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Hyperbolic and non-hyperbolic minimal sets of foliations

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The set-up:

M is a compact riemannian manifold;

$\mathcal{F}$ is a C^r foliation, for $r \geq 0$; usually $r \geq 1$.

q is the codimension of $\mathcal{F}$, p is the leaf dimension;

each leaf $L \hookrightarrow M$ is an immersed submanifold.

The "dynamics of $\mathcal{F}$" refers to the study of the recurrence and ergodic properties of the leaves of $\mathcal{F}$.

For $p = 1$ and $\mathcal{F}$ oriented, this is the study of the dynamics of the non-singular flow of the positively oriented vector field spanning $\mathcal{F}$.

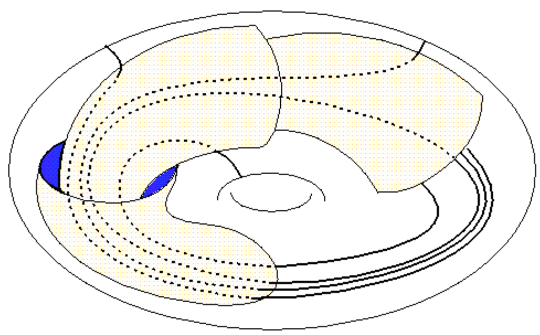
Given any finitely-generated group Γ and a C^r-action ϕ on a compact manifold N^q, there is a foliation $\mathcal{F}_\phi$ by surfaces of a compact manifold M^{q+2} with a section $\mathcal{T} \cong N$, whose dynamics is equivalent to that of the group action.

So for $p \geq 2$, the dynamics of $\mathcal{F}$ includes the study of the dynamics of finitely-generated group actions on compact manifolds. It also asks how the geometry of the leaves influences their recurrence and ergodic properties.

Definition: $X \subset M$ is a minimal set for $\mathcal{F}$ if

- X is a union of leaves
- X is closed
- every leaf $L \subset X$ is dense in X

If M is a compact manifold without boundary, for every leaf L of $\mathcal{F}$, its closure $\overline{L}$ always contains at least one minimal set.



The restriction of $\mathcal{F}$ to X defines a minimal *foliated space* or a *minimal lamination*.

Problem: What minimal laminations arise as the minimal sets for C^r-foliations?

Problem: Is there a restriction on the topological shape and embedding of a minimal lamination $\mathbf{K} \subset M$, imposed by the topology of M for example?

Problem: How do you characterize the dynamics of a foliated minimal lamination? (e.g., positive entropy; expansive; hyperbolic; distal; equicontinuous)

Problem: Given a complete Riemannian manifold L of bounded geometry, is it diffeomorphic (or quasi-isometric) to a leaf in a minimal set of a foliation?

In this talk, will discuss two types of examples recently discovered which realize Sierpinski spaces and solenoids as minimal sets of smooth foliations.

The examples suggests many questions.

Let $\mathcal{T}$ be a closed manifold of dimension q, possibly with boundary, and $\mathcal{T} \hookrightarrow M$ an embedding which is transverse to $\mathcal{F}$ and intersects each leaf of $\mathcal{F}$.

A minimal set X is *exceptional* if $\mathbf{K} = X \cap \mathcal{T}$ has no interior, and is not a finite set.

X is *exotic* if $\mathbf{K} = X \cap \mathcal{T}$ is exceptional, and not locally homeomorphic to the product of a manifold and a totally disconnected Cantor set.

Problem: What continua can be realized as $\mathbf{K} = X \cap \mathcal{T}$ for a minimal set X of a C^r-foliation?

Transversal pseudogroup: $\mathcal{T}$ a Riemannian manifold of dimension q.

Example 1: $\mathcal{T} = N$ a closed Riemannian q-manifold.

Example 2: $\mathcal{T} \subset \mathbb{R}^q$ is a disjoint union of closed disks.

Definition A C^r-pseudogroup of transformations $\mathcal{G}$ of $\mathcal{T}$ is *compactly generated* if

1. $\mathcal{T}$ contains a relatively compact open subset $\mathcal{T}_0$ meeting all the orbits of $\mathcal{G}$

2. there is a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}^r(\mathcal{T})$ such that $\mathcal{G}(\Gamma) = \mathcal{G}|\mathcal{T}_0$;

3. $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}^r(\mathcal{T})$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

$\overline{D(g)}$ the closure of the domain of g in $\mathcal{T}$

$\overline{R(g)}$ the closure of the range of g in $\mathcal{T}$

A C^r-foliation $\mathcal{F}$ of a compact manifold M always has a finite open covering $\{\phi_\alpha \colon U_\alpha \to \mathbb{D}^{p+q}\}$ by foliation charts which defines (via holonomy transformations) a compactly generated C^r-pseudogroup of transformations $\mathcal{G}$ so that the dynamics of $\mathcal{F}$ and of $\mathcal{G}$ are equivalent.

A closed subset $\mathbf{K} \subset \mathcal{T}$ is a minimal set for $\mathcal{G}$ if

- for each $x \in \mathbf{K}$, $\mathcal{G} \cdot x \subset \mathbf{K}$
- $\mathbf{K}$ is closed
- for each $x \in \mathbf{K}$, $\mathcal{G} \cdot x$ is dense in $\mathbf{K}$

Problem: What continua $\mathbf{K} \subset \mathcal{T}$ arise as the minimal sets for compactly generated, C^r-pseudogroups of transformations $\mathcal{G}$ of $\mathcal{T}$?

This problem is actually much easier than the analogous problem for foliations, as there is no requirement the dynamic can be completed to a system (foliation) on a compact manifold.

Every Iterated Function System is of this type, but it is generally unknown when a given IFS can be realized as the restriction of a holonomy pseudogroup of a foliation on a compact manifold.

Foliation entropy: For $g \in \mathcal{G}$, the word length $\|g\|_x$ is the least n such that

$$Germ_x(g) = Germ_x(g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_n}^{\pm 1})$$

Let $x \in U \subset \mathcal{T}$, U an open neighborhood.

$\mathcal{S} = \{x_1, \ldots, x_\ell\} \subset U$ is (n, ϵ)-separated if for all $x_i \neq x_j$

$\exists\ g \in \mathcal{G}|U$ such that $\|g\|_x \leq n$ and $d_{\mathcal{T}}(g(x_i), g(x_j)) \geq \epsilon$

$$h(U, n, \epsilon) = \max\ \#\{\mathcal{S} \mid \mathcal{S} \subset U \text{ is } (n, \epsilon) \text{ separated}\}$$

$B(x, \delta) \subset \mathcal{T}$ is open δ–ball about $x \in \mathcal{T}$

The *local entropy* of $\mathcal{G}$ at x is

$$h_{loc}(\mathcal{G}, x) = \lim_{\delta \to 0} \left\{ \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{\ln h(B(x, \delta), n, \epsilon)}{n} \right\} \right\}$$

Example: $\mathcal{G}$ generated by an expanding map $f\colon N \to N$ of a compact manifold N, then $h_{loc}(\mathcal{G}, x) = h_{loc}(f, x)$ is the usual local entropy of f.

$h_{loc}(\mathcal{G}, x) > 0$ means that looking out from the point x along orbits of $\mathcal{G}$, one sees exponentially growing chaos in every open neighborhood about x.

The geometric entropy of $\mathcal{G}$:

$$h(\mathcal{G}) = \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{\ln h(\mathcal{T}, n, \epsilon)}{n} \right\}$$

The geometric entropy of $\mathcal{F}$ is $h(\mathcal{F}) = h(\mathcal{G})$ where $\mathcal{G}$ is the pseudogroup determined by a good covering of M by foliations charts.

Proposition: $\mathcal{G}$ compactly generated $\Longrightarrow$

$$h(\mathcal{F}) \equiv h(\mathcal{G}) = \sup_{x \in \mathcal{T}} h_{loc}(\mathcal{G}, x)$$

Definition: A compactly generated pseudogroup $(\mathcal{M}, \Gamma_0)$, $\Gamma_0 = \{h_1, \ldots, h_k\}$ a generating set, is a Markov subpseudgroup for $\mathcal{G}$ if $k \geq 2$ and the generators satisfy:

1. each $h_i \in \Gamma_0$ is the restriction of an element $\widetilde{h}_i \in \mathcal{G}$ with $\overline{D(h_i)} \subset D(\widetilde{h}_i)$
2. (Open Set Condition) $R(h_i) \cap R(h_j) = \emptyset$ for $i \neq j$
3. if $R(h_i) \cap D(h_j) \neq \emptyset$ then $R(h_i) \subset D(h_j)$

Remarks:

- Can use $\widetilde{h}_i \colon \overline{D(h_i)} \to \overline{R(h_i)}$ in the definition.
- (2) does not assume $\overline{R(h_i)} \cap \overline{R(h_j)} = \emptyset$ for $i \neq j$.
- $(\mathcal{M}, \Gamma)$ is *discrete* if $\overline{R(h_i)} \cap \overline{R(h_j)} = \emptyset$ for $i \neq j$.

This last property corresponds to the *Strong Open Set Condition* for iterated function systems.

The *transition matrix* P for a Markov pseudogroup $(\mathcal{M}, \Gamma_0)$ is the $k \times k$ matrix with entries $\{0, 1\}$ defined by $P_{ij} = 1$ if $R(h_j) \subset D(h_i)$, and 0 otherwise.

• $(\mathcal{M}, \Gamma_0)$ is a *chaotic Markov pseudogroup* $\iff$ P is irreducible and aperiodic, so there exists $\ell > 0$ such that $(P^\ell)_{ij} \geq 1$ for all $1 \leq i, j \leq k$.

Definition: A $\mathcal{G}$-invariant minimal set $\mathbf{K} \subset \mathcal{T}$ is Markov if there is a chaotic Markov pseudogroup $(\mathcal{M}, \Gamma)$ such that

$$\mathbf{K} \subset \overline{R(h_1)} \cup \cdots \cup \overline{R(h_k)}$$

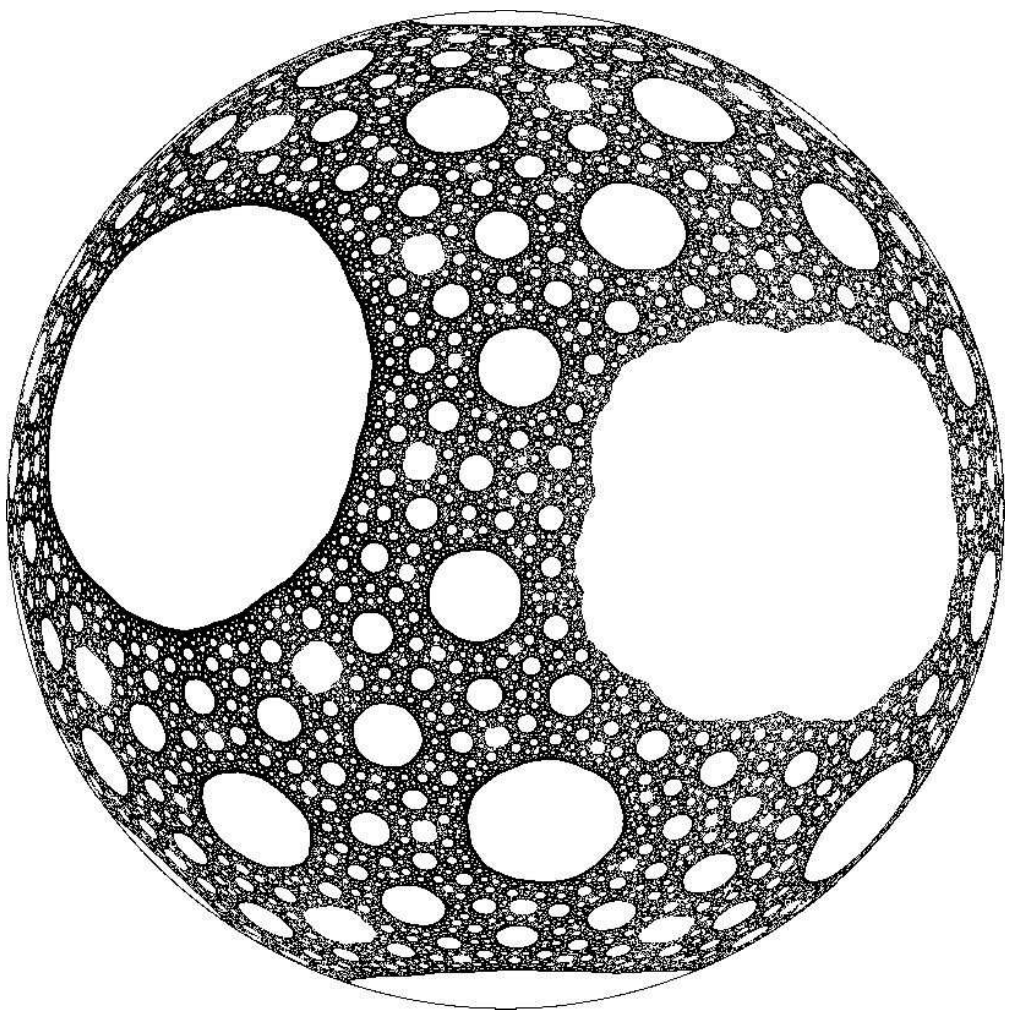
Here is a useful fact.

Theorem 1: If $\mathbf{K}$ is Markov then the relative foliation entropy $h(\mathcal{F}, K) > 0$.

Conjecture 1: Foliation entropy $h(\mathcal{G}, \mathcal{T}) > 0$ implies there is a Markov minimal set $\mathbf{K} \subset \overline{\mathcal{G} \cdot x}$.

Known examples includes McSwiggen's construction of a C^2-action of $\mathbb{Z}$ on $\mathbb{T}^q$ with dense wandering domains. [*Diffeomorphisms of the k-torus with wandering domains*, Ergodic Theory Dynam. Systems, 15:1189–1205, 1995.] These examples have entropy 0.

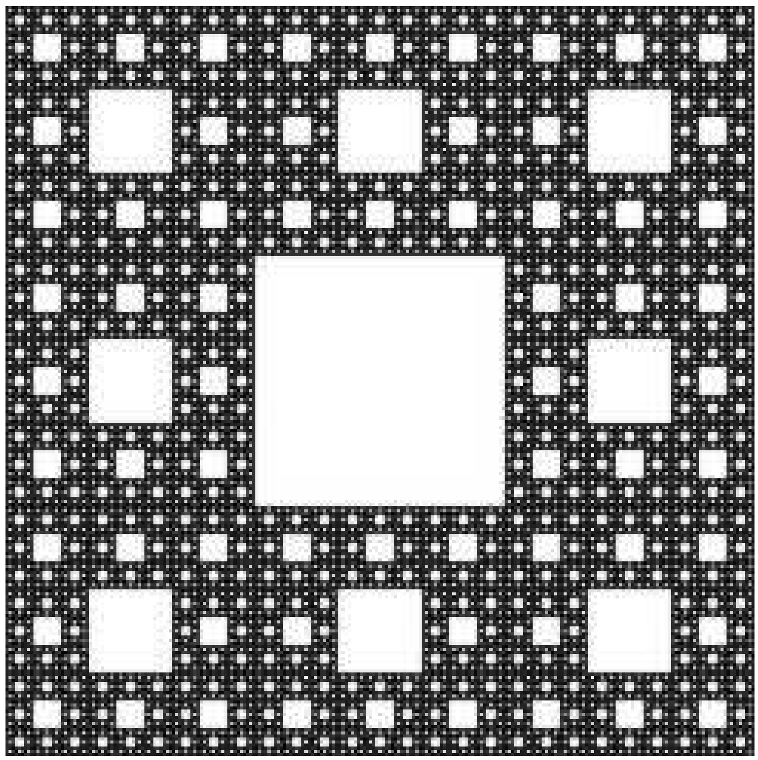
The suspension of Kleinian group actions on the sphere at $\infty \cong \mathbb{S}^q$ give examples were the minimal set is locally Sierpinski. These examples have positive entropy.



Graphic from [*The classification of conformal dynamical systems* by Curt McMullen]

Here are some recent results:

Theorem 1: [Bís, Hurder & Shive 2005] There exists a smooth foliation of a compact 4-manifold M, whose leaves are 2-dimensional, such that there is a unique chaotic Markov minimal set X that has intersection with a transversal $\mathbb{T}^2 \hookrightarrow M$ is a Sierpinski 2-torus:



The leaves inside of X are quasi-isometric to "tree-like" realizations of orbits of a generalized "Iterated Function System".

The above example is realized using a single covering map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ which is a deformation of the standard affine map $(x, y) \mapsto (2x, 2y)$.

The construction works more generally for the following data:

• N is a closed manifold of dimension q,

• $\Gamma_0 = \{f_1\colon N \to N, \ldots, f_k\colon N \to N\}$ a collection of self-covering maps.

Theorem 2: There exists a foliated manifold M whose whose dynamics is equivalent to that generated by the system of maps Γ.

The construction uses a generalization of the Hirsch example.

The dimension of the leaves is generally much greater than 2, and depends on the orders of the various covering groups.

As an application of Theorem 2, we have:

Theorem 3: [BHS 2005] For $q > 2$, there exists a smooth foliation $\mathcal{F}$ of a compact M such that $\mathcal{F}$ has a unique chaotic Markov minimal set X whose intersection $\mathbf{K} = X \cap \mathbb{T}^q$ with a transversal $\mathbb{T}^q \hookrightarrow M$ is a Sierpinski k-torus:

$$\pi_\ell(\mathbf{K}) = \begin{cases} \mathbb{Z}^\infty, & \ell = k-1, \\ 0, & \text{otherwise.} \end{cases}$$

There are many other constructions of exotic minimal sets for deformations of systems of expanding and of partially expanding self-maps of compact manifolds.

The use of self-coverings seems to have much more freedom to create wandering domains for smooth systems, than happens for systems of diffeomorphisms.

We can use this construction to obtain foliation of codimension q with transversally affine, local holonomy equivalent to $\mathbf{SL}(q,\mathbb{Z})$.

Let $\Gamma \subset \mathbf{SL}(q,\mathbb{Z})$ be a finitely generated subgroup; or rather, for matrices $\{A_1,\ldots,A_k\} \subset \mathbf{SL}(q,\mathbb{Z})$ let Γ denote the group they generate. For each index $1 \leq \ell \leq k$, let $\lambda_\ell \in \mathbb{N}$ be a positive integer, and let $\Lambda_\ell = \lambda_\ell \cdot Id$ be the diagonal matrix with all diagonal entries λ_ℓ. Let $B_\ell = \Lambda_\ell \cdot A_\ell$ be the integer matrix with inverse $B_\ell^{-1} \in \mathbf{SL}(q,\mathbb{Q})$.

$$\begin{array}{ccc} \mathbb{T}^q & \xrightarrow{A_\ell} & \mathbb{T}^q \\ \downarrow Id & & \downarrow \Lambda_\ell \\ \mathbb{T}^q & \xrightarrow{B_\ell} & \mathbb{T}^q \end{array}$$

The construction above yields a foliation $\mathcal{F}_{\mathcal{C}}$ of codimension q with transversal $\mathbb{T}^q$ whose global holonomy induced on the section $M_0 = \mathbb{T}^q$ is equivalent to the pseudogroup $\Gamma_{\mathcal{C}}$ generated by the maps $\{\widetilde{B}_\ell \colon \mathbb{T}^q \to \mathbb{T}^q \mid 1 \leq \ell \leq k\}$.

Let $\Gamma = \mathbf{SL}(q, \mathbb{Z})$ and $\{A_1, \ldots, A_k\}$ be a set of generators. Note that for any pair $1 \leq i, j \leq k$ we have that

$$[B_i, B_j] = B_i B_j B_i^{-1} B_j^{-1} = A_i A_j A_i^{-1} A_j^{-1} = [A_i, A_j]$$

as the factors Λ_i and Λ_j are multiples of the identity. Thus, the subgroup $\widehat{\Gamma} = \langle B_1, \ldots, B_k \rangle \subset \mathbf{SL}(q, \mathbb{Q})$ generated by the matrices $\{B_\ell\}$ contains a subgroup isomorphic to the commutator subgroup $[\Gamma, \Gamma] \subset \mathbf{SL}(q, \mathbb{Z})$.

$[\Gamma, \Gamma]$ is a normal subgroup of finite index in $\mathbf{SL}(q, \mathbb{Z})$.

While the commutator $[\widetilde{B}_i, \widetilde{B}_j]$ of maps is not well-defined as diffeomorphisms of $\mathbb{T}^q$, it is well-defined as local elements of the holonomy groupoid $\Gamma_{\mathcal{C}}$. Thus, the holonomy groupoid $\Gamma_{\mathcal{C}}$ contains a subgroupoid equivalent to that generated by the action of $[\Gamma, \Gamma]$ on $\mathbb{T}^q$.

Theorem 4: [Hurder 2006] The foliations $\mathcal{F}_{\mathcal{C}}$ are stable under C^1 deformations.

Question: Are the foliations $\mathcal{F}_{\mathcal{C}}$ stable under C^1 perturbations?

Theorem 5: [Clark & Hurder 2006] Let $\mathcal{F}$ be a C^r foliation of a compact manifold M, $r \geq 1$. If $\mathcal{F}$ has a compact leaf L such that:

- The holonomy of L is germinally C^r-flat,
- There is a surjective map $\rho\colon \pi_1(L) \to \mathbb{Z}^k$.

Then there exists a C^r-foliation $\mathcal{F}'$ which is C^r-close to $\mathcal{F}$ such that $\mathcal{F}'$ has a solenoidal minimal set X whose leaves are $\mathbb{Z}^k$-covers of L.

Moreover, X admits an $\mathcal{F}'$-holonomy invariant transverse measure, and the geometric entropy $h(\mathcal{F}', X) = 0$ so X is not Markovian.

[*Embedding solenoids*, Alex Clark & Robbert Fokkink, Fund. Math., Vol. 181, 2004. p. 111]

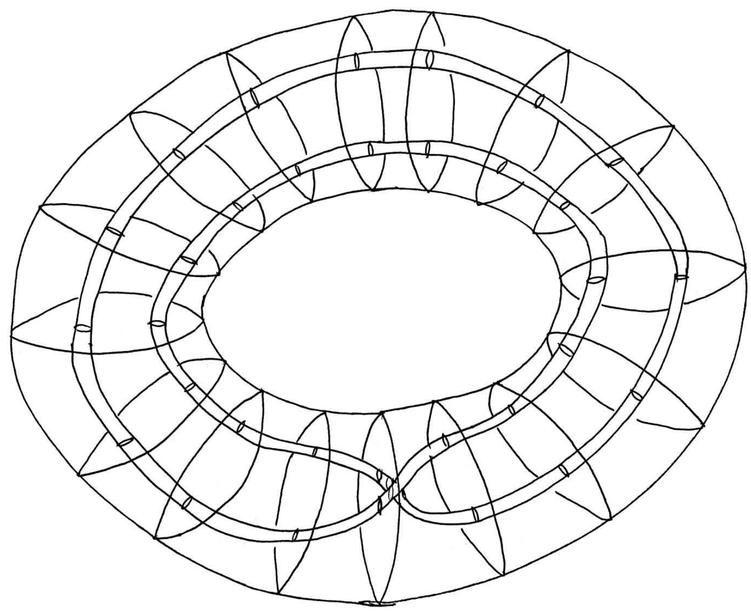
Remark: This construction can be made for the compact leaves in the generalized Hirsch foliations in Theorems 1 & 2. It produces a second foliation which is transverse to the "Hirsch foliation", and which contains an embedded solenoidal minimal set.

There is a "duality" between the generalized Hirsch foliations and embedded solenoids, in that the holonomy of the Hirsch foliations generate expanding maps of the solenoids, while the fiber dynamics of the solenoid capture the graph structure of the leaves of the Hirsch foliation. It is a sort of "resonance" between the dynamics of the two transverse foliations - one foliation is transversally hyperbolic, the other is transversally distal.

The resonance can be seen at the level of group actions, by considering the action of the fundamental group $\widehat{\Gamma} = \pi_1(M)$ on the universal covering $\widetilde{M} \to M$, where there are two transverse foliations and the action is partially hyperbolic.

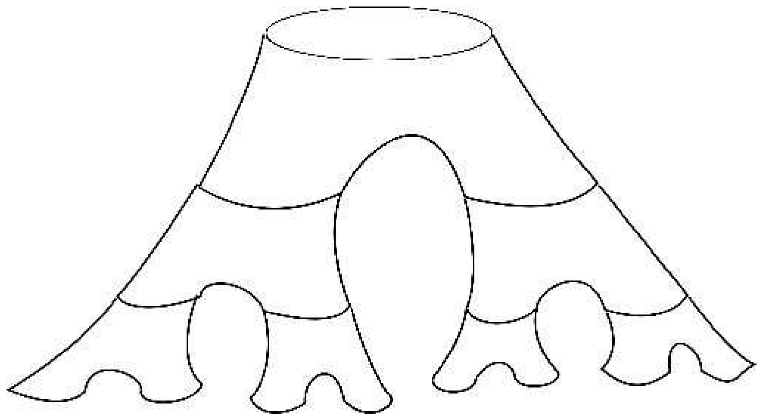
The Construction.

The traditional construction of the affine Hirsch example proceeds as follows. Choose an analytic embedding of S^1 in the solid torus $D^2 \times S^1$ so that its image is twice a generator of the fundamental group of the solid torus.

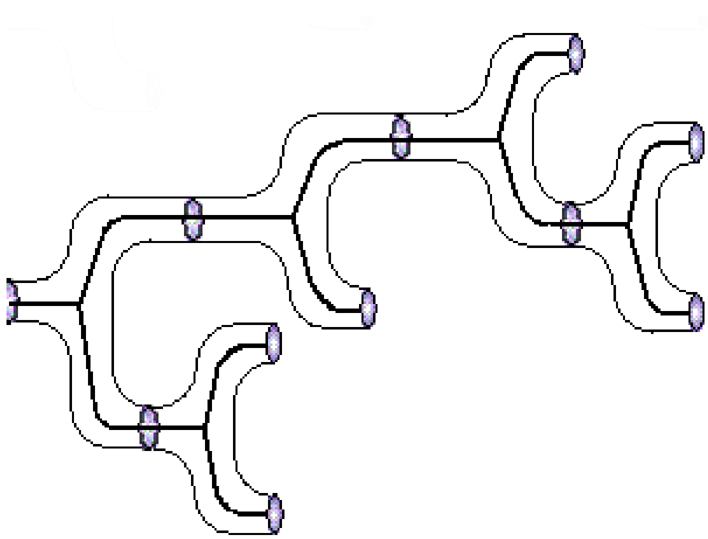


Remove an open tubular neighborhood of the embedded S^1. What remains is a three dimensional manifold N_1 whose boundary is two disjoint copies of T^2. $D^2 \times S^1$ fibers over S^1 with fibers the 2-disc. This fibration restricted to N_1 foliates N_1 with leaves consisting of 2-disks with two open subdisks removed.

Now identify the two components of the boundary of N_1 by a diffeomorphism which covers the map $z \mapsto z^2$ of S^1 to obtain the manifold N. Endow N with a Riemannian metric; then the punctured 2-disks foliating N_1 can now be viewed as pairs of pants.



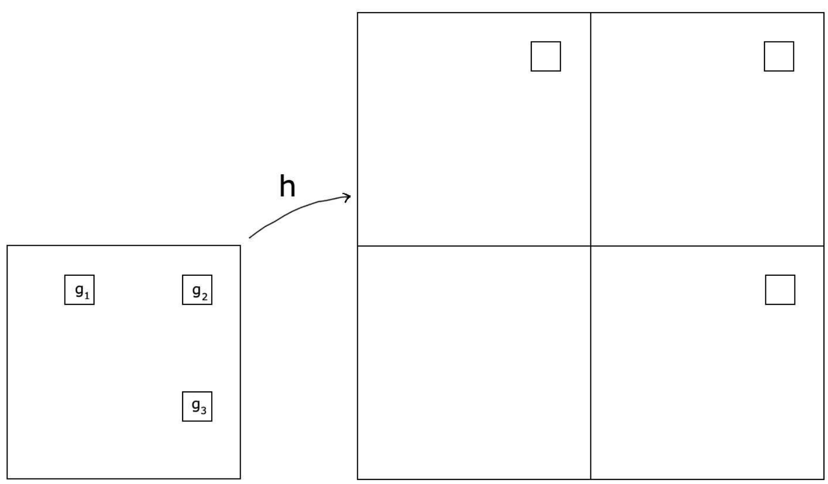
As the foliation of N_1 is transverse to the boundary, the punctured 2-disks assemble to yield a foliation of foliation $\mathcal{F}$ on N, where the leaves without holonomy (corresponding to irrational points for the chosen doubling map of S^1) are infinitely branching surfaces, decomposable into pairs-of-pants which correspond to the punctured disks in N_1.



The construction is generalized in two ways:

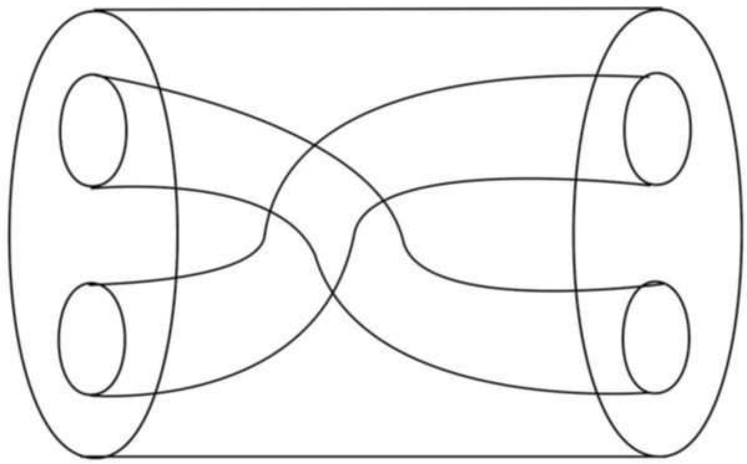
- Replace the map $z \mapsto z^2$ of $\mathbb{S}^1$ with a self-covering $f\colon N \to N$.

To obtain the example of Theorem 2, choose a map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ as pictured below, where $h = f^{-1}$, so that f has a sink on the square in the upper right square:



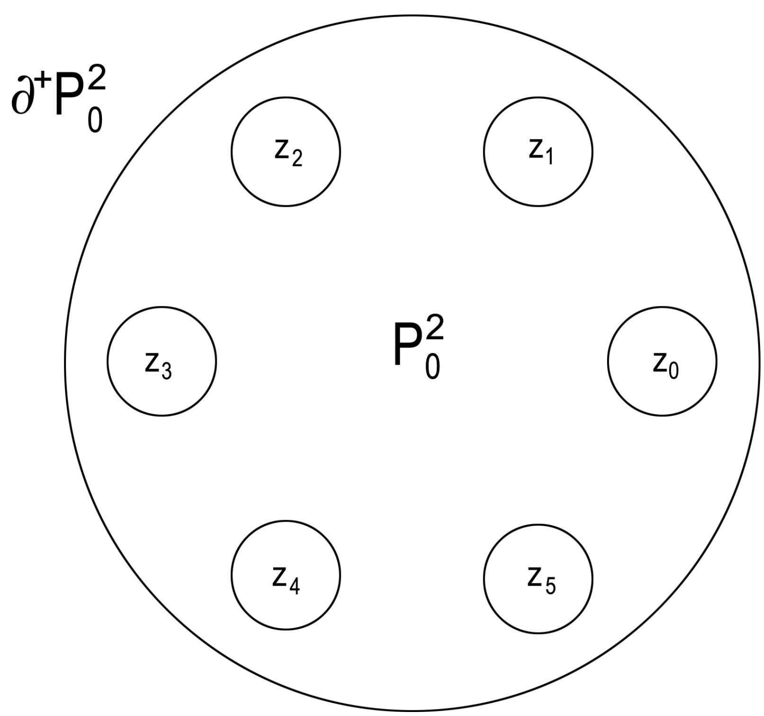
If the map f is a uniform expander, then the foliation $\mathcal{F}$ obtained from gluing will have all leaves dense. The idea in general is to introduce sinks of various shapes – then the foliation $\mathcal{F}$ will have a minimal set consisting of the regions "left over".

• Replace the simple braid of $\mathbb{S}^1 \times \mathbb{D}^2$ implicitly used in the original Hirsch construction



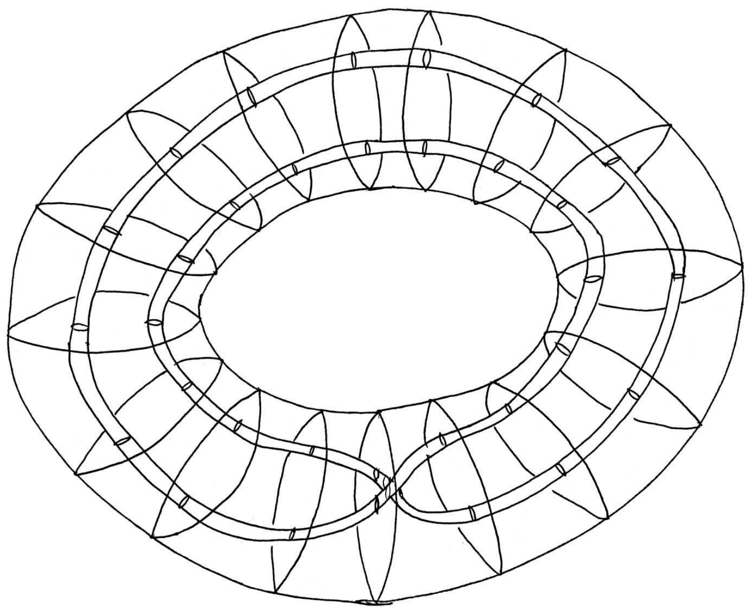
with a "flat-bundle braid" of $N \times \mathbb{S}^p$, $p \geq 2$ obtained from a representation,

$$\Pi = \pi_1(N)/f_*\pi_1(N), \ \rho \colon \Pi \to \mathbf{SO(p+1)}$$



Proof of Theorem 5: Step 1

Consider again the construction of the Hirsch example:



There is a core circle $\mathbb{S}^1$, and the torus is the boundary of a product bundle $\mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1$.

Given the leaf $L \subset M$, the radial "blow-up"' of its normal bundle at the zero section yields a product bundle $L \times \mathbb{S}^{q-1} \to L$. It is trivial since the linear holonomy of L is trivial, and the foliation is a product on this bundle with all leaves trivial coverings of the core L. The bundle is the boundary of the disk bundle $L \times \mathbb{D}^q$, and the idea is to recursively extend the foliation into this tube.

Proof of Theorem 5: Step 2

Choose a descending sequence of normal subgroups

$$\mathbb{Z}^k \supset n_1 \cdot \mathbb{Z}^k \supset n_1 n_2 \cdot \mathbb{Z}^k \supset n_1 n_2 n_3 \cdot \mathbb{Z}^k \supset \cdots$$

where all integers $n_\ell > 1$. Then $\pi_1(L) \to \mathbb{Z}^k$ induces a chain of normal subgroups

$$\pi_1(L) = \Gamma_0 \subset \Gamma_1 \supset \Gamma_2 \supset \cdots$$

such that $\bigcap_{\ell=1}^{\infty} \Gamma_\ell = \{e\}$. We get a corresponding sequence of finite coverings

$$L = L_0 \supset L_1 \supset L_2 \supset L_3 \supset \cdots$$

Each group Γ_ℓ acts via rotations on $\mathbb{S}^{2k-1} \subset \mathbb{C}^k$, giving representations $\rho_\ell \colon \Gamma_\ell \to \mathbf{SO(2k)}$. The representation ρ_ℓ is deformable to the identity,

$$\rho_{\ell,t} \colon \Gamma_k \to \mathbf{SO(2k)} \ , \ \rho_{\ell,0} = \rho_\ell \ , \ \rho_{\ell,1} = \mathrm{Id}$$

The family $\rho_{\ell,t}$ yields a foliation of $L \times \mathbb{D}^{2k-1} \to L$ which is a product near $L \times \mathbb{S}^{2k-2}$, and in a neighborhood of $L \times \{0\}$ is a flat bundle whose generic leaves are the coverings $L_\ell \to L$. The foliation is a tubular neighborhood of a generic leaf is a product.

This allows us to recursively insert each successive covering L_ℓ. The resulting limit foliation $\mathcal{F}'$ will contain a minimal set diffeomorphic to the solenoid defined by the sequence of covers above.

If the integers $n_\ell \to \infty$ sufficiently fast (eg, $n_\ell = \ell!$) then $\mathcal{F}'$ will be C^r.