Introduction to Matchbox Manifolds

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Report on works with Alex Clark and Olga Lukina

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Suppose that each $x \in \mathfrak{M}$ has an open neighborhood homeomorphic to $(-1,1)^n \times \mathfrak{T}_x$, where $\mathfrak{T}_x$ is a totally disconnected clopen subset of some Polish space $\mathfrak{X}$. $\Longrightarrow$ arc-components are locally Euclidean.

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Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* $\Longleftrightarrow$ $\mathfrak{M}$ admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$ where $\mathfrak{T}_i$ are is a totally disconnected clopen subsets of $\mathfrak{X}$.

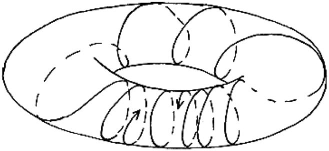
The transition functions are assumed to be C^r, for $1 \leq r \leq \infty$, along leaves, and the derivatives depend (uniformly) continuously on the transverse parameter.

Why Matchbox Manifolds?

L_0 is a connected, complete Riemannian manifold, "marked" with a metric, a net, a tiling, or other local structure.

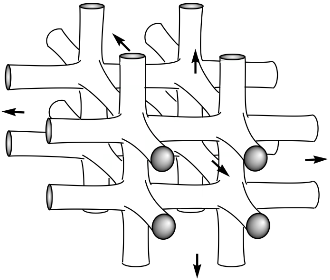
"Compactify" this data by looking for a *continuum* $\mathfrak{M}$ in which L_0 embeds as a leaf of a "foliation" and respecting the local structure.

A simple example which embeds in foliation on $\mathbb{T}^2$:



"Slinky model" gives even more compact model, and is continuum.

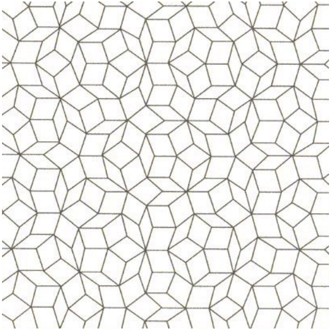
Triply periodic manifold



"Compactifying" gives a foliation of a compact 4-manifold $M = (L \times \mathbb{S}^1)/\mathbb{Z}^3$ with leaf L

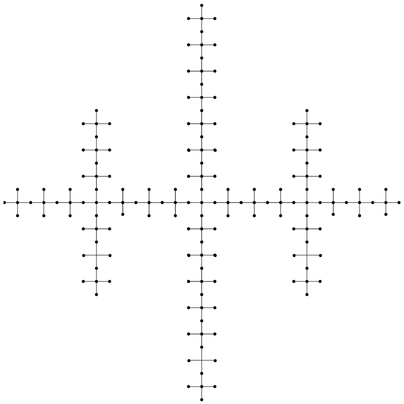
Penrose tiling stripped of decorations

Closure of $\mathbb{R}^2$-translates of the graph below yields a continuum $\mathfrak{M}$ foliated by action of $\mathbb{R}^2$:



Graph closures & Ghys-Kenyon examples

Closure of space of subtrees of given graph, yields a Cantor set with pseudogroup action, which generates a foliated continuum $\mathfrak{M}$.



Definition: An n-dimensional *matchbox manifold* is a continuum $\mathfrak{M}$ which is a smooth foliated space with codimension zero and leaf dimension n. Similar concept to laminations.

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• A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, and the pseudogroup dynamics of the foliation $\mathcal{F}$ on the transverse fibers $\mathfrak{T}_i$ represents *intrinsic* fundamental groupoid.

• They appear in study of tiling spaces, leaves of foliations, graph constructions, inverse limit spaces, pseudogroup actions on totally disconnected spaces, et cetera.

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- Restrict attention to one of two cases:

  $\mathfrak{M}$ is *transitive* if there exists a dense leaf.

  $\mathfrak{M}$ is *minimal* if every leaf in $\mathfrak{M}$ is dense.

Embeddings

If $f : \mathbf{K} \to \mathbf{K}$ is minimal action on Cantor set, then classical problem asks, when does this action arise as the restriction of a C^r-diffeomorphism of $f : \mathbb{S}^1 \to \mathbb{S}^1$?, where $r < 2$? How about as invariant set for some diffeomorphism on N^k where $k \geq 1$?

Solutions to this problem for solenoids modeled on $\mathbb{T}^n$, $n = 1$ by Gambaudo, Tressier, et al in 1990's. For $n \geq 1$ by Clark & Hurder, "Embedding solenoids in foliations", **Topology Appl.**, 2011.

The criteria for embedding depend on the degree of smoothness required. These are very special cases, and problem is wide open.

Homeomorphisms

Let $\mathfrak{M}$ be a matchbox manifold of dimension n.

Lemma: A homeomorphism $\phi\colon \mathfrak{M} \to \mathfrak{M}'$ of matchbox manifolds must map leaves to leaves $\Rightarrow$ is a foliated homeomorphism.

Proof: Leaves of $\mathcal{F} \iff$ path components of $\mathfrak{M}$

Corollary: $\mathbf{Homeo}(\mathfrak{M}) = \mathbf{Homeo}(\mathfrak{M}, \mathcal{F})$ – all homeomorphisms are leaf preserving.

Bing Bling

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Proofs vary in their degrees of "abstractness", suggesting:

Bing Conjecture: Suppose that $\mathfrak{M}$ is homogeneous continuum, and $\mathfrak{M}$ is a matchbox manifold of dimension $n \geq 1$. Then either $\mathfrak{M}$ is homeomorphic to a compact manifold, or to a McCord solenoid.

The Theorem

Theorem: [C & H, 2010] Bing Conjecture is true for all $n \geq 1$.

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Sketch of proof of this, introduces many ideas from foliation and topological dynamical systems of matchbox manifolds – *a.k.a. dynamics of pseudogroups acting on totally disconnected spaces*.

1. detour through weak solenoids
2. dynamics of pseudogroups
3. shape and transverse foliations
4. codings and solenoids
5. automorphisms...

Weak solenoids

Let B_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The p_ℓ are the *bonding maps* for the weak solenoid

$$\mathcal{S} = \varprojlim \; \{p_\ell \colon B_\ell \to B_{\ell-1}\} \;\subset\; \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: $\mathcal{S}$ has natural structure of a matchbox manifold, with every leaf dense.

From Vietoris solenoids to McCord solenoids

Basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(B_\ell, x_\ell)$.

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon B_\ell \longrightarrow B_0$.

Definition: $\mathcal{S}$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of G_{ℓ_0}.

Theorem [McCord 1965] Let B_0 be an oriented smooth closed manifold. Then a McCord solenoid $\mathcal{S}$ is an orientable, homogeneous, equicontinuous smooth matchbox manifold.

Classifying weak solenoids

A weak solenoid is determined by the base manifold B_0 and the tower equivalence of the descending chain

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

Theorem:[Pontragin 1934; Baer 1937] For $G_0 \cong \mathbb{Z}$, the homeomorphism types of McCord solenoids is uncountable.

Theorem:[Kechris 2000; Thomas2001] For $G_0 \cong \mathbb{Z}^k$ with $k \geq 2$, the homeomorphism types of McCord solenoids is not classifiable, *in the sense of Descriptive Set Theory*.

The number of such is not just huge, but indescribably large.

Pseudogroups

Covering of $\mathfrak{M}$ by foliation charts $\Longrightarrow$ transversal $\mathcal{T} \subset \mathfrak{M}$ for $\mathcal{F}$

Holonomy of $\mathcal{F}$ on $\mathcal{T}$ $\Longrightarrow$ compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$:

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$ such that $\langle \Gamma \rangle = \mathcal{G}_{\mathcal{F}} | \mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$, $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Dynamical properties of $\mathcal{F}$ formulated in terms of $\mathcal{G}_{\mathcal{F}}$; e.g.,

$\mathcal{F}$ has no leafwise holonomy if for $g \in \mathcal{G}_{\mathcal{F}}$, $x \in Dom(g)$, $g(x) = x$ implies $g|V = Id$ for some open neighborhood $x \in V \subset \mathcal{T}$.

Topological dynamics

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(h_{\mathcal{I}}) \text{ with } d_{\mathcal{T}}(x, x') < \delta \quad \Longrightarrow \quad d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(c')) < \epsilon$$

Theorem: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is minimal.

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Theorem: Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is minimal.

Theorem: If $\mathfrak{M}$ is a homogeneous matchbox manifold, then the pseudogroup $\mathcal{G}_{\mathcal{F}}$ is equicontinuous.

Topological dynamics of pseudogroups

Can also define and study pseudogroup dynamics which are distal, expansive, proximal, etc.

ADVERT: See

• **Lectures on Foliation Dynamics: Barcelona 2010**
S. H., [2011 arXiv]

• **Dynamics of foliations, groups and pseudogroups**,
P. Walczak, [2004, Birkhäuser, 2004]

Shape theory

The *shape* of a set $\mathfrak{M} \subset \mathcal{B}$ is defined by a co-final descending chain $\{U_\ell \mid \ell \geq 1\}$ of open neighborhoods in Banach space $\mathcal{B}$,

$$U_1 \supset U_2 \supset \cdots \supset U_\ell \supset \cdots \supset \mathfrak{M} \quad ; \quad \bigcap_{\ell=1}^{\infty} U_\ell = \mathfrak{M}$$

Such a tower is called a shape approximation to $\mathfrak{M}$.

Homeomorphism $h \colon \mathfrak{M} \to \mathfrak{M}'$ induces maps $h_{\ell,\ell'} \colon U_\ell \to U'_{\ell'}$ of shape approximations.

Main technical result

Theorem: Let $\mathfrak{M}$ be a transitive matchbox manifold. Then $\mathfrak{M}$ has a shape approximation such that each U_ℓ admits a quotient map $\pi_\ell \colon U_\ell \to B_\ell$ for $\ell \geq 0$ where B_ℓ is a "branched n-manifold", covered by a leaf of $\mathcal{F}$.

The system of induced maps $p_\ell \colon B_\ell \to B_{\ell-1}$ yields an inverse limit space homeomorphic to $\mathfrak{M}$.

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This study is part of sequence of papers by Clark, H. & Lukina:

- *Voronoi tessellations for matchbox manifolds*, July 2011 (arXiv).
- *Shape of matchbox manifolds*, September 2011, to appear.
- *Classification of matchbox manifolds*, 2011, to appear.

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• For $\mathfrak{M}$ with foliation defined by free G-action and tiling on orbits, as in Benedetti & Gambaudo, same as their result.

For general $\mathfrak{M}$, the problem is to find good local product structures, which are stable under transverse perturbation. The leaves are not assumed to have flat structures, so this adds an extra level of difficulty, as compared to the methods in paper of Giordano, Matui, Hiroki, Putnam, & Skau: "Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems", Invent. Math. 179 (2010)

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In terms of *leaf dimensions*, we have the fundamental observation:

$$1 \ll 2 \ll 3 < n$$

Coding orbits

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E. Thomas in 1970 paper for 1-dimensional matchbox manifolds applied these ideas to matchbox manifolds.

Theorem: [Clark & Hurder] Suppose that $\mathcal{F}$ is equicontinuous. Then for all $\epsilon > 0$, there is a decomposition into k_ϵ disjoint clopen sets, for $k_\epsilon \gg 0$,

$$\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{k_\epsilon}$$

such that $diam(\mathcal{T}_i) < \epsilon$ for all i, and the sets $\mathcal{T}_i$ are permuted by the action of $\mathcal{G}_{\mathcal{F}}$.

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We obtain a "good coding" of the orbits of the pseudogroup $\mathcal{G}_{\mathcal{F}}$.

Moreover, the coding respects the inverse limit structure defined by shape approximations.

Theorem: [C & H, 2010] Let $\mathfrak{M}$ be a equicontinuous matchbox manifold. Then $\mathfrak{M}$ is minimal, and homeomorphic to a weak solenoid.

Corollary: Let $\mathfrak{M}$ be a equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to the suspension of an minimal action of a countable group on a Cantor space $\mathbb{K}$.

Homogeneous matchbox manifolds

Definition: A matchbox manifold $\mathfrak{M}$ is *homogeneous* if the group of homeomorphisms of $\mathfrak{M}$ acts transitively.

Theorem: [C & H, 2010] Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then $\mathfrak{M}$ is equicontinuous, minimal, without holonomy; and $\mathfrak{M}$ is homeomorphic to a McCord solenoid.

Corollary: Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to the suspension of an minimal action of a countable group on a Cantor *group* $\mathbb{K}$.

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Theorem: True for $n = 1$, i.e., for oriented flows.

J.M. Aarts and M. Martens, "Flows on one-dimensional spaces", Fund. Math., 131:3958, 1988.

co-Hopfian

Example of Alex Clark shows this is false for $n = 2$!

False even for solenoids built over a surface B_0 of higher genus.

The problem comes up from the fact that covers of the base B_0 need not be homeomorphic to the base.

Problem: Understand equivalence between matchbox manifolds in terms of their holonomy pseudogroups, and other invariants of their dynamics and geometry.

Long way to go...