$ for the flow is a 2-dimensional lamination "with dense boundary" and $\mathcal{Z}$ equals the non-wandering set of Φ_t.*

Theorem (H & R, 2015) *The flow Φ_t has positive "slow entropy", for exponent $\alpha = 1/2$.*

Thus, a generic Kuperberg flow *almost* has positive entropy.

Theorem (H & R, 2015) *The minimal set $\mathcal{Z}$ has unstable shape; but may be moveable.*

This implies that it is wildly embedded in $\mathbb{R}^3$.

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Problem: *How do the dynamical properties of the non-generic Kuperberg flows differ from those of the generic flows?*

We also considered "Derived from Kuperberg" constructions, which deform a generic Kuperberg flow $\Phi_t = \Phi_t^0$ on a plug $\mathbb{K}$.

Theorem (H & R, 2016) *There is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $-1 < \epsilon \leq 0$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ has two periodic orbits, and all orbits are properly embedded.*

Theorem (H & R, 2016) *There is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $0 \leq \epsilon < a$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ admits countably many families of "horseshoes" with dense periodic orbits, and so has positive entropy.*

Conclusion: *The generic Kuperberg flows lie at the boundary of chaos (entropy > 0) and the boundary of tame dynamics.*

There is still much about the dynamics of *Derived from Kuperberg* flows that we still don't understand.

- H. & R., *Aperiodic flows at the boundary of chaos*, *arXiv:1603.07877*.
- H. & R., *Perspectives on Kuperberg flows*, *arXiv:1607.00731*.

For the rest of this talk, will try to explain why the Kuperberg flows have unstable minimal sets, and what this means for the dynamical properties of these flows.

The hidden dynamics of Kuperberg flows

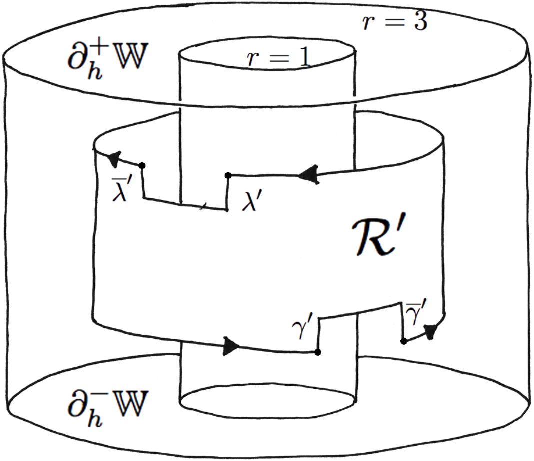
Define the closed subsets of $\mathbb{W} = [1,3] \times \mathbb{S}^1 \times [-2,2] \cong \mathbf{R} \times \mathbb{S}^1$

$\mathcal{D}_i = \sigma_i(D_i)$ for $i = 1, 2$ are solid 3-disks embedded in $\mathbb{W}$.

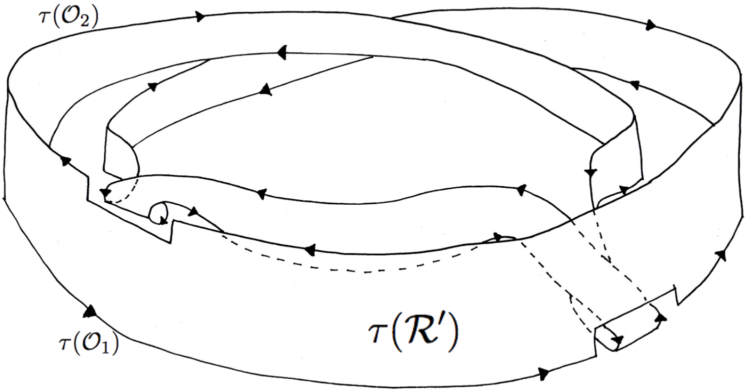
$$\mathbb{W}' \equiv \mathbb{W} - \{\mathcal{D}_1 \cup \mathcal{D}_2\} \quad , \quad \widehat{\mathbb{W}} \equiv \overline{\mathbb{W} - \{\mathcal{D}_1 \cup \mathcal{D}_2\}} \; .$$

$$\begin{array}{lcl} \mathcal{C} & \equiv & \{r = 2\} \quad [\textit{Full} \quad \textit{Cylinder}] \\ \mathcal{R} & \equiv & \{(2, \theta, z) \mid -1 \leq z \leq 1\} \quad [\textit{Reeb} \quad \textit{Cylinder}] \\ \mathcal{R}' & \equiv & \mathcal{R} \cap \widehat{\mathbb{W}} \quad [\textit{Notched} \quad \textit{Reeb} \quad \textit{Cylinder}] \\ \mathcal{O}_i & \equiv & \{(2, \theta, (-1)^i)\}, i = 1, 2 \quad [\textit{Periodic Orbits}] \end{array}$$

$\mathcal{O}_1$ is the lower boundary circle of the Reeb cylinder $\mathcal{R}$,
$\mathcal{O}_2$ is the upper boundary circle.


$r = 3$
$\partial_h^+ \mathbb{W}$
$r = 1$
$\overline{\lambda}'$
λ'
$\mathcal{R}'$
γ'
$\overline{\gamma}'$
$\partial_h^- \mathbb{W}$

The image $\tau(\mathcal{R}') \subset \mathbb{K}$



Consider the $\mathcal{K}$-orbit of the image $\tau(\mathcal{R}') \subset \mathbb{K}$ and its closure

$$\mathfrak{M}_0 \equiv \{\Phi_t(\tau(\mathcal{R}')) \mid -\infty < t < \infty\} \quad , \quad \mathfrak{M} \equiv \overline{\mathfrak{M}_0} \subset \mathbb{K} \ .$$

Then $\mathcal{O}_i \cap \widehat{\mathbb{W}} \subset \mathcal{R}'$ hence the minimal set $\mathcal{Z} \subset \mathfrak{M}$.

The non-wandering set of Φ_t in $\mathbb{K}$ is denoted Ω.

Theorem: *Let Φ_t be a generic Kuperberg flow, then $\mathcal{Z} = \Omega = \mathfrak{M}$.*

Theorem: *For a generic Kuperberg flow, the space $\mathfrak{M}$ has the structure of a zippered lamination with 2-dimensional leaves.*

Proof: The closures of the boundary orbits of $\mathfrak{M}_0$ are dense in $\mathfrak{M}$. Hence, the boundary of each leaf of $\mathfrak{M}$ is dense in itself.

Definition: Let $\mathfrak{Z} \subset X$ be a continuum embedded in a metric space X. A *shape approximation* of $\mathfrak{Z}$ is a sequence $\mathfrak{U} = \{U_\ell \mid \ell = 1, 2, \ldots\}$ satisfying the conditions:

1. each U_ℓ is an open neighborhood of $\mathfrak{Z}$ in X which is homotopy equivalent to a compact polyhedron;
2. $U_{\ell+1} \subset U_\ell$ for $\ell \geq 1$, and their closures satisfy $\bigcap_{\ell \geq 1} \overline{U}_\ell = \mathfrak{Z}$.

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Proposition: *The shape of $\mathcal{Z}$ is equal to the shape of $\mathfrak{M}_0$.*

Shape homotopy groups:

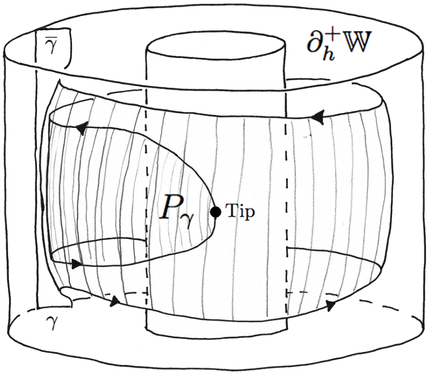
$$\widehat{\pi}_1(\mathcal{Z}, x_*) \equiv \varprojlim \; \{\pi_1(U_{\ell+1}, x_*) \to \pi_1(U_\ell, x_*)\}$$

Strategy: Use "almost closed paths" in $\mathfrak{M}_0$ to calculate $\widehat{\pi}_1(\mathcal{Z}, x_*)$.

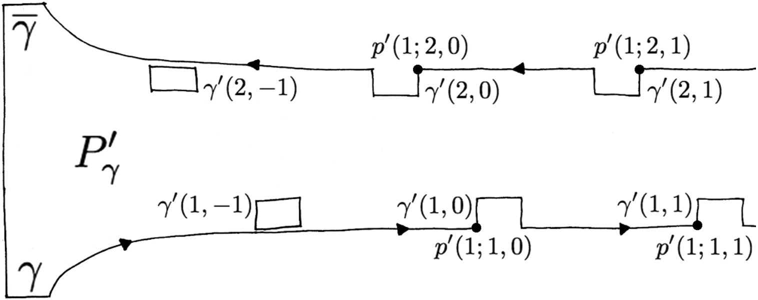
Problem: Give an explicit description of the embedded space $\mathfrak{M}_0$.

This is done in the Astérisque volume. The key observation is that $\mathfrak{M}_0$ is an infinite union of propellers at increasing levels. We give a sequence of illustrations to suggest how this works.

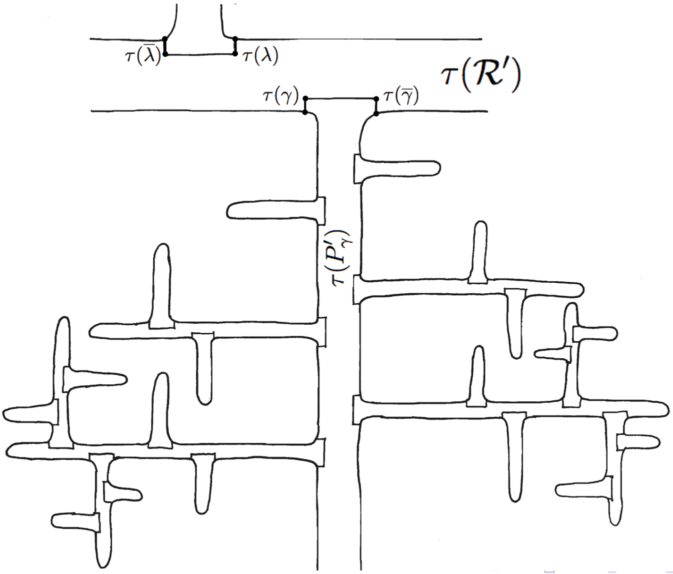
Finite propeller = flow of entry arc γ in region $r > 2$



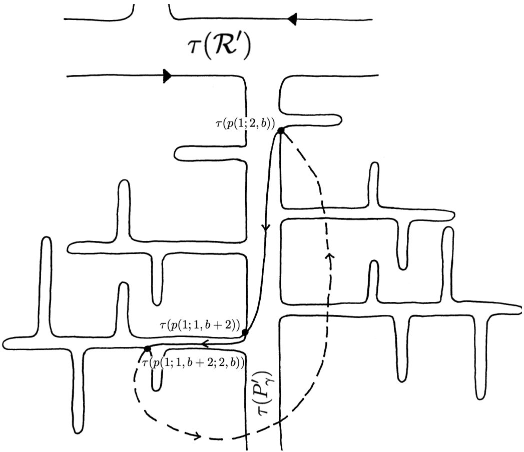
Infinite propeller with infinite sequence of notches to which finite propellers are attached.



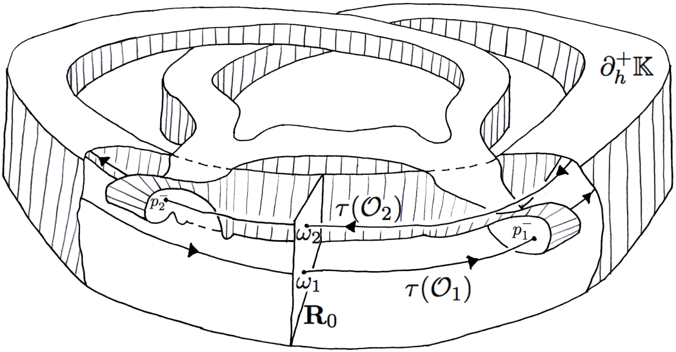
The surface $\mathfrak{M}_0$ is embedded in $\mathbb{K}$, where each propeller wraps around the Reeb cylinder $\tau(\mathcal{R}')$.



An approximate cycle is an “almost closed” path with endpoints sufficiently close. This yields a class in $\widehat{\pi}_1(\mathcal{Z}, x_*)$ by “translating the loop out to infinity in open neighborhoods of $\mathfrak{M}_0$”.



Need a systematic method to label the classes that arise from almost closed paths in $\mathfrak{M}_0$. **Idea:** Introduce a section $\mathbf{R}_0 \subset \mathbb{K}$ to the flow Φ_t and study the intersections of $\mathfrak{M}_0 \cap \mathbf{R}_0$.



The return map to the section defines a pseudogroup $\mathcal{G}_\Phi$

The flow of Φ_t is tangent to $\mathbf{R}_0$ along the center plane $\{z = 0\}$, so the action of the pseudogroup has singularities along this line.

Critical difficulty: *There is not always a direct relation between the continuous dynamics of the flow* Φ_t *and the discrete dynamics of the action of the pseudogroup* $\mathcal{G}_\Phi$.

None the less, the introduction of the Kuperberg pseudogroup $\mathcal{G}_\Phi$ is a fundamental tool for the study of the dynamics of the flow Φ_t.

The semi-group formed by the generators of $\mathcal{G}_\Phi$ are used to give a complete description of the embedded space $\mathfrak{M}_0$.

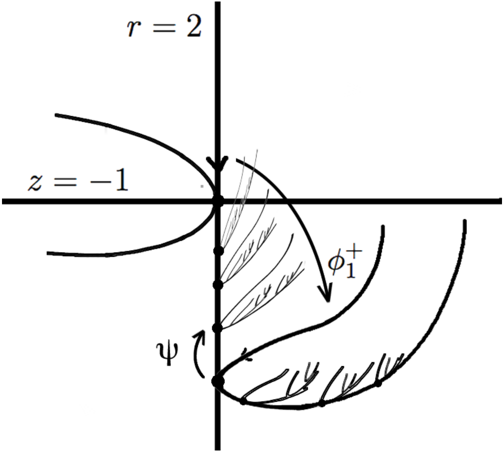
We consider two maps with domain in $\mathbf{R}_0$

- ψ which is the return map of the *Wilson flow* Ψ_t
- ϕ_1 which is the return map of the *Kuperberg flow* Φ_t for orbits that go through the entry region E_1

They generate a pseudogroup $\widehat{\mathcal{G}} = \langle \psi, \phi_1 \rangle$ acting on $\mathbf{R}_0$.

Proposition: *The restriction of* $\widehat{\mathcal{G}}$ *to the region* $\{r > 2\} \cap \mathbf{R}_0$ *is a sub-pseudogroup of* $\mathcal{G}_\Phi$ *The action of* $\widehat{\mathcal{G}}$ *on the line segment* $\mathcal{C} \cap \mathbf{R}_0$ *yields families of nested ellipses containing* $\mathbf{R}_0 \cap \mathfrak{M}_0$*.*

Corollary: *The elements of the pseudogroup* $\widehat{\mathcal{G}}$ *labels the lower branches of the tree structure of* $\mathfrak{M}_0$*.*

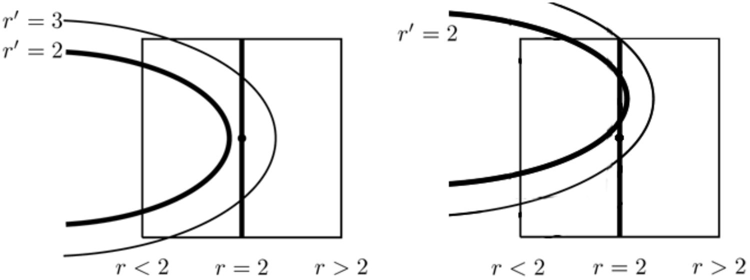


This looks like a ping-pong game, except that the play action is too slow to generate entropy.

Definition: A *Derived from Kuperberg* (DK) flow is obtained by choosing the embeddings so that we have:

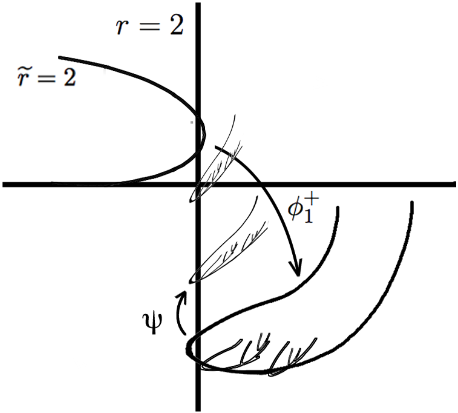
Parametrized Radius Inequality: For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^{\epsilon}(r', \theta', -2) \in \mathcal{L}_i$, then $r < r' + \epsilon$ unless $x' = (2, \theta_i, -2)$ and then $r = 2 + \epsilon$.

The modified radius inequality for the cases $\epsilon < 0$ and $\epsilon > 0$:

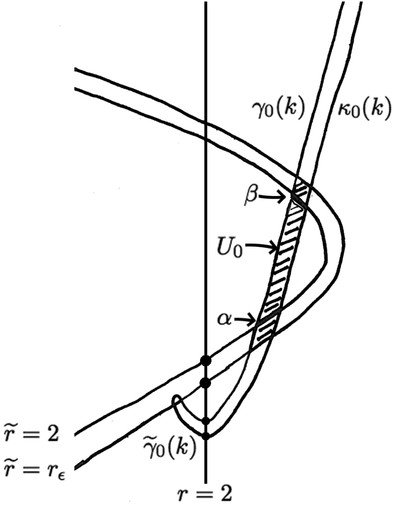


Meta-Principle: For $\epsilon > 0$ and "most" classes $[\gamma] \in \widehat{\pi}_1(\mathfrak{M}, \omega_0)$, there is a horseshoe subdynamics for the pseudogroup $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$ acting on $\mathbf{R}_0$ with a periodic orbit defining $[\gamma]$.

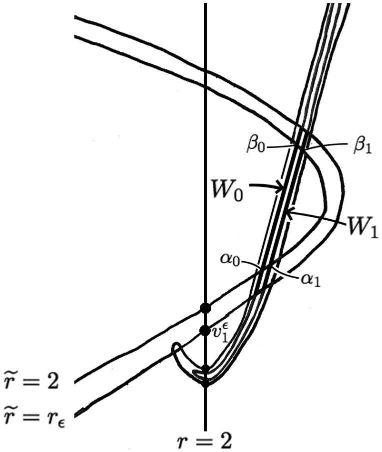
Action of $\widehat{\mathcal{G}}_\epsilon$ on the line $r = 2$ for $\epsilon > 0$.



Define a compact region $U_0 \subset \mathbf{R}_0$ which is mapped to itself by the map $\varphi = \psi^k \circ \phi_1^\epsilon$ for k sufficiently large. "k large" corresponds to translating the almost closed path far out in the surface $\mathfrak{M}_0$.

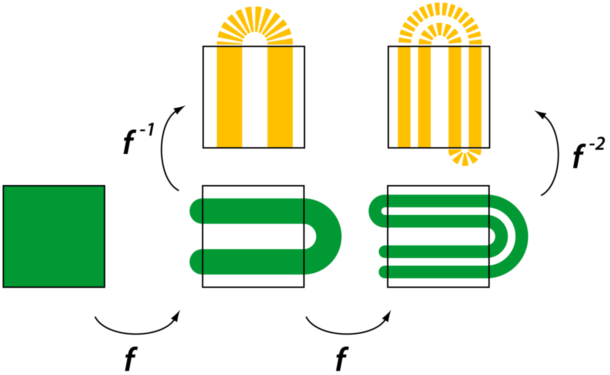


The images of the powers φ^{ℓ} form a δ-separated set for the action of the pseudogroup $\widehat{\mathcal{G}}_{\epsilon}$.



This is a horseshoe dynamical system for the *pseudogroup action*!

Compare to the usual illustration of a horseshoe dynamical system.



Concluding remarks:

- For $\epsilon < 0$, the dynamics of a DK flow is tame, and completely predictable, except that as $\epsilon \to 0$ the dynamics approaches that of the Kuperberg flow.

- For $\epsilon > 0$, the dynamics of a DK flow is chaotic, but making calculations of entropy for example, is only possible in special instances. Also, we have no intuition, for example, of how to describe the nonwandering sets for DK flow with $\epsilon > 0$.

- There are many more open questions and results to prove about Kuperberg flows and their variations!

Thank you Krystyna for your flows!!